SINGULAR VECTORS IN $\mathbb{F}_q((T^{-1}))^n$

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Abstract. Diophantine approximation has been studied extensively in mathematics. Traditionally it has been used to study problems in number theory. In this report, we will start by reviewing on some classical results in reals. With this in hand, we will look at its generalizations to $\mathbb{R}^n$, and real analytic manifolds. We managed to generalize some of these properties to ultrametric set-up of $\mathbb{F}_q((T^{-1}))^n$ and an analogy of analytic manifolds over $\mathbb{F}_q((T^{-1}))$.

Introduction

We know that $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e., for every $\alpha \in \mathbb{R}$ and for every $\varepsilon > 0$ there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that

$$|\alpha - \frac{p}{q}| < \varepsilon.$$

So a natural question is how well various real numbers can be approximated by rational numbers.

The starting point is the Dirichlet Approximation Theorem:

Theorem 0.1. *For any $\alpha \in \mathbb{R}$ and any $Q > 0$, there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that*

$$|q\alpha - p| < \frac{1}{Q}.$$
$$q \leq Q$$

Proof. Let $n = \lfloor Q \rfloor$, and assume that α is irrational. Then $0, \{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}$ are pairwise distinct, and by pigeonhole principle there exist at least two pairs of (q_1, p_1) and (q_2, p_2) with $|q_i| \leq Q$ such that $|(q_1\alpha - p_1) - (q_2\alpha - p_2)| \leq \frac{1}{n+1} < \frac{1}{Q}$. This finishes the proof. $\square$

We are interested in analogous properties in the ultrametric case. More specificially, we will study $K = \mathbb{F}_q((T^{-1}))$, where $\mathbb{F}_q$ is the finite field. So let us recall the definition of K first. We consider the polynomial ring $\mathbb{F}_q[T]$ of one variable over $\mathbb{F}_q$. The fraction field of $\mathbb{F}_q[T]$ is $\mathbb{F}_q(T)$, the rational functions. The norm of $f \in \mathbb{F}_q[T]$ is defined as

$$\|f\| = \begin{cases} 0 & f = 0, \\ \exp(\deg(f)) & \text{otherwise} \end{cases}.$$

For $\frac{f}{g} \in \mathbb{F}_q(T)$, we define $\|\frac{f}{g}\| = \exp\left(\deg(f) - \deg(g)\right)$. The completion of $\mathbb{F}_q(T)$ is denoted $\mathbb{F}_q((T^{-1}))$, which can also be understood as the field of Laurent series. Note that $\mathbb{F}_q[T] \subset$

Date: Jul. 03, 2021.

$\mathbb{F}_q(T) \subset \mathbb{F}_q((T^{-1}))$ are analogous to $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively. In the multidimensional case the norm is taken to be the maximum norm.

We will begin with reviewing some theorems in $\mathbb{R}$, and then talk about what we have done to show there analogs in K.

1. Properties of Interest in $\mathbb{R}$

1.1. **Definitions and background.** Looking at the Dirichlet's theorem it is natural to ask how, and where $\frac{1}{Q}$ can be further improved. Roughly speaking, we are interested in the following set of numbers.

Definition 1.1. *We define singular numbers as those numbers for which Dirichlet's theorem can be improved indefinitely.*

More explicitly,

$$\text{Sing} := \left\{ x \in \mathbb{R} \left| \begin{array}{l} \forall \epsilon > 0, \exists\, Q_0 > 0 \text{ s.t. } \forall Q \geq Q_0, \\ |qx - p| \leq \dfrac{\epsilon}{Q}, |q| \leq Q \\ \text{has nonzero integer solution} \end{array} \right. \right\}$$

. It is easy to observe that $\mathbb{Q} \subseteq \text{Sing}$.

To investigate the approximation of irrationals further, we make use of continued fractions. A continued fraction is of the form $a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$, where a_i's are integers. The integers a_i's can be determined by using Euclidean Algorithm. We may use continued fractions to approximate a real number by rationals. The finite part of a continued fraction is written as $\frac{p_i}{q_i}$.

Definition 1.2. *Best Approximation: we say a rational $\frac{a}{b}$ is a best approximation to a real number α if for all $\frac{c}{d}$ such that $|d| \leq |b|$ we have $|b\alpha - a| \leq |d\alpha - c|$.*

Theorem 1.1. *Best Approximation Theorem* [1]*: the finite continued fraction approximation described above gives the best approximation of real numbers.*

1.2. **Singular numbers in $\mathbb{R}$.** In this subsection we show that there is no nontrivial singular number in $\mathbb{R}$.

