AIM Qualifying Review Exam: Probability and Discrete Mathematics

August 16, 2024

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

The complete undirected graph, K_n, on n vertices has edges between all unordered pairs of distinct vertices.

(a) Suppose each edge of K_n is colored either maize or blue with equal probability. In terms of j and n, what is the probability that some j-clique is monochromatic (all edges are the same color)?

(b) Explain why this shows that, for even somewhat large n as a function of j, there is a coloring of K_n in which no j-clique is monochromatic.

(c) Estimate the threshold value of such n to within a constant factor when $1 \ll j \ll n$, using $\binom{n}{r} \approx \frac{n^r}{r!}$ and $r! \approx \sqrt{2\pi r}(r/e)^r$, that hold in this regime of j and n.

Problem 1

Problem 1

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Problem 2

Suppose that if you are s minutes early for an appointment you incur cost cs and if you are s minutes late then you incur ks. Suppose the travel time is a continuous random variable with density f (so $\int_{-\infty}^{+\infty} f(t)\,dt = 1$). Determine (in terms of f and/or $F(s) = \int_{-\infty}^{s} f(t)dt$ and their inverses) the time to leave to minimize your expected cost.

Problem 2

Problem 2

Problem 2

<u>Problem 3</u> A collection of subsets of $\{1, 2, \ldots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets.

Problem 3

Problem 3

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Problem 4

Solve the recurrence relation

$$h_n = \begin{cases} 5h_{n-1} - 6h_{n-2} + 1, & n \geq 2 \\ 1, & n = 0, 1. \end{cases}$$

Partial credit awarded for a solution that simplifies the first line to $h_n = 5h_{n-1} - 6h_{n-2}$.

Problem 4

Problem 4

Problem 4

Problem 5

An *inversion* in a sequence $(a_1, a_2, a_3, \ldots)$ is a pair (i, j) such that $i < j$ but $a_i > a_j$. For example, in $(10, 30, 40, 20)$, the pair $(2, 4)$, for entries $(30, \ldots, 20)$, is an inversion. For simplicity, assume all sequence elements are distinct in what follows.

(a) For general n, find a sequence of length n with $\Omega(n^2)$ inversions. (That is, the number of inversions is at least proportional to n^2.)

(b) Sketch an algorithm that counts the number of inversions in a sequence of length n and runs in time $O(n \log n)$. Sketch a proof of correctness and runtime. *(Hint: modify the MERGESORT algorithm, that sorts the left and right halves recursively and then merges the two sorted lists.)*

(c) Why can we assume that the sequence length is a power of 2? (Or easily reduce the problem with general sequence lengths to the problem where sequence lengths are a power of 2 -?)

Problem 5

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