AIM Qualifying Review Exam in ACCV: Solutions

January 2025

Problem 1

A $\mathcal{C}^2$ convex function $f : \mathbb{R}^n \to \mathbb{R}$ is one for which the Hessian $\frac{\partial^2 f}{\partial x_i \, \partial x_j}$ is non-negative, i.e., for every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$,

$$\sum_{i,j} \frac{\partial^2 f}{\partial x_i \, \partial x_j} \xi_i \xi_j \geq 0.$$

(a.) Show that for any $a, b \in \mathbb{R}^n$, if f is a $\mathcal{C}^2$ convex function, then

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b), \text{ for all } t \in [0,1].$$

(b.) Sketch a picture of the result in part (a.), in the case $n = 1$.

(c.) If

$$\sum_{i,j} \frac{\partial^2 f}{\partial x_i \, \partial x_j} \xi_i \xi_j \geq c \sum_i \xi_i^2, \text{ for all } x \text{ where } c > 0,$$

and $f(a) = f(b) = 0$, show that

$$f(\frac{a+b}{2}) \leq -\frac{c}{8} \, |a-b|^2.$$

Hint: Show $f - \frac{c}{2}(x-a) \cdot (x-b)$ is convex, where "$\cdot$" is the usual dot product in $\mathbb{R}^n$.

Solution: (a.) The question is really about a function in one variable, t, for any fixed a and b in $\mathbb{R}^n$, and is equivalent to showing

$$F(t) = f((1-t)a + tb) - (1-t)f(a) - tf(b) \leq 0,$$

for all $t \in [0,1]$. Notice that F is also convex. For any $\delta > 0$, consider the function

$$F_\delta(t) = F(t) + \delta \, t(t-1),$$

which is strictly convex, i.e., $F_\delta^{''} \geq 2\delta > 0$, while $F_\delta(0) = F_\delta(1) = 0$. Since $F_\delta^{''} > 0$, F_δ assumes its maximum on [0,1] at the endpoints, i.e., $F_\delta(t) \leq 0$, for all $t \in [0,1]$ and any positive δ. Fixing t, let $\delta \to 0$ and get $F(t) \leq 0$.

(b.) The graph of f lies below any chord joining two points on the graph.

(c.) Let $F = f - \frac{c}{2}(x-a) \cdot (x-b)$. Following the hint,

$$\sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial x_j} \xi_i \xi_j = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \xi_i \xi_j - c \sum_i \xi_i^2 \geq 0,$$

by assumption, so F is convex. Hence,

$$F(\frac{a+b}{2}) = F(\frac{a}{2} + \frac{b}{2}) \leq \frac{1}{2}F(a) + \frac{1}{2}F(b) = \frac{1}{2}f(a) + \frac{1}{2}f(b) = 0.$$

Thus,

$$f(\frac{a+b}{2}) - \frac{c}{2}(\frac{a+b}{2} - a) \cdot (\frac{a+b}{2} - b) \le 0,$$

and hence

$$f(\frac{a+b}{2}) \le \frac{c}{2}(\frac{a+b}{2} - a) \cdot (\frac{a+b}{2} - b) = -\frac{c}{8}|a-b|^2.$$

Problem 2

Assume the standard integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = 1.$$

(a.) Evaluate the complex line integral

$$I(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\{Im\, z = \xi\}} e^{-z^2/2}\, dz,$$

where the contour is oriented from $-\infty + i\xi$ to $+\infty + i\xi$.

(b.) Show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2}\, e^{ix\xi}\, dx = e^{-\xi^2/2}.$$

Solution: (a.) Consider the integral $\int_{C_r} e^{\frac{-z^2}{2}}\, dz$, where C_r is the rectangular contour, oriented counter-clockwise, with four straight line pieces:

$$\begin{aligned} C_{r,1} &= \{z = x \in \mathbb{R} \subset \mathbb{C} \,|\, -r \le x \le r\}; \\ C_{r,2} &= \{z = r + it, t \in [0,\xi] \text{ or } [\xi, 0]; \\ C_{r,3} &= \{z \in \mathbb{C} \,|\, z = x + i\xi, -r \le x \le r\}; \\ C_{r,4} &= \{z = -r + it, t \in [0,\xi] \text{ or } [\xi, 0]\}. \end{aligned}$$

