Computing The Constant In The Left-tail Asymptotic Of Maximum Eigenvalue Distribution Of Finite GUE

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Abstract

1 Introduction

In the field of nuclear Physics, Eugene Wigner first utilized random matrices to model the nuclei of heavy atoms in the 1950s[12]. He proposed that the spacing between the eigenvalues on the real line represents difference between energy levels. After that, Freeman Dyson also encountered radndom matrices, GUE specifically, in his Brownian motion model of Coulomb gas[2]. After that, Mehta and Gaudin refined his work to offer a precise expression for the asymptotic behavior of eigenvalue density functions[5]. The model is also widely adopted in studies of fermions in harmonic trap and queueing theory. Later on, scholars like Tracy and Widom[10], as well as Forrester and Witte[3], furthur explored the connection between random matrix properties and differential systems, including the Painlevé system.

This report will introduce the definitions of the GUE first, with a brief derivation of its eigenvalue distribution function. Our research focuses on the finite dimensional case, where entry distribution has an impact on the eigenvalue distribution. More specifically, our discussion concerns the maximum eigenvalues. We first calculate the normalization constant for the maximum eigenvalue distribution function, then discuss the asymptotic behaviors. For the left tail, both Hankel determinant and Toda equation offer complete solution. The righttail, on the other hand, can be easily solved by the Fredholm determinant. After that, we will briefly discuss a related ensemble named thinned ensemble. Finally,

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we resort to Brownian motion simulation to verify our computational results, then showcase two physics models where GUE eigenvalue distribution plays a major role.

2 Building Blocks of GUE

Start with these definitions,

- **Hermitian matrices** a matrix M is Hermitian if $M^T = \overline{M}$

- **Unitary matrices** a matrix M is Unitary if $M^T = M^{-1}$

- **Gaussian random variable**

Random variable X is Gaussian if it has distribution function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

where μ is the mean and σ is the standard deviation, denoted as $X \sim N(\mu, \sigma^2)$.

- **Gaussian Unitary Ensemble**

Collection of Hermitian matrices H whose entries are selected independently from Gaussian random distribution; their probabilistic distributions are preserved with conjugation by unitary matrices, i.e.

$$Prob(H \in A) = Prob(UHU^{-1} \in A)$$

More explicitly, it has the form:

$$H_{ii} \sim N(0, \tfrac{1}{\sqrt{2}}) \text{ and } Re(H_{ij}), Im(H_{ij}) \sim N(0, \tfrac{1}{2}) \text{ where } i \neq j$$

It's also worth noting that all eigenvalues of GUE matrices are real.

- **Eigenvalue Distribution**

We can express the probability distribution function of the eigenvalues in R^n as in Forrester's book on Log-Gases [4]

$$Prob((\lambda_1, \cdots, \lambda_n) \in B) = \frac{1}{Z_n} \int\limits_B e^{-\sum_1^n \lambda_i^2} \prod |\lambda_j - \lambda_k|^2 \prod d\lambda_l$$

This is the starting point of our research.

- **Largest Eigenvalue**

Given their probability distribution functions, we can calculate the distribution of the largest eigenvalues:

$$P(\lambda_{max} < t) = Prob((\lambda_1, \cdots, \lambda_n) \in (-\infty, t)^n$$

$$= \frac{1}{Z_n} \int\limits_{(-\infty,t)^n} e^{-\sum_1^n \lambda_i^2} \prod |\lambda_j - \lambda_k|^2 \prod d\lambda_l$$

We are mainly concerned about its right and left tail asymptotic behaviors.

• **Infinite Dimensional GUE Largest Eigenvalue Distribution**
Dubbed the Tracy Widom distribution, it is well-known that maximum eigenvalue of infinite-dimensional GUE follows the probability distribution function [11]

$$P(\lambda_{max}, N) \sim \sqrt{2}N^{2/3} f(\sqrt{2}N^{2/3}(\lambda_{max} - \sqrt{2}))$$

where f is some function that satisfies Painlevé II - a specific class of differential equations. This result is actually universal to all random Hermitian matrices, regardless of the distribution of their individual entries[9].

3 Normalization Constant

As we can see in the previous section, a normalization constant z_n is needed to ensure $\lim\limits_{t\to\infty} P(\lambda_{max} < t) = 1$. In this section we describe the steps to derive an explicit formula for this constant. To do this, we need to introduce a few definitions.

Hermite Polynomials
Hermite polynomials are formally defined as

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}$$

To list the first few of them:

$$\begin{aligned} H_0(x) =& 1 \\ H_1(x) =& 2x \\ H_2(x) =& 4x^2 - 2 \\ H_3(x) =& 8x^3 - 12x \end{aligned}$$

Hermite polynomials are orthogonal, i.e.

$$\int\limits_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2}\, dx = (2^m m! \sqrt{\pi}) \delta_{m,n}$$

Also note that the leading term of the nth Hermite polynomial is always $2^n x^n$.

