# AIM Qualifying Review Exam in Advanced Calculus & Complex Variables

*August 2025*

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. If you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

**Problem 1** Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$|f(x) - f(y)| \leq (\cos(x-y))^2 \quad \forall\, x, y \in \mathbb{R}.$$

(a) Show that $f$ is $\pi/2$-periodic, i.e., $f(x + \pi/2) = f(x)$ for all $x$.

(b) Now show that $f$ is constant. (Hint: differentiate.)

**Solution**

(a) By assumption, $f(x+h) = f(x)$ if $\cos h = 0$, i.e., if $h = (2n+1)\frac{\pi}{2}$ for $n \in \mathbb{Z}$. In particular, $f$ is $\pi/2$-periodic.

(b) Since $f$ is $\pi/2$-periodic, and given the assumption,

$$|f(x+h) - f(x)| = |f(x + \frac{\pi}{2} + h) - f(x)| \leq \Big(\cos(\frac{\pi}{2} + h)\Big)^2 = O(h^2)$$

as $h \to 0$. This holds for all fixed $x$, hence $f' = 0$.

**Problem 2** Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth and convex, meaning that

$$\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \xi_i \xi_j \geq 0$$

for all $x, \xi \in \mathbb{R}^n$.

(a) Integrate $\frac{d}{dt}[\nabla f\,(tx + (1-t)y)]$ to prove that

$$(\nabla f(x) - \nabla f(y)) \cdot (x - y) \geq 0 \quad \forall\, x, y \in \mathbb{R}^n.$$

(b) Assume now that

$$(\nabla f(x) - \nabla f(y)) \cdot (x - y) = 0 \quad \forall\, x, y \in \mathbb{R}^n.$$

What can you say about $f$?

**Solution**

(a) By the chain rule,

$$\frac{d}{dt}\nabla f\,((1-t)x + ty) = \nabla\nabla f\,((1-t)x + ty)\,(y - x).$$

Integrating from $t = 0$ to 1 and dotting against $y - x$ gives

$$(\nabla f(y) - \nabla f(x)) \cdot (y - x) = \int_0^1 (y - x) \cdot \nabla\nabla f\,((1-t)x + ty)\,(y - x)\,dt.$$

The integrand is non-negative by assumption, so we're done.

(b) $f$ is affine, meaning that $f(x) = a \cdot x + b$ for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Indeed, the additional assumption gives that

$$0 = \int_0^1 (y - x) \cdot \nabla\nabla f\,((1-t)x + ty)\,(y - x)\,dt$$

for all $x, y$, and as the integrand is non-negative it vanishes identically. In particular,

$$\xi \cdot \nabla\nabla f(x)\xi = 0$$

for all $x, \xi \in \mathbb{R}^n$. Hence, $\nabla\nabla f = 0$.

**Problem 3** Let $\Gamma \subset \mathbb{C} \setminus \{-1, 0\}$ be a simple closed contour. Find all possible values of the integral

$$\oint_\Gamma \frac{e^{10z}}{z^2(z+1)}\,dz$$

using a counterclockwise orientation.

**Solution**

Apply Cauchy's residue theorem. The integrand has a simple pole at $-1$ and a double pole at 0. To get the residues, expand about the poles. For $-1$, write that

$$\begin{aligned}\frac{e^{10z}}{z^2(z+1)} &= \frac{1}{z+1}\left[\frac{e^{10z}}{z^2}\Big|_{z=-1} + O(z+1)\right]\\ &= \frac{e^{-10}}{z+1} + \cdots\end{aligned}$$

where the dots are the regular part of the expansion. The residue is $e^{-10}$. For 0, write that

$$\begin{aligned}\frac{e^{10z}}{z^2(z+1)} &= \frac{1}{z^2}\left[\frac{1+10z+O(z^2)}{1+z}\right] \\ &= \frac{1}{z^2}\left[1+10z+O(z^2)\right]\left[1-z+O(z^2)\right] \\ &= \frac{1}{z^2}+\frac{9}{z}+\cdots\end{aligned}$$

where again the dots are the regular part. The residue is 9. Finally, we deal with winding. Since it is simple, the contour $\Gamma$ winds around each pole at most once. There are four possibilities: (i) $\Gamma$ winds around no poles; (ii) $\Gamma$ winds around $-1$ but not around 0; (iii) $\Gamma$ winds around 0 but not around $-1$; (iv) $\Gamma$ winds around both poles. According to the residue theorem,

$$\oint_\Gamma \frac{e^{10z}}{z^2(z+1)}\,dz = 2\pi i \sum_k r_k$$

where the sum is over the residues $r_k$ of the poles about which $\Gamma$ winds. Therefore, finally, the integral in question can take on the following values:

$$0, \quad 2\pi i \cdot e^{-10}, \quad 2\pi i \cdot 9 = 18\pi i, \quad 2\pi i e^{-10} + 18\pi i.$$

**Problem 4** Given a continuous and compactly supported function $f : \mathbb{R} \to \mathbb{R}$, define

$$F(z) = \int_{\mathbb{R}} e^{izt} f(t)\,dt.$$

(a) Find the Taylor expansion of $F$ about $z = 0$. Be sure to justify all steps.

(b) Prove that $f$ is identically zero if and only if $F$ is identically zero.