Then, $0 = \int_{C_r} e^{-\frac{z^2}{2}}\, dz$, by Cauchy's theorem, while

$$\int_{C_r} e^{-\frac{z^2}{2}}\, dz = \sum_j \int_{C_{r,j}} e^{-\frac{z^2}{2}}\, dz.$$

Note that

$$|\int_{C_{r,2}} e^{-\frac{z^2}{2}}\, dz| \le e^{-\frac{r^2}{2}} |\xi| e^{-|\xi|^2} \to 0,$$

as $r \to +\infty$. Similarly for $\int_{C_{r,4}} e^{-\frac{z^2}{2}}\, dz$. Hence,

$$\int_{\{\mathrm{Im}\ z = \xi\}} e^{-\frac{z^2}{2}}\, dz = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}.$$

(b.)

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-x^2/2}\, e^{ix\xi}\, dx &= e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-(\frac{x^2-\xi^2}{2}+ix\xi)}\, dx \\ &= e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{\{\mathrm{Im}\ z=\xi\}} e^{-\frac{z^2}{2}}\, dz \\ &= e^{-\frac{x^2}{2}},\ \text{by part (a.).}\end{aligned}$$

Problem 3

(a.) Use complex variable techniques to show that the Taylor series of the real function $g(x) = \frac{1}{1+x^2}$ at $x = 0$ has radius of convergence equal to 1.

(b.) Let f be the function analytic for $|z| > 1$ given by

$$f(z) = \sum_{j\geq 1} \frac{1}{z^j}.$$

Evaluate the line integral

$$\frac{1}{2\pi i}\int_{\{|z|=2\}} f(z)\, dz,$$

where the circle $\{|z| = 2\}$ is parametrized in the counter-clockwise direction.

Solution: (a.) g is the restriction to $\mathbb{R}$ of the meromorphic function $G(z) = \frac{1}{1+z^2}$. G has poles only at $z = \pm i$, so G is holomorphic on the unit disk $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$, and hence the radius of convergence of its Taylor series at $z = 0$ is 1. Therefore, the radius of convergence of the Taylor series of g is 1, since its Taylor series is the same as that for G.

(b.) There are two simple ways to figure this. First, f is analytic for $|z| > 1$, with an essential singularity at $z = 0$. Nevertheless, $\frac{1}{2\pi i}\int_{\{|z|=2\}} f(z)\, dz$ is just the residue of f at $z = 0$, which is just the coefficient of $\frac{1}{z}$ in the Laurent expansion of f at 0. Hence,

$$\frac{1}{2\pi i}\int_{\{|z|=2\}} f(z)\, dz = 1.$$

Or, one could change variable to $\zeta = \frac{1}{z}$ and get

$$\frac{1}{2\pi i}\int_{\{|z|=2\}} f(z)\, dz = -\frac{1}{2\pi i}\int_{\{|\zeta|=\frac{1}{2}\}} \frac{\zeta}{1-\zeta}\cdot -\frac{1}{\zeta^2}\, d\zeta = \frac{1}{2\pi i}\int_{\{|\zeta|=\frac{1}{2}\}} \frac{1}{\zeta(1-\zeta}\, d\zeta = 1,$$

again by the residue theorem at $\zeta = 0$.

Problem 4

Let Δ be Laplace's operator in $\mathbb{R}^3$, i.e.,

$$\Delta = \sum_{i=1}^{i=3} \frac{\partial^2}{\partial x_i^2},$$

and let u be in $\mathcal{C}^2(\mathbb{B}^3(0,1) \times \mathbb{R}_+)$, where $\mathbb{B}^3(0,1) \times \mathbb{R}_+ = \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} \,|\, |x| \leq 1, t \geq 0\}$. Assume that the function $u(x,t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = \Delta u,$$

as well as the boundary-initial conditions

$$\begin{aligned} u(x,0) &= h(x), \; x \in \mathbb{B}^3(0,1), \\ \tfrac{\partial u}{\partial n}(x,t) &= 0, \; x \in \partial\mathbb{B}^3(0,1), t \geq 0, \end{aligned}$$

where h is a differentiable function on $\mathbb{B}^3(0,1)$, and $\frac{\partial}{\partial n}$ is the outward unit normal along the boundary $\partial\mathbb{B}^3$.