Theorem 1.2. *The only singular numbers are rationals, i.e. Sing $= \mathbb{Q}$.*

Proof. It is clear that $\mathbb{Q} \subseteq \text{Sing}$. Suppose that x singular and irrational number and $\frac{p_n}{q_n}$ is the best approximantions of x. Let denote $\langle\cdot\rangle$ as the distance from nearest integer. Then we know that there exists no integer solution to the following system $\langle qx \rangle < \langle q_n x \rangle, 0 < q < q_{n+1}$. We also have that $\langle q_n x \rangle > \frac{1}{2q_{n+1}}$. If you take $Q = q_{n+1}$ we have $\langle q_n x \rangle \leq \frac{\varepsilon}{q_{n+1}}$. Hence we have $\frac{\varepsilon}{q_{n+1}} \geq \frac{1}{2q_{n+1}}$ for every $\varepsilon > 0$. This is a contradiction. Hence Sing $= \mathbb{Q}$. □

1.3. **Singular vectors in $\mathbb{R}^n$.** In $\mathbb{R}^n$, we take the norm $\|\cdot\|$ to be the maximum norm. We will start by recalling some definitions.

Definition 1.3. *A vector $\zeta \in \mathbb{R}^n$ is totally irrational if $1, \zeta_1, \zeta_2, \dots, \zeta_n$ are linearly independent over $\mathbb{Q}$.*

Definition 1.4. *An affine rational hyperplane of* $\mathbb{R}^n$ *is in the form of* $\{(y_1, \cdots, y_n) \in \mathbb{R}^n \mid c_1 y_1 + \cdots + c_n y_n = c_0\}$ *where* $c_i \in \mathbb{Q}$.

Definition 1.5. *For arbitrary* $\Phi : (\mathbb{Q}^n \setminus \{0\}) \to \mathbb{R}_+$ *being a proper function, i.e., the set* $\{q \in (\mathbb{Q}^n \setminus \{0\}) \ : \Phi(q) \leq C\}$ *being finite for any* $C > 0$, *we define the irrationality measure function* $\psi_{\Phi,\epsilon}(t) = \min_{q \in \mathbb{Z}^n \setminus \{0\}, \Phi(q) \leq t} |\langle q \cdot \epsilon \rangle|$

Singular vector in $\mathbb{R}^n$ can be defined similarly as in the case $\mathbb{R}$. It is easy to see that a vector which is not totally irrational is singular. For obvious reason, we will refer them as 'trivial' singulars.

Theorem 1.3. [2] *Let* $S \subset \mathbb{R}^n$ *be a nonempty locally closed subset. Let* $\{L_1, L_2, \dots\}$ *and* $\{L'_1, L'_2, \dots\}$ *be disjoint collections of distinct closed subsets of* S, *each of which is contained in a rational affine hyperplane in* $\mathbb{R}^n$, *and for each* i *let* A_i *be a rational affine hyperplane containing* L_i, *assume the following hold:*

(a)

$$\bigcup_i L_i \cup \bigcup_j L'_j = \{x \in S \ : x \text{ is contained in a rational affine hyperplane}\};$$

(b) For each i *and each* $T > 0$,

$$L_i = \overline{\bigcup_{|A_j| > T} L_i \cap L_j};$$

(c) For each i, *and for any finite subsets of indices* F, F' *with* $i \notin F$, *we have*

$$L_i = \overline{L_i - (\bigcup_{k \in F} L_k \cup \bigcup_{k' \in F'} L'_{k'})};$$

(d) $\bigcup_i L_i$ *is dense in* S.

Then for arbitrary $\Phi : (\mathbb{Q}^n \setminus \{0\}) \to \mathbb{R}_+$ *being a proper function, i.e., the set* $\{q \in (\mathbb{Q}^n \setminus \{0\}) \ : \Phi(q) \leq C\}$ *being finite for any* $C > 0$, *and for any non-increasing function* $\phi \ : \mathbb{R}_+ \to \mathbb{R}_+$, *there exists uncountably many totally irrationals* $\epsilon \in S$ *such that* $\psi_{\Phi,\epsilon}(t) \leq \phi(t)$ *for all large enough* t.

This immediately gives that there exists uncountably many totally irrationals $\epsilon \in S$ that are singular if we let $\Phi(q) = |q|$. We call the combination of the four properties above as property A'.

