AIM Qualifying Review Exam in Advanced Calculus and Complex Variables

August 2024

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

A function $f : \mathbb{C} \to \mathbb{R}$ is said to satisfy the mean value property if for every $z_0 \in \mathbb{C}$, and *every* $r > 0$, we have

$$f(z_0) = \frac{1}{\pi r^2} \int_{B_r(z_0)} f(z)\, dA(z),$$

where $B_r(z_0) = \{z \in \mathbb{C} \,|\, |z - z_0| < r\}$, the disk of radius r about z_0. In other words, $f(z_0)$ is the average of the values of f over any disk in $\mathbb{C}$ centered at z_0. Suppose that f satisfies the mean value property, and in addition is $\mathcal{C}^\infty$ on $\mathbb{C}$ (such high regularity is not really necessary). Show that f satisfies

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

on all of $\mathbb{C}$.

[20]

Solution

Since f is $\mathcal{C}^2$, we can use Taylor's theorem at order two:

$$\begin{aligned} f(z) &= f(z_0) + \tfrac{\partial f}{\partial x}(z_0)(x - x_0) + \tfrac{\partial f}{\partial y}(z_0)(y - y_0) \\ &\quad + \tfrac{1}{2}\left(\tfrac{\partial^2 f}{\partial x^2}(z_0)(x - x_0)^2 + 2\tfrac{\partial^2 f}{\partial x \partial y}(z_0)(x - x_0)(y - y_0) + \tfrac{\partial^2 f}{\partial y^2}(z_0)(y - y_0)^2\right) \\ &\quad + o(|z - z_0|^2). \end{aligned}$$

Average this equation over $B_r(z_0)$ and use the mean value property to get

$$f(z_0) = f(z_0) + \frac{1}{\pi r^2} \int_{B_r(z_0)} \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(z_0)(x - x_0)^2 + \frac{\partial^2 f}{\partial y^2}(z_0)(y - y_0)^2\right) dA(z) + o(r^2),$$

where the integrals of odd functions of $x - x_0$ or $y - y_0$ over $B_r(z_0)$ have given 0. Thus,

$$0 = \frac{1}{\pi r^2}(\frac{\partial^2 f}{\partial x^2}(z_0)\int_{B_r(z_0)}(x-x_0)^2\,dA(z) + \frac{\partial^2 f}{\partial y^2}(z_0)\int_{B_r(z_0)}(y-y_0)^2\,dA(z)) + o(r^2).$$

But

$$\int_{B_r(z_0)}(x-x_0)^2\,dA(z) = \int_{B_r(z_0)}(y-y_0)^2\,dA(z),$$

by change of variable, and so we have

$$\begin{aligned} 0 &= \tfrac{1}{\pi r^2}\Delta f(z_0)\textstyle\int_{B_r(z_0)}|z|^2 dA(z) + o(r^2) \\ &= \Delta f(z_0)\tfrac{1}{r^2}\cdot\tfrac{r^4}{2} + o(r^2) \\ &= \Delta f(z_0)(\tfrac{r^2}{2} + o(r^2)). \end{aligned}$$

Since $\frac{r^2}{2} + o(r^2) = r^2(\frac{1}{2} + \frac{o(r^2)}{r^2}) > 0$, for r sufficiently small, we get $\Delta f(z_0) = 0$. Since z_0 was arbitrary, $\Delta f \equiv 0$.

Another way to see this starts from setting

$$\pi r^2 f(z_0) = \int_{B_r(z_0)} f(z)dA(z),$$

because of the mean value property, where r, z_0 are arbitrary. Fixing z_0, by differentiating with respect to r, we see:

$$\begin{aligned} 2\pi r f(z_0) &= \tfrac{d}{dr}\textstyle\int_0^r\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,\rho\,d\theta\,d\rho \\ &= \textstyle\int_0^{2\pi} f(z_0 + re^{i\theta})rd\theta, \end{aligned}$$

a mean value statement over circles. Differentiate this new equation again with respect to r and get

$$\begin{aligned} 2\pi f(z_0) &= \textstyle\int_0^{2\pi}\frac{\partial f}{\partial r}\,r\,d\theta + \int_0^{2\pi} f\,d\theta \\ &= \textstyle\int_{\partial B_r(z_0)}\frac{\partial f}{\partial r} rd\theta + 2\pi f(z_0). \end{aligned}$$

But $\frac{\partial}{\partial r}$ is the unit normal to the boundary of $B_r(z_0)$. Hence, by the divergence theorem, we conclude that

$$0 = \int_{B_r(z_0)}\Delta f(z)\,dA(z).$$

Since Δf is continuous and this last equation is true for all $r > 0$, we get $\Delta f(z_0) = 0$. Since z_0 was arbitrary, $\Delta f \equiv 0$.