Christoffel-Darboux kernel

This kernel is defined in terms of Hermite polynomials

$$K_n(x,y) = e^{-\frac{x^2}{2}-\frac{y^2}{2}} \sum_{k=0}^{N-1} \frac{H_k(x)H_k(y)}{2^k k! \sqrt{\pi}}$$

It has the following properties

1. $\int_{-\infty}^{\infty} K_N(x,x)\,dx = N$
2. $\int_{-\infty}^{\infty} K_N(x,y)K_N(y,z)\,dy = K_N(x,z)$

Now we're ready to compute the normalization constants.

$$\begin{aligned} z &= \lim_{t\to\infty} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-\sum \lambda_k^2} \prod d\lambda_k \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-\sum \lambda_k^2} \prod d\lambda_k \end{aligned}$$

starting with the probability distribution integrated over the entire real line, we express $(\lambda_i - \lambda_j)^2$ in the form of Vandermonde determinants

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \begin{vmatrix} 1 & \lambda_1 & \dots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{N-1} \\ \vdots & & & \vdots \\ 1 & \lambda_N & \dots & \lambda_N^{N-1} \end{vmatrix} \begin{vmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_N \\ \vdots & & \vdots \\ \lambda_1^{N-1} & \dots & \lambda_N^{N-1} \end{vmatrix} e^{-\sum \lambda_k^2} \prod d\lambda_k$$

then we re-write them in terms of Hermite polynomials

$$= \prod_{k=0}^{N-1} 2^{-2k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \begin{vmatrix} H_0(\lambda_1) & H_1(\lambda_1) & \dots & H_{N-1}(\lambda_1) \\ \vdots & \vdots & & \vdots \end{vmatrix} \begin{vmatrix} H_0(\lambda_1) & \dots \\ \vdots & \\ H_{N-1}(\lambda_1) & \dots \end{vmatrix} e^{-\sum \lambda_k^2} \prod d\lambda_k$$

Here K_N refers to the Christoffel-Darboux kernel.

$$= \prod_{k=0}^{N-1} 2^{-2k} \prod_{k=0}^{N-1} 2^k k! \sqrt{\pi} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \det(K_N(\lambda_i, \lambda_j))_{i,j=1,\dots,N} \prod d\lambda_k$$

Now we expand the determinant with respect to the last row of matrix, then expand again with respect to the last column. Denote as $\det\left(K_N^{(N,i;N,j)}\right)$ the sub-matrix given by removing the Nth and ith row, as well as the Nth and ith column.

$$
\begin{aligned}
\det(K_N(\lambda_i,\lambda_j))_{i,j=1,\ldots,N} &= \sum_{j=1}^{N}(-1)^{j+N}K_N(\lambda_N,\lambda_j)\det\Big(K_N^{(N;j)}\Big) \\
&= K_N(\lambda_N,\lambda_N)\det\Big(K_N(\lambda_i,\lambda_j)_{i,j=1,\ldots,N-1}\Big) \\
&+ \sum_{j=1}^{N-1}(-1)^{j+N}K_N(\lambda_N,\lambda_j)\sum_{i=1}^{N-1}(-1)^{i+N-1}K_N(\lambda_i,\lambda_N)\det\Big(K_N^{(N,i;N,j)}\Big)
\end{aligned}
$$

Recall the two properties about Christoffel-Darboux kernel:

$$
\int_{-\infty}^{\infty} K_N(x,x)\,dx = N
$$

$$
\int_{-\infty}^{\infty} K_N(x,y)K_N(y,z)\,dy = K_N(x,z)
$$

Integrate with respect to the last eigenvalue λ_N and use these properties to simplify the kernels. Then convert the double summation back to the determinant form with respect to $i = N-1$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \det(K_N(\lambda_i,\lambda_j))_{i,j=1,\ldots,N}\,d\lambda_N \\
&= N\det\Big(K_N(\lambda_i,\lambda_j)_{i,j=1,\ldots,N-1}\Big) - \int_{-\infty}^{\infty}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}(-1)^{i+j}K_N(\lambda_i,\lambda_N)K_N(\lambda_N,\lambda_j)\det\Big(K_N^{(N,i;N,j)}\Big)\,d\lambda_N \\
&= N\det\Big(K_N(\lambda_i,\lambda_j)_{i,j=1,\ldots,N-1}\Big) - \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}(-1)^{i+j}K_N(\lambda_i,\lambda_j)\det\Big(K_N^{(N,i;N,j)}\Big) \\
&= (N-(N-1))\det\Big(K_N^{(N;N)}\Big)
\end{aligned}
$$

By induction on the matrix size, we have:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \det(K_N(\lambda_1,\lambda_j)\,d\lambda_1\ldots d\lambda_N) &= (N-(N-1))(N-(N-2))\ldots(N-(N-1)) \\
&= N!
\end{aligned}
$$

Therefore we found the normalization constant:

$$
\begin{aligned}
z &= N!\prod_{k=0}^{N-1}2^{-k}\sqrt{\pi}k! \\
&= \pi^{\frac{N}{2}}2^{-\frac{N(N-1)}{2}}\prod_{k=0}^{N}k!
\end{aligned}
$$

3.1 Verifications

We can expand the distribution into Vandermonde determinant form and quickly calculate the distribution and normalization constants for small values of n like $n = 1, 2, 3$ to verify our computation above.

