Algebra 2 QR Solutions — August 2023

Problem 1. Let

$$1 \to A \xrightarrow{\alpha} G \xrightarrow{\beta} B \to 1$$

be a short exact sequence of groups, with A and B abelian. Suppose that $\alpha(A)$ is central in G, and let h be an element of G. Show that $g \mapsto hgh^{-1}g^{-1}$ is a group homomorphism from G to G.

Solution. We may as well identify A with its image, and thus regard it as a cenral subgroup of G. Fix $h \in G$ and let $\phi(g) = hgh^{-1}g^{-1}$. Since B is abelian, the image of $\phi(g)$ in B is trivial, meaning that $\phi(g)$ actually belongs to A. We have

$$\phi(gg') = hgg'h^{-1}(g')^{-1}g^{-1} = (hgh^{-1})\phi(g')g^{-1} = \phi(g)\phi(g'),$$

where in the final step we commuted $\phi(g')$ with g^{-1}, which is allowed since $\phi(g') \in A$ is central.

Problem 2. Let r, s and t be positive integers, and let G be the group generated by elements a and b modulo the relations $a^r = b^s = 1$, $aba^{-1} = b^t$. Show that G is finite.

Solution. An element of G is represented by a word in a and b (we do not need inverses since a and b have finite order). The second relation can be rewritten as $ab = b^t a$, which shows that we can move all a's to the right, that is, every element has the form $b^i a^j$. By the condition on the orders of a and b, we can take $0 \le i < r$ and $0 \le j < s$. Thus G is finite.

Problem 3. Let G be a group of order $4 \cdot 3^n$. Show that G is solvable.

Solution. The number of 3-Sylows divides 4 and is 1 mod 3, so is therefore 1 or 4. If there is a unique 3-Sylow N then it is normal and solvable (since it is a p-group), and G/N is also solvable (since it has order 4), and so G is solvable.

Suppose that there are four 3-Sylows. The conjugation action of G on the set of 3-Sylows defines a homomorphism $f\colon G \to S_4$. The kernel of f cannot contain any 2-Sylow, for then it would normalize all 3-Sylows and they would be normal. So $\ker(f)$ has order 3^m or $2 \cdot 3^m$. If $\ker(f)$ has order 3^m then it is a p-group, and thus solvable. If it has order $2 \cdot 3^m$ then its 3-Sylow has index 2 and is thus normal, and so $\ker(f)$ is solvable (as in the first paragraph). Since $\operatorname{im}(f)$ is also solvable (as S_4 is solvable), it follows that G is solvable.

Problem 4. Let Ω/F be a field extension, let E_1 and E_2 be distinct subfields of Ω containing F with $[E_1 : F] = [E_2 : F] = d$, and let K be the subfield of Ω generated by E_1 and E_2. Show that $2d \le [K : F] \le d^2$, and give examples where the extreme values $2d$ and d^2 each occur.

Solution. Since E_1 and E_2 are algebraic extensions of F, every element of K can be written in the form $\sum_{i=1}^{i} a_i b_i$ with $a_i \in E_1$ and $b_i \in E_2$. It follows that an F-basis for E_2 will span K as an E_1-vector space, i.e., $[K : E_1] \leq [E_2 : F] = d$. Multiplying by $[E_1 : F] = d$ and using the tower law for degrees, we find $[K : F] \leq d^2$. On the other hand, K is a proper extension of E_1 (since E_1 and E_2 are distinct), and so $[K : F] = [K : E_1][E_1 : F] = ed$, where $e = [K : E_1] > 1$. Thus $[K : F] \geq 2d$.

Suppose $F = \mathbb{C}(x, y)$ and $E_1 = \mathbb{C}(x^{1/d}, y)$ and $E_2 = \mathbb{C}(x, y^{1/d})$; these are degree d extensions of F. In this case, $K = \mathbb{C}(x^{1/d}, y^{1/d})$ is a degree d^2 extension of F.

Next, let K/F be a Galois extension with Galois group the dihedral group of order $2d$. For example, one can take $K = \mathbb{C}(x^{1/d})$ and $F = \mathbb{R}(x)$. If E_1 and E_2 are the fixed fields of two different reflections then they are degree d extensions that generate K, which has degree $2d$.

Problem 5. Let p be an odd prime. Let K be a subfield of $\mathbb{C}$ that is Galois over $\mathbb{Q}$ of degree p^n. Show that $K \subset \mathbb{R}$.

Solution. Since K is Galois it is stable under complex conjugation c. Since $c|_K$ is an element of $\mathrm{Gal}(K/\mathbb{Q})$ that squares to the identity and this group has odd order, it follows that $c|_K$ is already the identity. Thus every element of K is fixed by c, and so $K \subset \mathbb{R}$.