Problem 2

Consider the analytic function $f(z) = \frac{1}{2}(z + \frac{1}{z})$, which is defined and analytic on $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Show that f maps the exterior of the unit circle $\{z \in \mathbb{C}\,|\,|z| > 1\}$ to the region $\mathbb{C}\setminus[-1,1]$ in a one-to-one manner, and calculate its inverse function.

[20]

Solution

Let us construct the analytic inverse of f. For $w \in \mathbb{C} \setminus [-1, 1]$, if we have

$$w = \frac{1}{2}(z + \frac{1}{z}),$$

then

$$2zw = z^2 + 1, \text{ or } z^2 - 2wz + 1 = 0.$$

By the quadratic formula,

$$z = \frac{2w \pm \sqrt{4w^2 - 4}}{2} = w \pm \sqrt{w^2 - 1} = h(w).$$

The problem remains: is the square root single-valued on $\mathbb{C} \setminus [-1, 1]$? First let us fix its value to be positive on the real interval $(1, +\infty)$. Choose the + sign in the quadratic formula for $h(w)$. Consider $w^2 - 1 = (w + 1)(w - 1)$. We can construct $\sqrt{w^2 - 1}$ as $\sqrt{w + 1}\sqrt{w - 1}$. But as we analytically continue $\sqrt{w - 1}$ counterclockwise once around the circle $C = \{|z| = 2\}$, say, it comes back to the real axis changed by multiplication by -1. Similarly for $\sqrt{w + 1}$, so the product is unchanged after analytic continuation around C and therefore is well defined. As $w \to \infty$, $h(w) \to \infty$. As $w \to [-1, 1]$, $h(w) \to$ the unit circle, since $f(\text{unit circle}) = [-1, 1]$, and h is an inverse for f on $\mathbb{C} \setminus [-1, 1]$.

Problem 3

Consider the surface $S \subset \mathbb{R}^3$ given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \,|\, q(x, y, z) = x^2 + 2xy + 3y^2 + 2xz + 2z^2 = 10\}.$$

(a) Let $f(x, y, z)$ be the linear function

$$f(x, y, z) = 4x + 7y + 5z + 3.$$

Using the method of Lagrange multipliers, find the maximum of f *restricted to* S.

[15]

(b) Explain geometrically why the method works.

[5]

Solution

(a) The method of Lagrange multipliers says to solve this system of 4 equations in 4 unknowns, x, y, z, λ, to find the points where f might be maximal on S:

$$\begin{aligned} \nabla q &= \lambda \nabla f, \\ q(x, y, z) &= 10. \end{aligned}$$

Explicitly, the first three equations are

$$\begin{aligned} 2(x + y + z) &= 4\lambda, \\ 6y + 2x &= 7\lambda, \\ 4z + 2x &= 5\lambda. \end{aligned}$$

Solve these equations to get

$$x = -\frac{5}{2}\lambda,\ y = 2\lambda,\ z = \frac{5}{2}\lambda.$$

Substituting in the fourth equation, we get

$$10 = (\frac{\lambda}{2})^2 q(-5, 4, 5) = (\frac{\lambda}{2})^2\, 33,$$

or

$$\lambda = \pm 2\sqrt{\frac{10}{33}}.$$

(Sorry for the radical!) So the maximum occurs at one of the two points

$$[x, y, z]^T = \pm\sqrt{\frac{10}{33}}\,[-5, 4, 5]^T.$$

Evaluating f on these two points gives $f = \pm 33 \cdot \sqrt{\frac{10}{33}} + 3$. Therefore the maximum is

$$f = +\sqrt{330} + 3,$$

at

$$[x, y, z]^T = +2\sqrt{\frac{10}{33}}\,[-5, 4, 5]^T.$$

(b) If $\nabla q(x, y, z), \nabla f(x, y, z)$, and λ are all non-zero, then the equation $\nabla q(x, y, z) = \lambda \nabla f(x, y, z)$ says that for every vector $\vec{v}$ tangent to S at (x, y, z), then the directional derivative $\nabla_{\vec{v}} f(x, y, z) = \nabla f(x, y, z) \cdot \vec{v}$ (by the chain rule) $= 0$.

Problem 4

Define the n-th Laguerre polynomial $L_n(x)$ as

$$L_n(x) = \frac{1}{n!}(\frac{d}{dx} - 1)^n x^n,\ \text{for } x \in \mathbb{R},$$

so the first few are given by

$$\begin{aligned} L_0(x) &= 1, \\ L_1(x) &= -x + 1, \\ L_2(x) &= \tfrac{1}{2}x^2 - 2x + 1, \end{aligned}$$

and so on.

