Algebra I QR August 2024

Problem 1. Let R be a commutative ring with 1. Let $R^\star \subset R$ be the set of invertible elements and $\mathfrak{m} := R \setminus R^\star$.
(1) Show that if $\mathfrak{m}$ is an abelian group, then it is the unique maximal ideal of R.
(2) Conversely, suppose that R has a unique maximal ideal. Show that this maximal ideal is equal to $\mathfrak{m}$.

Problem 2. Let V denote the vector space of real polynomials ax^2+bx+c of degree less than or equal to 2. Define

$$(p(x), q(x)) = (p(x)q(x))'|_{x=0}.$$

Here $f(x)'$ denotes the derivative of f. Check that $(.,.)$ is a symmetric bilinear form, find its signature, and find an orthogonal basis for $(.,.)$.

Problem 3. How many elements does each of the following groups have?

(1) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/20\mathbb{Z})$
(2) $(\mathbb{Z}/3\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$
(3) $(\mathbb{Z} \times \mathbb{Z})/M$, where M is the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $(3, 2)$ and $(2, 5)$

Problem 4.
(1) Let $\mathbb{F}_2$ denote the field with two elements. For $(a, b) \in \mathbb{F}_2 \times \mathbb{F}_2$, define the ring

$$R_{a,b} := \mathbb{F}_2[x]/(x^2 + ax + b).$$

For which distinct pairs (a, b) and (c, d) do we have a ring isomorphism $R_{a,b} \cong R_{c,d}$? Which of these rings are fields? Which of these rings are integral domains?
(2) For each of the rings in (1), list all the prime ideals.

Problem 5. Let F be a field and V be a vector space of dimension n over F. For $1 \leq k \leq n$, consider the set

$$X_k := \{(W, U) \mid W, U \subset V \text{ and } \dim(W) = k = \dim(U)\}$$

of ordered pairs of k-dimensional subspaces of V.
(1) The diagonal action of $\mathrm{GL}_n(F)$ on X_k is given by $g \cdot (W, U) = (g \cdot W, g \cdot U)$, for $g \in \mathrm{GL}_n(F)$. How many orbits are there of the diagonal action of $\mathrm{GL}_n(F)$ on X_k?
(2) Suppose that $F = \mathbb{F}_q$ is a finite field with q elements. What is the cardinality of X_k?