AIM Qualifying Review Exam in Advanced Calculus & Complex Variables

August 2025

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. If you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1 Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$|f(x) - f(y)| \leq (\cos(x - y))^2 \quad \forall\, x, y \in \mathbb{R}.$$

(a) Show that f is $\pi/2$-periodic, i.e., $f(x + \pi/2) = f(x)$ for all x.

(b) Now show that f is constant. (Hint: differentiate.)

Problem 1

Problem 1

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Problem 2 Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth and convex, meaning that

$$\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\xi_i \xi_j \geq 0$$

for all $x, \xi \in \mathbb{R}^n$.

(a) Integrate $\frac{d}{dt}[\nabla f\,(tx + (1-t)y)]$ to prove that

$$(\nabla f(x) - \nabla f(y)) \cdot (x - y) \geq 0 \quad \forall\, x, y \in \mathbb{R}^n.$$

(b) Assume now that

$$(\nabla f(x) - \nabla f(y)) \cdot (x - y) = 0 \quad \forall\, x, y \in \mathbb{R}^n.$$

What can you say about f?

Problem 2

Problem 2

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Problem 3 Let $\Gamma \subset \mathbb{C} \setminus \{-1, 0\}$ be a simple closed contour. Find all possible values of the integral

$$\oint_\Gamma \frac{e^{10z}}{z^2(z+1)}\, dz$$

using a counterclockwise orientation.

Problem 3

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Problem 4 Given a continuous and compactly supported function $f : \mathbb{R} \to \mathbb{R}$, define

$$F(z) = \int_{\mathbb{R}} e^{izt} f(t)\, dt.$$

(a) Find the Taylor expansion of F about $z = 0$. Be sure to justify all steps.

(b) Prove that f is identically zero if and only if F is identically zero.

Problem 4

Problem 4

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Problem 5 This problem walks you through the proof of the identity

$$\frac{1}{(\sin z)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z - \pi n)^2}. \tag{1}$$

(a) Check that the series on the right-hand side of (1) converges absolutely and locally uniformly on $\mathbb{C} \setminus \pi\mathbb{Z}$. This shows that it defines a meromorphic function on $\mathbb{C}$.

(b) Show that the difference

$$D(z) = \frac{1}{(\sin z)^2} - \sum_{n=-\infty}^{\infty} \frac{1}{(z - \pi n)^2}$$

is π-periodic, i.e.,

$$D(x + iy) = D(x + \pi + iy) \quad \forall\, x, y \in \mathbb{R}.$$

(c) Show that D is entire.

(d) Finally, apply Liouville's theorem to conclude that $D = 0$.

Problem 5

Problem 5

Problem 5