Show that if a continuous function $f : [0, 1] \to \mathbb{R}$ satisfies

$$\int_0^1 L_n(x)\, f(x)\, dx = 0,\ \text{for all}\, n,$$

then $f \equiv 0$ on [0,1].

[20]

Solution

First notice that $L_n(x) = \frac{(-1)^n x^n}{n!} +$ lower order terms, the linear span of the polynomials $L_i(x), i = 0, \ldots, L_n(x)$ is the same as the linear span of $1, \ldots, x^n$, i.e., all polynomials of degree $\leq n$. Hence, our function f satisfies $\int_{[0,1]} P(x)f(x)dx = 0$, for any polynomial $P(x)$. But by the Weierstrass approximation theorem, there is a sequence of polynomials $P_m(x)$ such that $\lim_{m\to\infty} P_m(x) = f(x)$ uniformly in x on [0,1]. By a theorem about convergence of sequences of integrals

$$0 = \lim_{m\to\infty} \int_{[0,1]} P_m(x) \cdot f(x)\, dx = \int_{[0,1]} (\lim_{m\to\infty} P_m(x)) \cdot f(x)\, dx = \int_{[0,1]} f^2(x)\, dx.$$

Since f is continuous, this implies $f \equiv 0$ on [0,1].

<u>Problem 5</u>

Consider the curve (circle) $C \subset \mathbb{C}$ given by $|z - z_0| = 1$. Let $\phi(\zeta), \zeta \in C$, be a differentiable function, not necessarily analytic.

(a) Show that the function

$$\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(\zeta)}{\zeta - z}\, d\zeta,$$

defined for $z \notin C$, is analytic in z.

[15]

(b) If $\phi(\zeta)$ has an extension to all of $\mathbb{C}$ as an analytic function, call it still $\phi(z)$, what is $\Phi(z)$?

[5]

Solution

(a) There are two cases; $|z - z_0| < 1$ and $|z - z_0| > 1$. Treat the first case first.

If $|z - z_0| < 1$, then $\zeta - z = (\zeta - z_0) - (z - z_0) = (\zeta - z_0)(1 - \frac{z - z_0}{\zeta - z_0})$. Since

$$|\frac{z - z_0}{\zeta - z_0}| = \frac{|z - z_0|}{|\zeta - z_0|} = |z - z_0| < 1,$$

we have

$$\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0} (\frac{z - z_0}{\zeta - z_0})^n,$$

uniformly in z if $|z - z_0| \leq r < 1$. Therefore,

$$\begin{aligned}
\Phi(z) &= \tfrac{1}{2\pi i} \textstyle\int_C \frac{\phi(\zeta)}{\zeta - z}\, d\zeta \\
&= \tfrac{1}{2\pi i} \textstyle\int_C \frac{1}{\zeta - z_0} \cdot (\sum_{n=0} \frac{(z - z_0)^n}{(\zeta - z_0)^n}) \phi(\zeta)\, d\zeta \\
&= \textstyle\sum_{n=0} [\frac{1}{2\pi i} \int_C \frac{\phi(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta] \cdot (z - z_0)^n \\
&= \textstyle\sum_{n=0} a_n (z - z_0)^n,
\end{aligned}$$

where $a_n = \frac{1}{2\pi i} \int_C \frac{\phi(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta$. Since $|a_n| \leq \frac{1}{2\pi} \int_0^{2\pi} |\phi(\zeta)| \, d\theta \leq M$, where $M = \max_{\zeta \in C} |\phi(\zeta)|$, $\Phi(z)$ is given by a convergent power series in $z - z_0$ and so is analytic on $\{|z - z_0| < 1\}$.

The case $|z - z_0| > 1$ is treated the same way once you make the (analytic) change of variables $w = \frac{1}{z - z_0}$, which takes the exterior of $\{|z - z_0| = 1\}$ to the interior of $\{|w| = 1\}$.

There are many other ways of approaching this problem!

(b) There are two cases. If $z \in \{z \,|\, |z - z_0| < 1\}$, then $\Phi(z) = \phi(z)$ by Cauchy's integral formula. If $|z - z_0| > 1$, then $\frac{\phi(\zeta)}{\zeta - z}$ is analytic in ζ on $\zeta \in \{\zeta \,|\, |\zeta - z_0| < 1\}$. Hence, $\Phi(z) = 0$, for $|z -_0 | > 1$, by Cauchy's theorem.