First, an $n \times n$ Vandermonde matrix V has the form

$$V = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix} \quad or \quad V_{(i,j)} = x_i^{j-1}$$

Induction on matrix size n easily gives

$$det(V) = \prod_{1 \le i < j \ leqn} (x_j - x_i)$$

We can calculate the normalization constant explicitly by employing the Vandermonde determinant described above.The largest eigenvalue is no greater than t means that all eigenvalues fall into the interval $(-\infty, t)$, so we integrate the probability function we found earlier on this interval.

$$P(\lambda_{max} < t) = \frac{1}{z} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \prod (\lambda_i - \lambda_j)^2 e^{-\sum \lambda_k^2} \, d\lambda_1 \cdots d\lambda_N$$

We replace each of the product in $\prod(\lambda_i - \lambda_j)^2 = \prod(\lambda_i - \lambda_j) \prod(\lambda_i - \lambda_j)$ with the corresponding matrices according to the Vandermonde determinant. Using the transpose has no effect on the determinant.

$$= \frac{1}{z} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \begin{vmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ \lambda_1^{N-1} & \dots & \lambda_N^{N-1} \end{vmatrix} \begin{vmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ \lambda_1^{N-1} & \dots & \lambda_N^{N-1} \end{vmatrix} e^{-\sum \lambda_k^2} \, d\lambda_1 \cdots d\lambda_N$$

Calculate the determinant for the second matrix, but this time using permutations,

$$= \frac{1}{z} \sum_{\sigma} sign(\sigma) \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \begin{vmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ \lambda_1^{N-1} & \dots & \lambda_N^{N-1} \end{vmatrix} \prod \lambda_i^{\sigma(i)-1} e^{-\sum \lambda_k^2} \, d\lambda_1 \cdots d\lambda_N$$

Multiply the permuted λs from the second matrix into the first matrix,

$$= \frac{1}{z} \sum_{\sigma} sign(\sigma) \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \begin{vmatrix} \lambda_1^{\sigma(1)-1} & \cdots & \lambda_N^{\sigma(N)-1} \\ \vdots & \ddots & \vdots \\ \lambda_1^{\sigma(1)+N-2} & \dots & \lambda_N^{\sigma(N)+N-2} \end{vmatrix} e^{-\sum \lambda_k^2} \, d\lambda_1 \cdots d\lambda_N$$

Multiply each of the $e^{\lambda_k^2}$ into the kth column of the first matrix. Now each column in the matrix contains only one λ, so we can bring in the n-integral. Integrating the ith column with respect to λ_i,

$$= \frac{1}{z} \sum_{\sigma} sign(\sigma) \begin{vmatrix} \int_{-\infty}^{t} \lambda_1^{\sigma(1)-1} e^{-\lambda_1^2}\, d\lambda_1 & \dots & \int_{-\infty}^{t} \lambda_N^{\sigma(N)-1} e^{-\lambda_N^2}\, d\lambda_N \\ \vdots & \ddots & \vdots \\ \int_{-\infty}^{t} \lambda_1^{\sigma(1)+N-2} e^{-\lambda_1^2}\, d\lambda_1 & \dots & \int_{-\infty}^{t} \lambda_N^{\sigma(N)+N-2} e^{-\lambda_N^2}\, d\lambda_N \end{vmatrix}$$

For each of the permutations, swap columns so that the ith λ lands in the ith column. This will give another $sign(\sigma)$, which cancels out the sign in front. Therefore:

$$= \frac{n!}{z} \det\{ \int\limits_{-\infty}^{t} \lambda^{i+j} e^{-\lambda^2}\, d\lambda \quad {}_{i,j=0,\dots,N-1} \}$$

For $n = 1, 2, 3$, we can calculate the probability relatively easily with this method:

$$Prob_1(t) = \frac{1}{z_1} \int\limits_{-\infty}^{t} e^{-\lambda^2\, d\lambda} = \frac{1}{2}(1 + erf(t))$$

where $z_1 = \sqrt{\pi}$.

$$\begin{aligned} Prob_2(t) &= \frac{1}{z_2} \det \begin{vmatrix} \int_{-\infty}^{t} e^{-\lambda^2}\, d\lambda & \int_{-\infty}^{t} \lambda e^{-\lambda^2}\, d\lambda \\ \int_{-\infty}^{t} \lambda e^{-\lambda^2}\, d\lambda & \int_{-\infty}^{t} \lambda^2 e^{-\lambda^2}\, d\lambda \end{vmatrix} \\ &= \frac{1}{z_2} (\int\limits_{-\infty}^{t} e^{-\lambda^2}\, d\lambda \int\limits_{-\infty}^{t} \lambda^2 e^{-\lambda^2}\, d\lambda - (\int\limits_{-\infty}^{t} \lambda e^{-\lambda^2}\, d\lambda)^2) \\ &= \frac{1}{z_2} ((\frac{\sqrt{\pi}}{2}(1 + erf(x)))(\frac{\sqrt{\pi}}{4} + \frac{\sqrt{\pi}}{4} erf(t) - \frac{e^{-t^2}}{2}) - (-\frac{e^{-t^2}}{2})^2) \end{aligned}$$

With the help of Wolfram, we find $z_2 = \frac{\pi}{2}$.

