AIM Qualifying Review Exam in Advanced Calculus & Complex Variables

January 2025

There are five (5) problems in this examination. Each is worth 20 points.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

A $\mathcal{C}^2$ convex function $f : \mathbb{R}^n \to \mathbb{R}$ is one for which the Hessian $\frac{\partial^2 f}{\partial x_i \, \partial x_j}$ is non-negative, i.e., for every $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$,

$$\sum_{i,j} \frac{\partial^2 f}{\partial x_i \, \partial x_j} \xi_i \xi_j \geq 0.$$

(a.) Show that for any $a, b \in \mathbb{R}^n$, if f is a $\mathcal{C}^2$ convex function, then

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b), \text{ for all } t \in [0,1].$$

(b.) Sketch a picture of the result in part (a.), in the case $n = 1$.

(c.) If

$$\sum_{i,j} \frac{\partial^2 f}{\partial x_i \, \partial x_j} \xi_i \xi_j \geq c \sum_i \xi_i^2, \text{ for all } x \text{ where } c > 0,$$

and $f(a) = f(b) = 0$, show that

$$f(\frac{a+b}{2}) \leq -\frac{c}{8} \, |a-b|^2.$$

Hint: Show $f - \frac{c}{2}(x-a) \cdot (x-b)$ is convex, where "$\cdot$" is the usual dot product in $\mathbb{R}^n$.

Problem 1

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Problem 2

Assume the standard integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = 1.$$

(a.) Evaluate the complex line integral

$$I(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\{\mathrm{Im}\ z = \xi\}} e^{-z^2/2}\, dz,$$

where the contour is oriented from $-\infty + i\xi$ to $+\infty + i\xi$.

(b.) Show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2}\, e^{ix\xi}\, dx = e^{-\xi^2/2}.$$

Problem 2

Problem 2

Problem 2

Problem 3

(a.) Use complex variable techniques to show that the Taylor series of the real function $g(x) = \frac{1}{1+x^2}$ at $x = 0$ has radius of convergence equal to 1.

(b.) Let f be the function analytic for $|z| > 1$ given by

$$f(z) = \sum_{j \geq 1} \frac{1}{z^j}.$$

Evaluate the line integral

$$\frac{1}{2\pi i} \int_{\{|z|=2\}} f(z)\, dz,$$

where the circle $\{|z| = 2\}$ is parametrized in the counter-clockwise direction.

Problem 3

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Problem 4

Let Δ be Laplace's operator in $\mathbb{R}^3$, i.e.,

$$\Delta = \sum_{i=1}^{i=3} \frac{\partial^2}{\partial x_i^2},$$

and let u be in $\mathcal{C}^2(\mathbb{B}^3(0,1) \times \mathbb{R}_+)$, where $\mathbb{B}^3(0,1) \times \mathbb{R}_+ = \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} \,|\, |x| \leq 1, t \geq 0\}$. Assume that the function $u(x,t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = \Delta u,$$

as well as the boundary-initial conditions

$$\begin{aligned} u(x,0) &= h(x),\ x \in \mathbb{B}^3(0,1), \\ \tfrac{\partial u}{\partial n}(x,t) &= 0,\ x \in \partial\mathbb{B}^3(0,1), t \geq 0, \end{aligned}$$

where h is a differentiable function on $\mathbb{B}^3(0,1)$, and $\frac{\partial}{\partial n}$ is the outward unit normal along the boundary $\partial\mathbb{B}^3(0,1)$.

(a.) If u is interpreted as a distribution of heat on $\mathbb{B}^3 \times \mathbb{R}_+$, interpret the boundary-initial conditions.

(b.) Calculate

$$\lim_{t \to +\infty} \int_{\mathbb{B}^3(0,1)} u(x,t)\, dx.$$

Problem 4

Problem 4

Problem 4

Problem 5

(a.) Let F be an analytic function on $\mathbb{C}$ such that $|F(z)| \leq A|z|^N + B$, for some positive constants A, B, and positive integer N. Show that F is a polynomial in z, of degree at most N.

(b.) Let $f : D \to D$ be an analytic mapping of the unit disk D to itself, which is continuous on the closed disk. Assume that $f(z) \neq 0$ for $|z| < 1$ and that $|f(z)| = 1$ if $|z| = 1$. Show that f is a constant function.

Problem 5

Problem 5

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