AIM Preliminary Exam: Probability and Discrete Mathematics

January 6, 2020

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Suppose X, Y, and Z are uniform and identically distributed on $[0, 1]$. Compute $\Pr[X \geq YZ]$.

Problem 1

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Problem 2

(a) Suppose $X \sim \mathfrak{N}(0,1)$ is a standard normal random variable, i.e., with density function $\frac{e^{-x^2/2}}{\sqrt{2\pi}}$. Compute $M_X(t) = \mathbb{E}[e^{tX}]$ as a function of t.

(b) Suppose Y is also a standard normal random variable, independent of X. Compute $M_{X+Y}(t)$.

(c) Compute M_{cZ}, where Z is also a standard normal, independent of the others, and c is a positive real number.

(d) Use your answer to Part (c) to conclude about the distribution of $X + Y$.

Problem 2

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Problem 3

Consider the following graph parameters:

- The **clique** number of a graph, G, is the largest k such that G has an induced complete subgraph of size k.
- The **independence** number of G is the largest ℓ such that G has an induced subgraph of ℓ vertices and no edges.
- The **domination** number of G is the smallest j such that, for some set C of j vertices of G, every vertex of G is in C or adjacent to C. (The set C is also called a **vertex cover** of G.)

(a) Give relationships among k, ℓ, j for an arbitrary graph G and related graphs, and briefly prove it by counting and identifying vertices in G and related graphs.

(b) Suppose we have an algorithm that **approximates** one of these parameters to within a factor 2. Can we use the algorithm to approximate the other two parameters to within a useful factor? Discuss, and give an example, where appropriate, to show that an approximation for one parameter does not yield a useful approximation algorithm for another parameter.

Problem 3

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Problem 4

How many ways are there to cover a $2 \times n$ checkerboard with 2×1 and 1×2 dominoes? (*Hint: Is the upper left cell covered by a horizontal or vertical dominoe?*)

Problem 4

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Problem 5

Alice and Bob agree to use the following error-correcting code. Working modulo a large prime q, Alice picks a polynomial p of k terms (i.e., degree at most $k-1$, though p may have leading zeros), and sends Bob $p(0), p(1), \ldots, p(n-1)$. Mallory intercepts the transmission and may change up to t of the n symbols.

(a) How big can t be while still allowing Bob to detect that Mallory has caused an error? Justify.

(b) How big can t be while still allowing Bob to recover Alice's message? Justify.

(*Hint*: If p and p' agree in k places, then $p = p'$.)

Problem 5

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