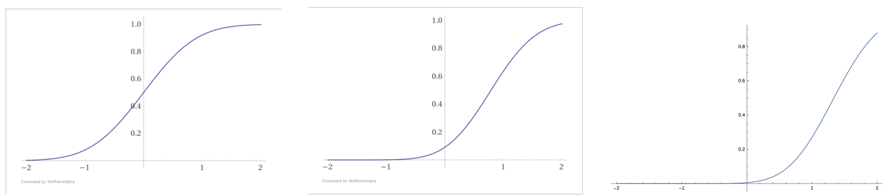
$$Prob_3(t) = \frac{1}{z_3} \det \begin{vmatrix} \int_{-\infty}^{t} e^{-\lambda^2\, d\lambda} & \int_{-\infty}^{t} \lambda e^{-\lambda^2\, d\lambda} & \int_{-\infty}^{t} \lambda^2 e^{-\lambda^2\, d\lambda} \\ \int_{-\infty}^{t} \lambda e^{-\lambda^2\, d\lambda} & \int_{-\infty}^{t} \lambda^2 e^{-\lambda^2\, d\lambda} & \int_{-\infty}^{t} \lambda^3 e^{-\lambda^2\, d\lambda} \\ \int_{-\infty}^{t} \lambda^2 e^{-\lambda^2\, d\lambda} & \int_{-\infty}^{t} \lambda^3 e^{-\lambda^2\, d\lambda} & \int_{-\infty}^{t} \lambda^4 e^{-\lambda^2\, d\lambda} \end{vmatrix}$$

Mathematica gives the result as

$$-\frac{1}{8}\sqrt{\pi}\, e^{-2t^2} t^2 \operatorname{erf}(t) - \frac{3}{32}\pi e^{-t^2} t \operatorname{erf}(t)^2 - \frac{3}{16}\pi e^{-t^2} t \operatorname{erf}(t) - \frac{3}{32}\sqrt{\pi}\, e^{-2t^2} \operatorname{erf}(t) - \frac{1}{16}\pi e^{-t^2} t^3 \operatorname{erf}(t)^2 - \frac{1}{8}\pi e^{-t^2} t^3 \operatorname{erf}(t) + \frac{1}{32}\pi^{3/2} \operatorname{erf}(t)^3 + \frac{3}{32}$$
$$\pi^{3/2} \operatorname{erf}(t)^2 + \frac{3}{32}\pi^{3/2} \operatorname{erf}(t) - \frac{1}{8}\sqrt{\pi}\, e^{-2t^2} t^2 - \frac{1}{16} e^{-3t^2} t - \frac{3}{32}\pi e^{-t^2} t - \frac{3}{32}\sqrt{\pi}\, e^{-2t^2} - \frac{1}{16}\pi e^{-t^2} t^3 + \frac{\pi^{3/2}}{32}$$

And the corresponding $z_3 = \frac{\pi^{\frac{3}{2}}}{4}$.

We briefly sketched out these cumulative probability distributions:



(a) $n = 1$ (b) $n = 2$ (c) $n = 3$

Figure 1: Cumulative Distribution Function For Small n

They confirmed the intuition that having more eigenvalues will shift the distribution to the right since it's more likely to yield a larger maximum.

4 Left-Tail Asymptotic

4.1 Hankel Determinant Method

Theorem 1. *At negative infinity, Big O for the maximum eigenvalue distribution for $N \times N$ GUE is*

$$Prob(\lambda_{max} < t) \simeq (-1)^N \frac{\prod_{k=0}^{N-1} k!}{\pi^{\frac{N}{2}} 2^{\frac{N(N+1)}{2}}} \frac{e^{-Nt^2}}{t^{N^2}}, \quad t \to -\infty$$

Proof. We start with the general distribution from the introduction and integrate over $(-\infty, t)^N$ as the probability that all eigenvalues fall into this interval. Shift the integral to eliminate variable t from the interval we're integral on.

$$\begin{aligned} Prob(\lambda_{max} < t) &= \frac{1}{z} \int\limits_{(-\infty,t)^N} \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)^2 \prod e^{-\lambda_k^2}\, d\lambda_i \\ &= \frac{1}{z} \int\limits_{(-\infty,0)^N} \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)^2 \prod e^{-(\lambda_k + t)^2}\, d\lambda_i \end{aligned}$$

Furthur separate λ_k from t since the t term remains constant when integrating with respect to λ.

$$\begin{aligned} Prob(\lambda_{max} < t) &= \frac{1}{z} \int\limits_{(-\infty,0)^N} \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)^2 \prod e^{-\lambda_k^2} e^{-2\lambda_k t} e^{-t^2}\, d\lambda_i \\ &= \frac{e^{-Nt^2}}{z} \int\limits_{(-\infty,0)^N} \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)^2 \prod e^{-\lambda_k^2} e^{-2\lambda_k t}\, d\lambda_i \end{aligned}$$