**Solution**

(a) One way to do this is to use the Taylor expansion

$$e^{izt} = \sum_{n=0}^{\infty} \frac{(izt)^n}{n!}$$

in the definition of $F$. We can exchange the orders of summation and integration since the expansion converges uniformly on compact sets, and because $f$ is compactly supported. Thus,

$$F(z) = \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{(izt)^n}{n!} f(t)\,dt = \sum_{n=0}^{\infty} a_n z^n$$

where

$$a_n = \frac{i^n}{n!}\int_{\mathbb{R}} t^n f(t)\,dt.$$

(b) Evidently, if $f$ is zero then so is $F$. To prove the converse, let $F$ be zero and note that its Taylor coefficients vanish: $a_n = 0$ for all $n$. But then $0 = \int_{\mathbb{R}} t^n f(t)\,dt$ for all $n$, hence $0 = \int pf$ for all (real) polynomials $p$. The only way this can happen is if $f$ is identically zero. Indeed, $f$ has compact support by assumption, meaning that we can apply the Weierstrass approximation theorem to find a sequence of polynomials $p_n \to f$ uniformly on its support. Then,

$$\int f^2 = \lim_{n\to\infty} \int f p_n = 0$$

so $f = 0$ as claimed.

**Problem 5** This problem walks you through the proof of the identity

$$\frac{1}{(\sin z)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-\pi n)^2}. \tag{1}$$

(a) Check that the series on the right-hand side of (1) converges absolutely and locally uniformly on $\mathbb{C} \setminus \pi\mathbb{Z}$. This shows that it defines a meromorphic function on $\mathbb{C}$.

(b) Show that the difference

$$D(z) = \frac{1}{(\sin z)^2} - \sum_{n=-\infty}^{\infty} \frac{1}{(z-\pi n)^2}$$

is $\pi$-periodic, i.e.,

$$D(x+iy) = D(x+\pi+iy) \quad \forall\, x, y \in \mathbb{R}.$$

(c) Show that $D$ is entire.

(d) Finally, apply Liouville's theorem to conclude that $D = 0$.

**Solution**

(a) For large enough $n$ depending only on $z$ we have that

$$\left|\frac{1}{z-\pi n}\right| \le \frac{1}{n}.$$

This holds in particular if $n \ge (\pi-1)^{-1}|z|$. Since $\sum \frac{1}{n^2} < \infty$ we are done.

(b) By definition,

$$\begin{aligned}(\sin(z+\pi))^2 &= \left(\frac{e^{iz}e^{i\pi} - e^{-iz}e^{-i\pi}}{2i}\right)^2 = e^{2i\pi}\left(\frac{e^{iz} - e^{-iz}e^{-2i\pi}}{2i}\right)^2 \\ &= \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = (\sin z)^2.\end{aligned}$$

Also,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+\pi-\pi n)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z+\pi-\pi(n+1))^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-\pi n)^2}.$$

Thus $D$ is $\pi$-periodic.

(c) The two terms in the definition of $D$ are meromorphic, so we only need to check that the singular parts of their Laurent expansions agree. Since $D$ is $\pi$-periodic, it suffices to expand about $z = 0$. Note

$$\frac{1}{(\sin z)^2} = \frac{1}{\left(z - \frac{z^3}{6} + \cdots\right)^2} = \frac{1}{z^2} \frac{1}{\left(1 - \frac{z^2}{6} + \cdots\right)^2}$$
$$= \frac{1}{z^2} \frac{1}{1 - \frac{z^2}{3} + \cdots} = \frac{1}{z^2}\left[1 + \frac{z^2}{3} + \cdots\right] = \frac{1}{z^2} + f(z)$$

where $f(z) = \frac{1}{(\sin z)^2} - \frac{1}{z^2} = \frac{1}{3} + \cdots$ is analytic near $z = 0$. Also,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z - \pi n)^2} = \frac{1}{z^2} + g(z)$$

where $g(z) = \sum_{n \neq 0} \frac{1}{(z - \pi n)^2}$ is analytic near $z = 0$. Therefore $D = f - g$ is analytic near $z = 0$.

(d) We just showed that $D$ is entire. We must now show that it is bounded and therefore constant by Liouville's theorem; we also need to show that this constant is zero. Since $D$ is $\pi$-periodic both goals are achieved by showing that

$$\max_{x \in [\frac{-\pi}{2}, \frac{\pi}{2}]} D(x + iy) \to 0 \quad \text{as } |y| \to \infty.$$

We revisit the convergence proof from (a). Let $z = x + iy$ satisfy $x \in [-\pi/2, \pi/2]$ and divide the sum by whether $n \geq (\pi - 1)^{-1}|z|$ or not. This yields

$$\sum_{n=-\infty}^{\infty} \left|\frac{1}{z - \pi n}\right|^2 = \sum_{n < (\pi-1)^{-1}|z|} \left|\frac{1}{z - \pi n}\right|^2 + \sum_{n \geq (\pi-1)^{-1}|z|} \left|\frac{1}{z - \pi n}\right|^2$$
$$\leq \frac{(\pi - 1)^{-1}|z|}{\min_{n \in \mathbb{Z}} |z - \pi n|^2} + \sum_{n \geq (\pi-1)^{-1}|z|} \frac{1}{n^2}$$
$$\leq \frac{(\pi - 1)^{-1}\sqrt{\pi^2/4 + y^2}}{|y|^2} + \sum_{n \geq (\pi-1)^{-1}|y|} \frac{1}{n^2} \to 0$$

uniformly in $x \in [-\pi/2, \pi/2]$ as $|y| \to \infty$. Note in the last line we used the inequalities

$$|y| \leq |z| = \sqrt{x^2 + y^2} \leq \sqrt{\frac{\pi^2}{4} + y^2} \quad \text{and} \quad |z - \pi n|^2 = (x - \pi n)^2 + y^2 \geq y^2$$

to simplify.