THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Algebraic Topology

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1. The *cone* CX on a topological space X is the quotient of $X \times [0,1]$ by identifying all the points $(x,1)$ to a single point $*$. Suppose that there exists an open neighborhood U of the point $*$ in CX which is homeomorphic to $\mathbb{R}^n$. What are the possible sequences of groups
$$H_0(X),\ H_1(X),\ H_2(X),\ldots?$$

   Solution: For $n=0$, clearly, X has to be empty. For $n>0$, we have
$$H_k(U, U \smallsetminus \{*\}) \cong H_k(CX, CX \smallsetminus \{*\}) \cong \widetilde{H}_{k-1}(X)$$
(since $CX \smallsetminus \{*\} \simeq X$). The left-hand side is $\mathbb{Z}$ for $k=n$ and 0 else. Thus, X has to have the homology of the $(n-1)$-sphere. So the only possible sequence for $n=0$ is
$$0, 0, 0, \ldots,$$
for $n=1$, it is
$$\mathbb{Z} \oplus \mathbb{Z}, 0, 0 \ldots,$$
for $n=2$, it i
$$\mathbb{Z}, \mathbb{Z}, 0, \ldots,$$
etc.

2. Let F_n be the free group on n generators. For which $n = 1, 2, \ldots$ does there exist an injective homomorphism $h : F_n \to F_2$? For which $n = 1, 2, \ldots$ can h be chosen so the image has finite index in F_2?

   Solution: Yes on both counts for $n \geq 2$. We need to find a covering space with fundamental group F_n of the graph with a single vertex and two edges. Any $(n-1)$-fold connected covering would do. For $n=1$, the subgroup clearly exists, but not of finite index (since a covering has to have degree > 0).

3. Let X be a path-connected space with basepoint $*$. Let $x \neq y \in X$ be points, and let $Y = X/x \sim y$.

   (a) Is the homomorphism $p_* : \pi_1(X, *) \to \pi_1(Y, *)$ induced by the projection $p : X \to Y$ necessarily injective?

(b) Can p_* be an isomorphism?

Solution: One proves that π_1 remains the same if we replace Y by the space Y' obtained by attaching $[0,1]$ by identifying $0,1$ with x,y, respectively. Then $\pi_1(Y,*)$ is the free product of $\pi_1(X,*)$ with $\mathbb{Z}$ where p_* is the injection to the first factor. Thus, p_* is always injective, and never an isomorphism.

4. Describe all the homotopy equivalence classes of CW-complexes which have exactly three cells in dimension $0, 1, n$, respectively, for a given $n > 1$.

   Solution: The 1-skeleton is S^1. For $n = 2$, the possibly different homotopy types arise by different choices of the homotopy class of the 2-cell attaching map, which is classified by degree $\in \mathbb{Z}$. However, degree $n \in \mathbb{Z}$ is identified with $-n$ by reversing the orientation of the cell. Thus, we get different homotopy equivalence classes for attaching maps of degrees $0, 1, 2, \ldots$ (which are proved different using the first homology group).

   For $n > 2$, the attaching map is $S^{n-1} \to S^1$, which is homotopic to a constant map (since it lifts to the universal cover of S^1, which is contractible). Thus, there is only one class.

5. Compute the homology of the space $\mathbb{R}P^3/\mathbb{R}P^1$ (where $\mathbb{R}P^1$ is embedded in the standard way).

   Solution: By using the long exact sequence in homology, we get homology groups $\mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}$. (One can also see directly that the quotient space is homotopy equivalent to $S^2 \vee S^3$.)