Use change of variable $\lambda_i t = x_i$ and adjust the signs accordingly

$$\begin{aligned}Prob(\lambda_{max} < t) &= (-1)^N \frac{1}{z} e^{-Nt^2} \int\limits_{(0,\infty)^N} \prod_{1\leq i<j\leq N} (\frac{x_i}{t} - \frac{x_j}{t})^2 \prod e^{-\frac{x_k^2}{t^2}-2x_k} (\frac{1}{t}\, dx_i) \\ &= (-1)^N \frac{1}{z} \frac{e^{-Nt^2}}{t^{N^2}} \int\limits_{(0,\infty)^N} \prod_{1\leq i<j\leq N} (x_i - x_j)^2 \prod e^{-\frac{x_k^2}{t^2}-2x_k}\, dx_i\end{aligned}$$

At $t \to -\infty$, $x_i \to \infty$ while $\frac{x_k^2}{t^2}$ is some constant, therefore

$$\prod e^{-\frac{x_k^2}{t^2}-2x_k} \simeq \prod e^{-2x_i} = e^{\sum -2x_i}$$

Substitute back into the integral,

$$Prob(\lambda_{max} < t) \simeq (-1)^N \frac{1}{z} \frac{e^{-Nt^2}}{t^{N^2}} \int\limits_{(0,\infty)^N} \prod_{1\leq i<j\leq N} (x_i - x_j)^2 e^{\sum -2x_i}\, dx_i, \quad t \to -\infty$$

Use change of variable $y_i = 2x_i$ to get rid of 2 in the exponent

$$\begin{aligned}Prob(\lambda_{max} < t) &\simeq (-1)^N \frac{1}{z} \frac{e^{-Nt^2}}{t^{N^2}} \int\limits_{(0,\infty)^N} \prod_{1\leq i<j\leq N} (\frac{y_i}{2} - \frac{y_j}{2})^2 e^{-\sum y_k} (\frac{1}{2}\, dy_i) \\ &\simeq (-1)^N \frac{1}{z} \frac{e^{-Nt^2}}{(2t)^{N^2}} \int\limits_{(0,\infty)^N} \prod_{1\leq i<j\leq N} (y_i - y_j)^2 e^{\sum -y_k}\, dy_i\end{aligned}$$

Follow a process similar to the one introduced in the normalization constant section, we express the term $\prod_{1\leq i<j\leq N}(y_i - y_j)^2$ in the form of Vandermonde determinant and expand the second matrix by permutation,

$$\begin{aligned}Prob(\lambda_{max} < t) &\simeq (-1)^N \frac{1}{z} \frac{e^{-Nt^2}}{(2t)^{N^2}} \int\limits_{(0,\infty)^N} \begin{vmatrix} 1 & \dots & y_1^{n-1} \\ & \dots & \\ 1 & \dots & y_n^{n-1} \end{vmatrix} \begin{vmatrix} 1 & \dots & 1 \\ & \dots & \\ y_1^{n-1} & \dots & y_n^{n-1} \end{vmatrix} e^{\sum -y_k}\, dy_i \\ &\simeq (-1)^N \frac{1}{z} \frac{N! e^{-Nt^2}}{(2t)^{N^2}} \det\left(\int_0^\infty y^{i+j} e^{-y}\, dy \quad {}_{i,j=0,\dots,N-1} \right)\end{aligned}$$

Substitute $z = \pi^{\frac{N}{2}} 2^{-\frac{N(N-1)}{2}} \prod_{k=0}^N k!$ into our equation,

$$Prob(\lambda_{max} < t) \simeq \frac{(-1)^N}{\pi^{\frac{N}{2}} 2^{\frac{N(N+1)}{2}} \prod_{k=0}^{N-1} k!} \frac{e^{-Nt^2}}{t^{N^2}} \det\left(\int_0^\infty y^{i+j} e^{-y}\, dy \quad {}_{i,j=0,\dots,N-1} \right)$$

Notice that the determinant part gives a constant, denoted as z_L. We can compute this constant following a procedure analogous of constant computation in the last section. Instead of Hermite polynomials, this time we use the Laguerre polynomials with its corresponding kernel.

Laguerre Polynomials

Define the nth Laguerre polynomial L_n as

$$L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x}x^n$$

List the first few L_n

$$\begin{aligned}
L_0 &= 1 \\
L_1 &= -x + 1 \\
L_2 &= \frac{1}{2}x^2 - 2x + 1 \\
L_3 &= -\frac{1}{6}x^3 + \frac{3}{2}x^2 - 3x + 1
\end{aligned}$$

Notice that the coefficient of the leading term in L_n is $\frac{(-1)^n}{n!}$. They have orthogonality condition

$$\int_0^\infty L_i(x)L_j(x)e^{-x}\,dx = \delta_{i,j}$$

Associated Christoffel-Darboux kernel

Define the kernel in terms of Laguerre polynomials

$$K_n(x,y) = \sum_{i=0}^{n-1} L_i(x)L_i(y)$$

This kernel has the property that

$$\int_0^\infty K_n(x,x)e^{-x}\,dx = \sum_{i=0}^{n-1}\int_0^\infty L_i(x)L_i(x)e^{-x}\,dx = \sum_{i=0}^{n-1} 1 = n$$

Now we are ready to tackle this constant. Follow the same steps to expand the squared term, re-write them in the form of Laguerre polynomials, then calculate the new determinant with property of the kernel.

