Overlaps of a spherical spin glass model with external field

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Introduction

What are spin glasses?

Spin glasses are disordered magnetic alloys. Physicists use probabilistic models to study their behavior. These models are also useful in computer science, biology, economics, etc.

Components of a spin glass model

- **Spin variable** $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_N)$ is a random vector in a high dimensional space.
- **Probability measure** $p(\boldsymbol{\sigma})$ gives the distribution of the spins.
- **Disorder variables** are random parameters in the definition of $p(\boldsymbol{\sigma})$ (for our model, they are M, g defined below). Because $p(\boldsymbol{\sigma})$ has random parameters, it is a *random measure*.

Key questions:

1. How does an **external magnetic field** affect spin distribution?
2. What happens in the **transition** between a model with an external field and one without?

Spherical Sherrington-Kirkpatrick model

This project focuses on the Spherical Sherrington-Kirkpatrick (SSK) model for spin glasses.

- **Spin variable for SSK**: $\boldsymbol{\sigma} \in S_{N-1}$, random vector in the N-sphere.
- **Interaction matrix** M: an $N \times N$ matrix from the Gaussian Orthogonal Ensemble (GOE) (i.e. M is symmetric and M_{ij} are independent, normal random variables for $i \leq j$).
- **External field** $\mathbf{g}$: a Gaussian random vector $\mathbf{g} \in \mathbb{R}^N$ with external field strength $h \geq 0$.
- **Ground state:** eigenvector $\mathbf{u}_1$ associated with largest eigenvalue of M.
- **Gibbs measure** (defines spin distribution):

$$p(\boldsymbol{\sigma}) = \frac{1}{\mathscr{Z}_N} e^{\beta \mathscr{H}(\boldsymbol{\sigma})} \quad \text{where} \quad \mathscr{H}(\boldsymbol{\sigma}) = \frac{1}{2} \boldsymbol{\sigma}^T M \boldsymbol{\sigma} + h \mathbf{g}^T \boldsymbol{\sigma}$$

$\beta = \frac{1}{T} > 0$ is "inverse temperature" and $\mathscr{Z}_N$ is a normalization factor. Our study focuses on the low temperature setting ($T < 1$)

Intuition:

Spins concentrate near vectors that maximize $\mathscr{H}$, such as $\mathbf{g}$ and $\pm\mathbf{u}_1$.

Overlaps

Overlaps help us analyze spin distribution. Let $\boldsymbol{\sigma}$ be a random spin and $\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}$ independent copies of $\boldsymbol{\sigma}$. We consider three types of overlap:

Overlap with...	Formula	Interpretation
...external field	$\mathfrak{O}_{EF} = \frac{1}{N} \mathbf{g} \cdot \boldsymbol{\sigma}$	cos(angle between $\boldsymbol{\sigma}$ and $\mathbf{g}$)
...ground state	$\mathfrak{O}_{GS} = \frac{1}{N} (\mathbf{u}_1 \cdot \boldsymbol{\sigma})^2$	$\cos^2$(angle between $\boldsymbol{\sigma}$ and $\mathbf{u}_1$)
...a replica	$\mathfrak{O}_R = \frac{1}{N} \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}$	cos(angle between $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$)

Results for SSK with & without external field [1]

We compare the overlaps for two different SSK models:

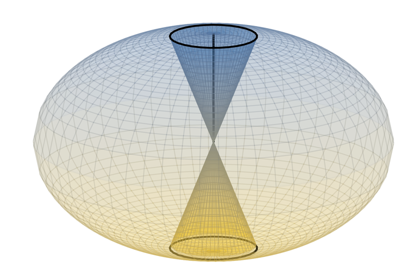
- model <u>without</u> external field ($h = 0$),
- model <u>with</u> external field ($h > 0$ constant).

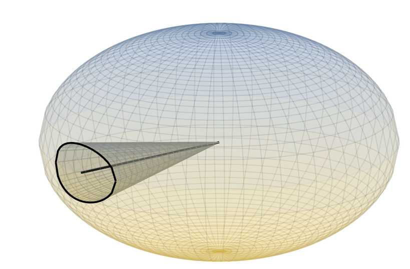
Overlap	$h = 0$ case	$h > 0$ case
$\mathfrak{O}_{EF}$	$O(N^{-1/2})$ mean=0	$A(h, T)$**
$\mathfrak{O}_{GS}$	$1 - T$	$O(N^{-1})$
$\mathfrak{O}_{Rep}$	$(1 - T)\text{Bern}(\frac{1}{2})$*	$B(h, T)$**

*Bern($\frac{1}{2}$) is a shifted Bernoulli, taking values ± 1 with equal probability.
**A, B are positive, deterministic, order 1 functions that increase with h.

