THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Algebraic Topology
Solutions

August 2024

1. Let S^3 be the one point compactification of $\mathbb{R}^3$ and let $T \subset S^3$ be the subset consisting of all points $(x, y, 0) \in \mathbb{R}^3$ where $x^2 + y^2 = 1$. Let X be the quotient of S^3 under the smallest equivalence relation which identifies $(x, y, 0) \in T$ with $(y, -x, 0)$. Calculate the homology of X.

   Solution: The space X is a homotopy pushout

$$\begin{array}{ccc} T & \xrightarrow{\subset} & S^3 \\ {\scriptstyle g}\downarrow & & \\ T & & \end{array}$$

   where g is a map of degree 4. Using the Mayer-Vietoris sequence in reduced homology, we see that $\widetilde{H}_1 X = \mathbb{Z}/4$, $\widetilde{H}_3 X = \mathbb{Z}$ while the other homology groups are 0. For unreduced homology, add $\mathbb{Z}$ in dimension 0.

2. Let F_n denote the free group on n elements.

   (a) For what pairs $(m, n) \in \mathbb{N}^2$ does there exist a subgroup of F_n isomorphic to F_m?

   (b) For what pairs $(m, n) \in \mathbb{N}^2$ does there exist a subgroup of F_n of finite index isomorphic to F_m?

   Solution: If $n = 1$, then $F_1 = \mathbb{Z}$, so we must have $m = 1$. We have $\chi(F_n) = n - 1$. For a subgroup G of index k, $\chi(G) = k(n-1)$, so $m = k(n-1) + 1$, $k \in \mathbb{N}$, thus answering (a). For (b), if $n > 1$, we can find a subgroup F_N of F_n of finite index with $N > m$, hence there is a subgroup isomorphic to F_m for all $m \in \mathbb{N}$

3. The *join* $X * Y$ of two topological spaces X, Y is defined as the quotient of

$$X \times Y \times [0, 1]$$

   by the smallest equivalence relation identifying $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$. Let A, B be two compact CW-complexes. Denoting by χ the Euler characteristic, find a formula expressing $\chi(A * B)$ in terms of $\chi(A)$, $\chi(B)$.

Solution: The cells of $A * B$ correspond to cells of A, cells of B, and cells of $A \times B$ shifted in dimension up by 1. Thus, the answer is

$$\chi(A * B) = \chi(A) + \chi(B) - \chi(A)\chi(B).$$

4. Let S, T be two disjoint images of the unit circle

$$\{(x, y, z) \mid x^2 + y^2 = 1,\ z = 0\}$$

under isometries $\mathbb{R}^3 \to \mathbb{R}^3$. Can the isomorphism class of the fundamental group $\pi_1(\mathbb{R}^3 \smallsetminus (S \cup T))$ depend on the reative position of S and T with respect to each other? Explain.

Solution: Since π_1 only depends on the 2-skeleton, we may replace $\mathbb{R}^3$ by its one-point compactification S^3. If S and T are unlinked, as

$$S = \{(x, y, z) \mid x^2 + y^2 = 1,\ z = 0\},\ T = \{(x, y, z) \mid (x - 3)^2 + y^2 = 1,\ z = 0\},$$

then $\pi_1(\mathbb{R}^3 \smallsetminus (S \cup T))$ is the free group on two elements by the Seifert-VanKampen theorem. If they are linked as in

$$S = \{(x, y, z) \mid x^2 + y^2 = 1,\ z = 0\},\ T = \{(x, y, z) \mid (x - 1)^2 + z^2 = 1,\ y = 0\},$$

then $S^3 \smallsetminus (S \cup T)$ is homotopy equivalent to a torus, so its fundamental group is $\mathbb{Z} \times \mathbb{Z}$. Thus, it does depend.

5. Descibe the universal covering of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

Solution: The fundamental group is $\mathbb{Z}/2 * \mathbb{Z}/2$ by the Seifert-VanKampen theorem, i.e. the infinite dihedral group. Therefore, a model of the universal covering is for example

$$\{(x, y, z) \mid (x - 2k)^2 + y^2 + z^2 = 1,\ k \in \mathbb{Z}\}.$$