AIM Qualifying Review Exam in Advanced Calculus & Complex Variables

August 2024

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

A function $f : \mathbb{C} \to \mathbb{R}$ is said to satisfy the mean value property if for every $z_0 \in \mathbb{C}$, and *every* $r > 0$, we have

$$f(z_0) = \frac{1}{\pi r^2} \int_{B_r(z_0)} f(z)\, dA(z),$$

where $B_r(z_0) = \{z \in \mathbb{C} \,|\, |z - z_0| < r\}$, the disk of radius r about z_0. In other words, $f(z_0)$ is the average of the values of f over any disk in $\mathbb{C}$ centered at z_0. Suppose that f satisfies the mean value property, and in addition is $\mathcal{C}^\infty$ on $\mathbb{C}$ (such high regularity is not really necessary). Show that f satisfies

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

on all of $\mathbb{C}$.

[20]

Problem 1

Problem 1

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Problem 2

Consider the analytic function

$$f(z) = \frac{1}{2}(z + \frac{1}{z}),$$

which is defined and analytic on $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Show that f maps the exterior of the unit circle $\{z \in \mathbb{C} \mid |z| > 1\}$ to the region $\mathbb{C} \setminus [-1, 1]$ in a one-to-one manner, and calculate its inverse function.

[20]

Problem 2

Problem 2

Problem 2

Problem 3

Consider the surface $S \subset \mathbb{R}^3$ given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid q(x, y, z) = x^2 + 2xy + 3y^2 + 2xz + 2z^2 = 10\}.$$

(a) Let $f(x, y, z)$ be the linear function

$$f(x, y, z) = 4x + 7y + 5z + 3.$$

Using the method of Lagrange multipliers, find the maximum of f *restricted to* S.

[15]

(b) Explain geometrically why the method works.

[5]

Problem 3

Problem 3

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Problem 4

Define the n-th Laguerre polynomial $L_n(x)$ as

$$L_n(x) = \frac{1}{n!}(\frac{d}{dx} - 1)^n x^n, \text{ for } x \in \mathbb{R},$$

so the first few are given by

$$\begin{aligned} L_0(x) &= 1, \\ L_1(x) &= -x + 1, \\ L_2(x) &= \tfrac{1}{2}x^2 - 2x + 1, \end{aligned}$$

and so on.

Show that if a continuous function $f : [0, 1] \to \mathbb{R}$ satisfies

$$\int_0^1 L_n(x)\, f(x)\, dx = 0, \text{ for all } n,$$

then $f \equiv 0$ on [0,1].

[20]

Problem 4

Problem 4

Problem 4

Problem 5

Consider the curve (circle) $C \subset \mathbb{C}$ given by $|z - z_0| = 1$.

(a) Let $\phi(\zeta), \zeta \in C$, be a differentiable function, not necessarily analytic.

Show that the function

$$\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(\zeta)}{\zeta - z}\, d\zeta,$$

defined for $z \notin C$, is analytic in z.

[15]

(b) If $\phi(\zeta)$ has an extension to all of $\mathbb{C}$ as an analytic function, call it still $\phi(z)$, what is $\Phi(z)$?

[5]

Problem 5

Problem 5

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