Interpretation of the results

Without ext. field ($h = 0$)	With ext. field ($h > 0$)
Spins concentrate on **double cone around** $\pm\mathbf{u}_1$ (angle=$\cos^{-1}\sqrt{1-T}$, see $\mathfrak{O}_{GS}$). Spins are equally likely to occur on either side of the double cone (see $\mathfrak{O}_{Rep}$).	Spins concentrate on a **cone around** $\mathbf{g}$. They get closer to $\mathbf{g}$ as h increases (see $\mathfrak{O}_{EF}$).





Note: In tables above and below, we only include the leading order term of each overlap, although we can calculate more. Results hold with high probability.

Results for transitional regimes [1]

To study the transition between $h = 0$ and $h > 0$ models, we consider the case where $h \to 0$ as $N \to \infty$. We find two transitional scalings:

- $h \sim N^{-1/2}$ "microscopic" external field,
- $h \sim N^{-1/6}$ "mesoscopic" external field.

Overlap	$h \sim N^{-1/2}$ case	$h \sim N^{-1/6}$ case
$\mathfrak{O}_{EF}$	$O(N^{-1/2})$ mean>0	h
$\mathfrak{O}_{GS}$	$1 - T$	$1 - T - K(h, T, \mathbf{g}, M)$**
$\mathfrak{O}_{Rep}$	$(1 - T)\text{Bern}(p > \frac{1}{2})$*	$1 - T$

*Bern($p > \frac{1}{2}$) takes value +1 more often than −1.
**$K(h, T, \mathbf{g}, M)$ is random, taking values between 0 and $1 - T$.

Interpretation of the results

Microscopic field ($h \sim N^{-\frac{1}{2}}$)	Mesoscopic field ($h \sim N^{-\frac{1}{6}}$)
Spins concentrate on double cone around $\pm\mathbf{u}_1$ (see $\mathfrak{O}_{GS}$), but are **more likely to be on the side of the double cone nearer g** (since $\mathfrak{O}_{EF}, \mathfrak{O}_{Rep}$ usually positive).	Spins concentrate on cone around $+\mathbf{u}_1$ or $-\mathbf{u}_1$ (see $\mathfrak{O}_{GS}$) but are **exclusively on the cone nearer g** (since $\mathfrak{O}_{EF}, \mathfrak{O}_{Rep} > 0$). The angle of the cone is wider than for $h = 0$ and is no longer deterministic (see $\mathfrak{O}_{GS}$).

Techniques

Computing the overlaps involves three key steps

1. **Use contour integral representation of $\mathscr{Z}_N$.**
   The Gibbs measure involves partition function $\mathscr{Z}_N$. Due to specific properties of SSK, we can rewrite this as a contour integral
   $$\mathscr{Z}_N = C_N \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\frac{N}{2} G(z)} dz$$
   where C_N is constant and $G(z)$ depends on M, g. A contour integral is easier to compute than an N-dimensional surface integral.
2. **Write a moment generating function for each overlap.** These can be expressed as a ratio of contour integrals like the one above.
3. **Analyze using random matrix theory.**
   GOE matrices exhibit *eigenvalue rigidity*, meaning the eigenvalues are usually very close to their predicted locations. This and other properties help to analyze the random integrals.

Further research

Other topics we investigated:

- **Free energy of SSK** has transitional h regimes at all temperatures (unlike overlaps, where transition is only at low temperature).
- **Susceptibility** or "magnetization per external field strength," is an application of our results.
- **Precise fluctuation terms** are included in our paper, in addition to leading order terms.
- **Rigorous proofs** are omitted in some places in our paper [1]. Some proofs were obtained in separate papers by Landon & Sosoe [3] and by Collins-Woodfin [2].

Open questions:

- **Spin distribution within the double cone** is uniform for $h = 0$ but not at transitional scalings. Can we further analyze those?
- **Other spin glass models:** Do they exhibit similar transitions?

References

[1] J. Baik, E. Collins-Woodfin, P. L. Doussal, and H. Wu. Spherical spin glass model with external field. *arXiv:2010.06123*, 2020.

[2] E. Collins-Woodfin. Overlaps of a spherical spin glass model with microscopic field. *In preparation*, 2020.

[3] B. Landon and P. Sosoe. Fluctuations of the 2-spin ssk model with magnetic field. *arXiv:2009.12514*, 2020.