Theorem 1.4. [2] *Let* $n \geq 2$ *and let* $S_1, \dots, S_n$ *be perfect subsets of* $\mathbb{F}_q((T^{-1}))$ *such that* $(\mathbb{F}_q(T) \cap S_1)$ *is dense in* S_1 *and* $(\mathbb{F}_q(T) \cap S_2)$ *is dense in* S_2. *Let* $S = \prod_{j=1}^n S_j$. *Then there are collections* $\{L_i\}$, $\{L'_j\}$, $\{A_i\}$ *satisfying property A'.*

Lemma 1.1. [2] *Let* $k \geq 2$, *and let* $M \subset \mathbb{R}^n$ *be a connected* k*-dimensional real analytic submanifold which is not contained in a proper rational affine subspace of* $\mathbb{R}^n$. *Then* M *contains a bounded real analytic surface which is not contained in a proper rational affine subspace of* $\mathbb{R}^n$.

Theorem 1.5. [2] *Let* S *be a conencted real analytic submanifold of* $\mathbb{R}^n$ *of dimension at least* 2 *which is not contained in any proper rational affine subspace of* $\mathbb{R}^n$. *Then there are collections* $\{L_i\}$, $\{L'_j\}$, $\{A_i\}$ *satisfying condition A'.*

1.4. **Properties on ψ-Dirichlet Sets.** We are also very interested in more general ψ-Dirichlet sets.

Definition 1.6. *For a non-increasing $\psi : [t_0, +\infty) \to \mathbb{R}_+$, let $D(\psi)$ be the set of $x \in \mathbb{R}$ with a nontrivial integer solution for all large enough t for the system $|qx - p| < \psi(t)$, $|q| < t$. The elements of $D(\psi)$ are called ψ-Dirichlet.*

Lemma 1.2. [3] *Let ψ be as above. Then $x \in [0, 1]\backslash\mathbb{Q}$ is ψ-Dirichlet if and only if $\langle q_{n-1}x\rangle < \psi(q_n)$ for sufficiently large n.*

Theorem 1.6. [3] *Let ψ be as above. If $\psi(t) < \frac{1}{t}$ for sufficiently large t, then $D(\psi) \neq \mathbb{R}$.*

2. Analogous Properties in $K = \mathbb{F}_q((T^{-1}))$

2.1. **Basic Properties of $K = \mathbb{F}_q((T^{-1}))$.** We define rationals, continued fractions, singularities similarly as in $\mathbb{R}$. Notice that the continued fractions also has the best approximation theorem here [4].

Lemma 2.1. *K^n is Heine-Borel.*

Proof. Consider an arbitrary bounded closed subset $B \subseteq (\mathbb{F}_q(T^{-1}))^n$. For an arbitrarily infinite sequence $\{x_i\} \subseteq B$, we claim that there exists a subsequence of $\{x_i\}$ that converges to a point in B. It suffices to look at only one coordinate, say $\{y_i\}$ is the sequence of the first coordinates of $\{x_i\}$. By Heine-Borel in $\mathbb{R}$, there exists a subsequence k_i of $\{1, 2, \dots\}$ such that $\{||y_{k_i}||\}$ converges. If the limit point of the subsequence is 0, then of course $\{y_{k_i}\}$ would converge to 0. If the limit is not 0, notice that the "norm" can only take values in the form e^z where $z \in \mathbb{Z}$, we see that there is a subsequence $\{k_i'\}$ of $\{k_i\}$ such that $||y_{k_i'}|| = e^z$ for any i. Since $\mathbb{F}_q$ is a finite field, there is an infinite subsequence $\{m_{0,i}'\}$ of $\{k_i'\}$ such that the coefficients of the degree z terms of $\{y'_{m_{(0,i)}}\}$ are all equal. By induction, for any $j \in \mathbb{Z}, j > 0$, there exists an infinite subsequence $\{m_{j,i}'\}$ of $\{m_{j-1,i}'\}$ such that the coefficients of the degree $z-j$ terms of $\{y'_{m_{(j,i)}}\}$ are all equal. Therefore we may take a subsequence $\{m_i\}$ of $\{1, 2, \dots\}$ such that y_{m_i} converges. By doing this coordinate-wise, we see that there is a convergent subsequence for any infinite sequence of B, and by the closeness of B, the limit point of the subsequence is in B. This shows that B is sequentially compact. The "norm" defined above makes $(\mathbb{F}_q(T^{-1}))^n$ a valid metric space, so B is compact. $\square$

2.2. **Analogous Properties of Singularities in K^n where $K = \mathbb{F}_q((T^{-1}))$.** To begin with, we have the following theorem,

Theorem 2.1. *The singular numbers in $\mathbb{F}_q((T^{-1}))$ are all 'trivial' i.e.* $\mathrm{Sing} = \mathbb{F}_q(T)$.