$$\begin{aligned}
z_L &= \det\left(\int_0^\infty y^{i+j}e^{-y}\,dy \quad {}_{i,j=0,\dots,N-1}\right) \\
&= \frac{1}{N!}\prod_{k=0}^{N-1}\left(\frac{k!}{(-1)^k}\right)^2 \int_0^\infty \cdots \int_0^\infty \begin{vmatrix} L_0(y_1) & \dots & L_{N-1}(y_1) \\ \vdots & & \vdots \end{vmatrix} \begin{vmatrix} L_0(y_1) & \dots \\ \vdots & \\ L_{N-1}(y_1) & \dots \end{vmatrix} e^{-\sum y_k} \prod dy_k \\
&= \frac{1}{N!}\prod_{k=0}^{N-1}\left(\frac{k!}{(-1)^k}\right)^2 \int_0^\infty \cdots \int_0^\infty \det(K_N(y_i,y_j))_{i,j=0,\dots,N-1} e^{-\sum y_k} \prod dy_k
\end{aligned}$$

Expand this determinant with respect to the last row of the matrix, then the last column.

$$
\begin{aligned}
&\det(K_N(y_i, y_j))_{i,j=0,\dots,N-1} e^{-\sum y_k} \prod dy_k \\
&= \sum_{j=1}^{N} (-1)^{j+N} K_N(y_N, y_j) \det\Big(K_N^{(N;j)}\Big) \\
&= K_N(y_N, y_N) \det\Big(K_N(y_i, y_j)_{i,j=1,\dots,N-1}\Big) + \\
&\sum_{j=1}^{N-1} (-1)^{j+N} K_N(y_N, y_j) \sum_{i=1}^{N-1} (-1)^{i+N-1} K_N(y_i, y_N) \det\Big(K_N^{(N,i;N,j)}\Big) \\
&= K_N(y_N, y_N) \det\Big(K_N(y_i, y_j)_{i,j=1,\dots,N-1}\Big) + \\
&\sum_{i=1}^{N-1} \sum_{j=1}^{N-1} (-1)^{i+j} K_N(\lambda_i, \lambda_N) K_N(\lambda_N, \lambda_j) \det\Big(K_N^{(N,i;N,j)}\Big)\, d\lambda_N
\end{aligned}
$$

Very similar to the process introduced in the normalization constant section, we integrate with respect to y_N then collapse the double summation term into an $(N-1) \times (N-1)$ matrix determinant

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \det(K_N(y_i, y_j))_{i,j=1,\dots,N}\, dy_N \\
&= N \det\Big(K_N(y_i, y_j)_{i,j=1,\dots,N-1}\Big) - \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} (-1)^{i+j} K_N(y_i, y_j) \det\Big(K_N^{(N,i;N,j)}\Big) \\
&= (N - (N-1)) \det\Big(K_N^{(N;N)}\Big)
\end{aligned}
$$

Induction on matrix size gives

$$
\int_0^{\infty} \cdots \int_0^{\infty} \det(K_N(y_i, y_j))_{i,j=0,\dots,N-1} e^{-\sum y_k} \prod dy_k = (N-(N-1))(N-(N-2)) \cdots = N!
$$

Therefore

$$
z_L = \prod_{k=0}^{N-1} (k!)^2
$$

And we arrive at our conclusion

$$
Prob(\lambda_{max} < t) \simeq (-1)^N \frac{\prod_{k=0}^{N-1} k!}{\pi^{\frac{N}{2}} 2^{\frac{N(N+1)}{2}}} \frac{e^{-Nt^2}}{t^{N^2}}
$$

$\square$

4.2 τ Function Theory and Painlevé Equation

Building on the Fredholm determinant method, Tracy and Widom established a connection between the eigenvalue distribution to solutions of differential equations. More precisely, one property of the Fredholm determinant states

$$\frac{\partial}{\partial \lambda_k} \log \det(I-K) = (-1)^{k-1} K_N(\lambda_k, \lambda_k)$$

Here $K_N(\lambda_k, \lambda_k)$ refers to the Christoffel-Darboux kernel. Therefore,

$$Prob(\lambda_{max} < t) = \exp\left(-\int_t^\infty R(s)\right)$$

where $R(s) = K_N(s,s)$. Such a function $R(s)$ has been proven to satisfy the σ version of Painlevé IV with parameters $\mu_1 = 0, \mu_2 = 2N$ [7]

$$(R'')^2 + 4(R')^2(R' + 2N) - 4(tR' - R)^2 = 0$$

More explicitly, $R(s)$ can be written as

$$R(s) = Ny - \frac{s^2}{2}y - \frac{s}{2}y^2 - \frac{1}{8}y^3 + \frac{1}{8y}(y')^2$$

where y solve Painlevé IV with parameters $\alpha = 2N-1$, $\beta = 0$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} = \frac{1}{2y}(\frac{\mathrm{d}y}{\mathrm{d}s})^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y}$$

Finally, we conclude that

$$P(\lambda_{max} < t) = \exp\left\{-\int_t^\infty R(s)\,ds\right\}$$

Additionally, we can find the expanded form of $R(t)$ at infinity by specifying $\lim\limits_{t\to\infty} R(t)$:

$$R(s) = -2Ns - \frac{N^2}{s} + \frac{N^3}{s^3} - \frac{N^2(1+9N^2)}{4s^5} + \frac{N^3(10+27N^2)}{4s^7} + \dots$$

