Applied Functional Analysis QR Exam

August 14, 2024

Problem 1

Show that for every finite $L > 0$, the nonlinear integral equation

$$u(x) = 1 + \frac{1}{\pi} \int_{-L}^{L} \frac{\sin(u(y))\,\mathrm{d}y}{1 + (x-y)^2}, \quad x \in [-L, L]$$

has a unique solution in the class of bounded and continuous functions.

Problem 2

Determine whether the linear integral equation

$$u(x) = f(x) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(y)\,\mathrm{d}y}{1 + (x-y)^2}$$

has a solution $u \in L^2(\mathbb{R})$ for every given $f \in L^2(\mathbb{R})$ and prove it. If it does not have a solution for every $f \in L^2(\mathbb{R})$ determine necessary and sufficient additional conditions on f for there to exist a solution, and determine whether the solution for such f is unique.

Problem 3

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a test function, and for positive numbers $a, b > 0$, define

$$F_{a,b}[\varphi] := \int_{-\infty}^{-a} \frac{\varphi(x)\,\mathrm{d}x}{x} + \int_{-a}^{b} \frac{\varphi(x) - \varphi(0)}{x}\,\mathrm{d}x + \int_{b}^{+\infty} \frac{\varphi(x)\,\mathrm{d}x}{x}.$$

Prove that this defines a distribution in $\mathcal{D}'(\mathbb{R})$. Express it in terms of the principal value distribution $\mathrm{PV}\frac{1}{x}$ and any other necessary distributions.

Problem 4

Let $n = 1, 2, 3, \ldots$ be a parameter, and set $S_n := \mathrm{span}\{\sin(x), \sin(2x), \ldots, \sin(nx)\}$. If $g \in S_n \subset L^2(0, \pi)$ is the function closest in norm to $f \in L^2(0, \pi)$, find in *closed form* (no sums) the kernel $K_n(x, y)$ such that

$$g(x) = \int_0^{\pi} K_n(x, y) f(y)\,\mathrm{d}y.$$

Use this representation to prove that the right-hand side of this formula defines a self-adjoint operator P_n on $L^2(0, \pi)$. What identity would need to be satisfied by $K_n(x, y)$ to guarantee that P_n is a projection operator? Explain why $K_n(x, y)$ satisfies this identity.

Problem 5

Let $S \subset L^1(0, 1)$ be a closed subspace. Given $f \in L^1(0, 1)$, is there a unique element $g \in S$ closest in norm to f? If so, prove it. If not, construct a counterexample.