Proof. It suffices to show that $\mathrm{Sing} \subseteq \mathbb{F}_q(T)$. For any $x \in \mathbb{F}_q((T^{-1})) \cap \mathrm{Sing}$ such that $x \notin \mathbb{F}_q(T)$, fix the Q' and choose an n such that $|q_n| > Q'$. We will make advantage of continued fraction mentioned in the previous slide. Take $Q = |q_{n+1}|$. We have $\frac{1}{q_{n+1}} = \langle q_n x\rangle \leq \frac{\epsilon}{|q_{n+1}|}$ for every $\epsilon > 0$. This leads to a contradiction.

$\square$

Notice that there is no more $\frac{1}{2}$ in the estimation, which reminds us of the differences in the ultrametric case. Ultilizing the Heine-Borel property just proved above, we can prove the following theorem.

Theorem 2.2. *Let $S \subset (\mathbb{F}_q((T^{-1})))^n$ be a nonempty locally closed subset. Let $\{L_1, L_2, \dots\}$ and $\{L_1', L_2', \dots\}$ be disjoint collections of distinct closed subsets of S, each of which is contained in a rational affine hyperplane in $(\mathbb{F}_q((T^{-1})))^n$, and for each i let A_i be a rational affine hyperplane containing L_i, assume the following hold:*

(a)

$$\bigcup_i L_i \cup \bigcup_j L_j' = \{x \in S \ : x \text{ is contained in a rational affine hyperplane}\};$$

(b) For each i and each $T > 0$,

$$L_i = \overline{\bigcup_{|A_j|>T} L_i \cap L_j};$$

(c) For each i, and for any finite subsets of indices F, F' with $i \notin F$, we have

$$L_i = \overline{L_i - (\bigcup_{k\in F} L_k \cup \bigcup_{k'\in F'} L_{k'}')};$$

(d) $\bigcup_i L_i$ is dense in S.

Then for arbitrary $\Phi : ((\mathbb{F}_q[T])^n \setminus \{0\}) \to \mathbb{R}_+$ being a proper function, i.e., the set $\{q \in ((\mathbb{F}_q[T])^n \setminus \{0\}) \ : \Phi(q) \le C\}$ being finite for any $C > 0$, and for any non-increasing function $\phi \ : \mathbb{R}_+ \to \mathbb{R}_+$, there exists uncountably many totally irrationals $\epsilon \in S$ such that $\psi_{\Phi,\epsilon}(t) \le \phi(t)$ for all large enough t.

For our convenience we will refer to the combination of the four properties above as property A.

Proof. Let $B = \{\epsilon \in S \,\exists\, t_0$ such that $\forall t \ge t_0, \psi_{\Phi,\epsilon}(t) \le \phi(t)$ and ϵ is totally irrational$\}$, and suppose by contradiction that B is at most countably finite. Write $B = \{b_1, b_2, \dots\}$. Let W be an open subset of $(\mathbb{F}_q((T^{-1})))^n$ such that $S = \overline{S} \cap W$. Put $U_0 = W$, $q_0 = 0, p_0 = 0, i_0 = 0, \Phi(0) = 0$. We will show that for each $v \in \mathbb{N}$ there is a bounded open set $U_v \subset W$ and an index $i_v \in \mathbb{N}$ such that with the notation $(p_v, q_v) = m_{i_v}$, the following conditions are satisfied:

1. $\emptyset \ne \overline{S \cap U_v} \subset U_{v-1}$;
2. $i_v > i_{v-1}$ and $\Phi(q_v) > \Phi(q_{v-1})$ for all $v \in \mathbb{N}$;
3. For all $k < v$, U_v is disjoint from $L_k \cup L_k' \cup \{b_k\}$;
4. For all $v\mathbb{N}$ and all $\epsilon \in U_v$ we have $|\epsilon \cdot q_{v-1} - p_{v-1}| < \phi(\Phi(q_v))$;
5. For all $v \in \mathbb{N}$, $U_v \cap L_{i_v} \ne \emptyset$.