Thus we found a representation of the eigenvalue distribution with Painlevé transcendent. Based on this asymptotic expression of $R(s)$, we can find the leading term of lefttail distribution. According to Tracy and Widom,

$$Prob(\lambda_{max} < t) \simeq \frac{c_N e^{-Nt^2}}{(-t)^{N^2}}, \quad t \to -\infty \tag{1}$$

By explicitly calculation from section 3, we determined the c_N for some small values of N:

N	1	2	3	4
c_N	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{8\pi}$	$\frac{1}{32\pi^{\frac{3}{2}}}$	$\frac{3}{256\pi^2}$

This constant is very costly to compute for larger N due to the complexity of N-integrals. We can generalize it for larger N using the Toda Equation. To start with, Witte and Forrester [3] established the connection between the probability distribution and the τ function as noted in section 2.1:

$$P(\lambda_{max} < t) = C^* \tau_3[N](t)$$

where the τ functions satisfy the following condition

$$\frac{d^2}{dt^2} \log\Big(e^{nt^2} \tau_3[n]\Big) = C \frac{\tau_3[n+1]\tau_3[n-1]}{\tau_3[n]^2}$$

Given that at $t \to \infty$

$$\tau_3[N](t) = \frac{1}{C^*} P(\lambda_{max} < t) = \frac{1}{C^*}(1 + \mathcal{O}(t^{2N-3}e^{-t^2}))$$

we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \log\Big(e^{Nt^2} \tau_3[N]\Big) = 2Nt + \mathcal{O}(\frac{2N-3}{t} - 2t)$$

At $t \to \infty$, the equation simplifies to:

$$2N = C$$

Now that we know $C = 2N$, we can plug in the left tail estimate in (1) and find a recurrence relationship between c_N:

$$c_N^2 = \frac{2}{N} c_{N+1} c_{N-1}$$

Equivalently,

$$\begin{aligned} \frac{c_{N+1}}{c_N} &= \frac{N}{2} \frac{c_N}{c_{N-1}} \\ &= \frac{N!}{2^{N-1}} \frac{c_2}{c_1} \\ &= \frac{N!}{2^{N+1}\sqrt{\pi}} \end{aligned}$$

Therefore,

$$c_N = c_1 \prod_{i=1}^{N-1} \frac{c_{N+1}}{c_N} = \frac{\prod_{i=1}^{N-1} i!}{2^{\frac{N}{2}(N+1)} \pi^{\frac{N}{2}}}$$

The numerator is the superfactorial function $sf(N-1)$. Our computational results from the last section agree with this formula. Therefore, we solved the left-tail asyomptotic problem through a different approach.

5 Fredholm Determinant And Right Tail Asymptotic Behavior

We can derive another expression using the Fredholm determinant method, as shown below.

- **Fredholm Determinant**:

$$\det(I - \xi K_N \chi_J) = 1 + \sum_{n=1}^{N} \frac{(-\xi)^n}{n!} \int\limits_{J\times\cdots\times J} \det\left(K^{(n)}\right) \prod_{k=1}^{n} d\lambda_k$$

with matrix $K^{(n)}$ defined by the Christoffel-Darboux Kernel:

$$K^{(n)} = K_n(\lambda_i, \lambda_j), \quad i, j = 1, \ldots, n$$

Note that its form resembles the matrix given by the Vandermonde method.

This method was first developed by Mehta and Gaudin based on their theorem that the eigenvalue distribution of an $N \times N$ Hermitian random matrices can be described by the Fredholm determinant of integral operators associated with the weight function. Generally, given weight function $w(x)$, let $\{p_k(x)\}$ denote the sequence of polynomials orthonormal with respect to $w(x)$, then define

$$\phi_k(x) := p_k(x)\sqrt{w(x)}$$

then

$$\begin{aligned} E(n, J) &:= Prob(\text{there are n eigenvalues in the interval J}) \\ &= \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \det(I - \xi K_N) \end{aligned}$$

where K_N is the integral operator on interval J with kernel

$$K_N(x, y) := \sum_{k=0}^{N-1} \phi_k(x)\phi_k(y)$$

In the case of GUE, the weight function is $w(x) = e^{-x^2}$, the Hermite polynomials play the role of the orthonormal polynomials, and Christoffel-Darboux kernel serves as the kernel for the integral operator. Also, recall from our normalization constant calculation that integrating the Christoffel-Darboux kernel gives the probability on a certain interval. Staring with the inclusion-exclusion theorem, we deduce

$$
\begin{aligned}
Prob(\lambda_{max} < t) &= \text{Prob(there is no eigenvalue on } (t,\infty)) \\
&= 1 + \sum_{i=1}^{N} (-1)^i Prob(\text{there exists at least i eigenvalues on } [t,\infty)) \\
&= 1 + \sum_{i=1}^{N} (-1)^i \binom{N}{i} \frac{1}{z} \int\limits_{(t,\infty)^i} \int\limits_{R^{N-i}} \prod 2^{-k} k! \sqrt{\pi} \det\Big(K^{(N)}\Big) \prod_{k=1}^{N} d\lambda_k
\end{aligned}
$$