(a.) If u is interpreted as a distribution of heat on $\mathbb{B}^3(0,1) \times \mathbb{R}_+$, interpret the boundary-initial conditions.

(b.) Calculate

$$\lim_{t \to +\infty} \int_{\mathbb{B}^3(0,1)} u(x,t)\, dx.$$

Solution: (a.) The initial condition part just gives the distribution o f heat at time $t = 0$. The boundary term says that the flux of heat across the boundary (in space) is 0, i.e., the boundary is insulated.

(b.) We calculate:

$$\begin{aligned} \tfrac{d}{dt} \textstyle\int_{\mathbb{B}^3(0,1)} u(x,t)\, dx &= \textstyle\int_{\mathbb{B}^3(0,1)} \tfrac{\partial}{\partial t} u(x,t)\, dx \\ &= \textstyle\int_{\mathbb{B}^3(0,1)} \Delta u(x,t)\, dx \\ &= \textstyle\int_{\mathbb{B}^3(0,1)} 1 \cdot \, \mathrm{div}(\nabla u(x,t))\, dx, \quad \text{where } \nabla \text{ is the gradient in the } x\text{-variables only,} \\ &= \textstyle\int_{\{|x|=1\}} 1 \cdot \tfrac{\partial u}{\partial n}\, dS + \int_{\mathbb{B}^3(0,1)} \nabla 1 \cdot \nabla u\, dx \\ &= 0. \end{aligned}$$

Hence, $\int_{\mathbb{B}^3(0,1)} u(x,t)\, dx$ is constant in t, so

$$\lim_{t \to +\infty} \int_{\mathbb{B}^3(0,1)} u(x,t)\, dx = \int_{\mathbb{B}^3(0,1)} u(x,0)\, dx = \int_{\mathbb{B}^3(0,1)} h(x)\, dx.$$

Problem 5

(a.) Let F be an analytic function on $\mathbb{C}$ such that $|F(z)| \leq A|z|^N + B$, for some positive constants A, B, and positive integer N. Show that F is a polynomial of degree at most N.

(b.) Let $f : D \to D$ be an analytic mapping of the unit disk D to itself, which is continuous on the closed disk. Assume that $f(z) \neq 0$ for $|z| < 1$ and that $|f(z)| = 1$ if $|z| = 1$. Show that f is a constant function.

Solution: (a.) Fix any $z_0 \in \mathbb{C}$. By Cauchy's formula

$$\frac{d^{N+1}F}{dz^{N+1}}(z_0) = \frac{(N+1)!}{2\pi i} \int_{\{|z-z_0|=R\}} \frac{F(\zeta)}{(\zeta - z_0)^{N+2}}\, d\zeta,$$

and therefore

$$
\begin{aligned}
|\tfrac{d^{N+1}F}{dz^{N+1}}(z_0)| &\leq \tfrac{(N+1)!}{2\pi}\int_0^{2\pi}\tfrac{|F(\zeta)|}{R^{N+2}}\,R\,d\theta \\
&\leq \tfrac{(N+1)!}{2\pi}(\tfrac{AR^N+B+E(R)}{R^{N+2}})\,R\,2\pi,\ \text{where } E(R) \text{ is a polynomial in } R \text{ of degree} < N, \\
&\leq (N+1)!\tfrac{CR^{N+1}+D}{R^{N+2}},\ \text{where } C, D \text{ are positive constants.}
\end{aligned}
$$

Letting $R \to +\infty$, we get

$$\frac{d^{N+1}F}{dz^{N+1}}(z_0) = 0.$$

Since z_0 was arbitrary in $\mathbb{C}$, we must have that F is a polynomial of degree at most N.

(b.) Define a function F as follows:

$$F(z) = \begin{cases} f(z),\ \text{if } |z| \leq 1 \\ \\ 1/\overline{f(1/\bar{z})},\ \text{if } |z| \geq 1. \end{cases}$$

Since $f(z) \neq 0$, and $|f(z)| = 1$, if $|z| = 1$, this is well-defined for all $z \in \mathbb{C}$. By the Schwarz reflection principle, it is analytic on $\mathbb{C}$. Since there is a positive $c \leq 1$ such that $|f(z)| \geq c$ for $|z| \leq 1$, then $|F(z)| \leq \frac{1}{c} < +\infty$ for all $z \in \mathbb{C}$. Hence, by Liouville's theorem, F must be a constant.