To see this suffices, take a point $\epsilon \in S \cap \bigcap_v U_v = \bigcap_v \overline{S \cap U_v}$. This intersection is nonempty by Cantor intersection theorem. We will reach a contradiction by showing that both $\epsilon \in B$ and $\epsilon \notin B$. By (3), $\forall i, \epsilon \ne b_i$ and thus $\epsilon \ne B$. Also by (3), ζ is not contained in any of the sets in the collections L and L', and by (a) ζ is totally irrational. The irrationality measure function $\psi_{\Phi,\epsilon}$ is non-increasing, and the properness condition guarantees that $\Phi(q_v) \to \infty$ as $v \to \infty$. By (2), for any $t > t_0 = \Phi(q_1)$, there is a v with $t \in [\Phi(q_v), \Phi(q_{v+1})]$ and by (4) we have $\psi_{\Phi,\epsilon}(t) \le \psi_{\Phi,\epsilon}(\Phi(q_v)) \le \langle q_v \cdot \epsilon\rangle \le |q_v \cdot \epsilon - p_v| < \phi(\Phi(q_{v+1})) \le \phi(t)$. This shows that $\epsilon \in B$.

The inductive construction starts with $v = 1$. Choose $i_1 = \min\{i \in \mathbb{N} \ : L_i \ne \emptyset\}$. Define U_1 to be some open set containing a point in L_1 and such that $\overline{U_1} \subset W$. Then property $(1) - (5)$ follows from this choice. Now suppose we have constructed U_k and

i_k with the desired properties for $k = 1, \ldots, v$. Let $i = i_v$. By (5) for $k = v$ we have $U_v \cap L_i \neq \emptyset$. By hypothesis (b) there is an infinite subsequence of indices j such that along the subsequence, $U_v \cap L_i \cap L_j \neq \emptyset$ and $|A_j| \to_{j\to\infty} \infty$. For each such j, write $A_j = A_{m_j}$ where $m_j = (p'_j, q'_j)$. Then along this subsequence we have $||q'_j|| \to \infty$, and hence we can choose $j > i$ such that $\Phi(q'_j) > \Phi(q_v)$. We then set $i_{v+1} = j$. This choice ensures that (2) holds for $v+1$. Let $\epsilon_1 \in U_v \cap L_i \cap L_j$. The point belong to L_i and hence satisfies $\epsilon_1 \cdot q_v = p_v$. By continuity we can take a neighbourhood $V \subset U_v$ around ϵ_1, so that for all $\epsilon \in V$ we have $|\epsilon \cdot q_v - p_v| < \phi(\Phi(q_{v+1}))$. This is the inequality in (4) for $v+1$. Since $\epsilon_1 \in L_j = L_{i_{v+1}}$ we have $V \cap L_{v+1} \neq \emptyset$, so we can apply hypothesis (c) to find that there is $\epsilon \in L_j \cap V \setminus \bigcup_{k<v+1}(L_k \cup L'_k \cup \{b_k\})$. Furthermore, we can take a neighbourhood U_{v+1} of ϵ such that $\overline{U_{v+1}} \subset U_v$ and $U_{v+1} \cap \bigcup_{k<v+1}(L_k \cup L'_k \cup \{b_k\}) = \emptyset$. This completes the construction. □

As a corollary of the previous theorem we have the following theorem,

Theorem 2.3. *Let $n \geq 2$ and let $S_1, \ldots, S_n$ be perfect subsets of $\mathbb{F}_q((T^{-1}))$ such that $(\mathbb{F}_q(T) \cap S_1)$ is dense in S_1 and $(\mathbb{F}_q(T) \cap S_2)$ is dense in S_2. Let $S = \prod_{j=1}^n S_j$. Then there are collections $\{L_i\}$, $\{L'_j\}$, $\{A_i\}$ satisfying property A.*

Proof. Let $e_1, \ldots, e_n$ be standard base vectors, and let $\{A_i\}$ be the collection of all rational hyperplanes which are normal to one of e_1, e_2 and have nontrivial intersection with S. For each i define $L_i = S \cap A_i$, and let $\{L'_j\}$ denote the collection of non-empty intersections $S \cap A$, where A is a rational affine hyperplane and the sets L'_j does not appear in the list $\{L_i\}$. We claim that with these choices, hypotheses of theorem 1.1 are satisfied. The hypothesis (a) and (d) follows directly from our definition. For (b) and (c), let us first consider those L_i with $\zeta_1 = \frac{p_i}{q_i} \in \mathbb{F}_q(T)$. Then we can choose coprime p_j, q_j-s such that $||q_j|| \to \infty$ and $\frac{p_j}{q_j} \to \zeta_2$. Let $L_j = \{\zeta \in (\mathbb{F}_q((T^{-1})))^n \ : \zeta_2 = \frac{p_j}{q_j}\} \cap S$. Then we see that ζ is an accumulation point of $L_i \cap L_j$ for any $|A_J| > T$ as $|A_j| = ||q_j|| \to \infty$. This proves (b). To prove (c), notice that $S_2, \ldots, S_n$ are perfect, so the intersection of L_i with an arbitrary open subset of $(\mathbb{F}_q((T^{-1})))^n$ cannot lie in a union of finitely many proper affine subspaces of $(\mathbb{F}_q((T^{-1})))^n$ different from A_i.

□

We encountered difficulties to prove the existence of collections satisfying property A. However, we are able to simplify a little bit strengthened case into dimension 2.

Lemma 2.2. *Suppose that for $k \geq 3$, M is a k-dimensional K-analytic submanifold of K^n which is the image of I^k under a K-analytic immersion $f \ : I^k \to K^n$, where I is a bounded open and closed ball in K (which is possible because of the total-disconnectedness of K). Suppose also that M is totally not contained in any rational affine hyperplane. Then there exists $\alpha \in I$ such that the K-analytic submanifold $f_\alpha(I^{k-1})$ where $f_k \ : I^{k-1} \to K^n$, $f_\alpha(x_1, \ldots, x_{k-1}) = f(x_1, \ldots, x_{k-1}, \alpha)$ does not belong to any rational affine hyperplane.*

Proof. Suppose not. Then $\bigcup_i f^{-1}(A_i \cap M) = I^k$ where A_i is taken over all rational affine hyperplanes in K^n. The left is a countable union of closed subsets of I^k so by the Baire category theorem, at least one of them, say $f^{-1}(A_0 \cap M)$ has an nonempty interior. This means that there is a nonempty open subset $U \subset I^k$ such that $f(U) \subset A_0$. Then $M \cap A_0$ has nonempty interior in M, contradicting the assumption that M is totally not contained in A_0.

□

Lemma 2.3. *Let $k \geq 2$ and let $M \subset K^n$ be a k-dimensional K-analytic submanifold of K^n which is totally not contained in a proper rational affine subspace of K^n. Then M contains a bounded K-analytic surface (dimension 2 submanifold) which is not contained in a proper rational affine subspace of K^n.*

Proof. Induction on dimension using the previous lemma gives the result.

□

The above lemma reduces the problem to dimension 2. But there are some difficulties occurred to proceed further. We are currently working on it.

2.3. **ψ-Dirichlet on $K = \mathbb{F}_q((T^{-1}))$.** It is even more difficult to proceed with an analogy of these properties. However, we still have some intermediate results.

Lemma 2.4. *Let $\psi : [t_0, +\infty) \to \mathbb{R}_+$ be non-increasing. Then $x \in \mathbb{F}_q((T^{-1})) \setminus \mathbb{F}_q(T)$ is ψ-Dirichlet if and only if $|\langle q_{n-1}x\rangle| < \psi(|q_n|)$ for sufficiently large n.*

Proof. Suppose x is $x \in \mathbb{F}_q((T^{-1})) \setminus \mathbb{F}_q(T)$ is ψ=Dirichlet. Then for sufficiently large n there exists a $q \in \mathbb{F}_q[T]$ with $|\langle qx\rangle| < \psi(|q_n|)$, $|q| < |q_n|$. Since $|\langle q_{n-1}x\rangle| \leq |\langle qx\rangle|$ whenever $|q| < |q_n|$, we have $|\langle q_{n-1}x\rangle| < \psi(|q_n|)$ for sufficiently large n. Conversely, suppose $|\langle q_{n-1}x\rangle| < \psi(|q_n|)$ for $n \geq N$. Then for a real number $t > |q_N|$, take $|q_{n-1}| < t \leq |q_n|$. The inequality $|\langle q_{n-1}x\rangle| < \psi(t)$ follows since ψ is non-increasing, so x is ψ-Dirichlet.

□

3. Acknowledgements

The author wishes to express his gratitude to Professor Shreyasi Datta for her patience and efforts in this REU. The author also want to thank the Department of Mathematics, University of Michigan for giving him a great chance in engaging in mathematical research.

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