Recall that $z = N! \prod_{k=0}^{N-1} 2^{-k} \sqrt{\pi} k!$. Therefore we first solve the integral for the $N-i$ eigenvalues together with the binomial coefficient

$$
\binom{N}{i} \frac{1}{z} \int\limits_{R^{N-i}} \prod 2^{-k} k! \sqrt{\pi} \det\Big(K^{(N)}\Big) \prod_{k=i+1}^{N} d\lambda_k = \frac{1}{i!(N-i)!} \int\limits_{R^{N-i}} \det\Big(K^{(N)}\Big) \prod_{k=i+1}^{N} d\lambda_k
$$

integrate over the last row and column

$$
= \frac{1}{i!(N-i)!} (N-(N-1)) \int\limits_{R^{N-i}} \det\Big(K^{(N-1)}\Big) \prod_{k=i+2}^{N} d\lambda_k
$$

repeatedly integrate for $N-i$ times

$$
= \frac{1}{i!(N-i)!} (N-i)! \det K^{(i)} = \frac{1}{i!} \det K^{(i)}
$$

For the remaining i eigenvalues on the interval (t,∞),

$$
\begin{aligned}
Prob(\lambda_{max} < t) &= 1 + \sum_{i=1}^{N} \frac{(-1)^i}{i!} \int\limits_{(t,\infty)^i} \det\Big(K^{(i)}\Big) \prod_{k=1}^{i} d\lambda_k \\
&= \det\big(I - K_N \chi_{(t,\infty)}\big)
\end{aligned}
$$

As $t \to \infty$, the integration interval gets increasingly small. Integrating over more dimensions means multiplying with smaller numbers, so the first term with $n = 1$ dominates over all other terms. This form offers the unique advantage of easy access to its right tail asymptotic behavior. We have:

$$
Prob(\lambda_{max} < t \simeq 1 - \frac{2^{N-2} t^{2N-3} e^{-t^2}}{\sqrt{\pi}(N-1)!}, \quad t \to \infty
$$

6 Thinned Ensemble: A Side Note

When discussing the asymptotic behaviors of a standard GUE, we also briefly looked into the case of so called “thinned ensemble”. This “ensemble” is defined just like a standard GUE, except that every eigenvalue is assigned a probability $0 < \xi < 1$ to remain valid. Note that $\xi = 1$ gives exactly the standard case for GUE. However, when analyzing the left tails, computation by both the Fredholm determinant and Vendermone determinant yields

$$\lim_{t \to -\infty} Prob(\lambda_{min} \leq t) = (1 - \xi)^N \neq 0$$

which violates definition for probability distribution functions.

This, however, is understandable since we compute the probability by counting the cases where no eigenvalues resides in the interval $[t, \infty)$. In the case that no eigenvalue remains which happens with probability $(1 - \xi)^N$, there is definitely no eigenvalue in any interval so this brings up the lower probability limit from 0 to $(1 - \xi)^N$.

If we're only concerned about the cases when at least one eigenvalue remains, then we should subtract this number from the probability and re-scale it

$$P' = (P - (1 - \xi)^N) \frac{1}{1 - (1 - \xi)^N}$$

so that

$$\lim_{t \to -\infty} P(x < t) = 0, \quad \lim_{t \to \infty} P(x < t) = 1$$

Then we can perform left- and right-tail analysis just as usual and apply the same transformation to yield the adjusted asymptotic behaviors.

On the other hand, we can also study the second leading term only.

7 Applications In Physics

7.1 O'Connell-Yor Model

O'Connell and Yor built a model with Brownian motions that offers insight into polymer free energy and other physical properties[6]. Their model mainly concerns a directed random polymer in random media. Here we're mainly interested in the semi-discrete case. Consider a sequence of n Brownian queues $B^{(1)}, B^{(2)}, \ldots$ in tandem that are independent of each other. Let β denote $\frac{1}{temperature}$. Then the polymer partition function of the system can be expressed as

$$Z_N(t) = \int\limits_{0 < s_1 < \cdots < s_{N-1} < t} e^{\beta(B^{(1)}_{(s_1)} + B^{(2)}_{(s_1, s_2)} + \dots)} \, ds_1 \dots ds_{N-1}$$

And the polymer free energy $F_N(t)$ can be explicitly calculated as

$$F_N(t) = -\frac{\log(Z_N(t))}{\beta}$$

At zero temperature, i.e. $\lim \beta \to \infty$,

$$-F_N(t) = \sup_{0 \le s_1 \le \cdots \le s_{n-1}} B^{(1)}_{(0,s_1)} + \cdots + B^{(n)}_{(s_{n-1},1)}$$

This distribution with respect to t has the same law as the largest eigenvalue distribution of GUE per observation by Baryshnikov[1]. Therefore, we can deduce the polymer energy distribution at temperature close to zero with the asymptotic behaviors we found in our research.

7.2 GUEs and Queues

As mentioned in the last section,

$$D_n = \sup_{0 \le s_1 \le \cdots \le s_{n-1}} \sum_{i=0}^{n-1} B^{(i)}_{(s_i, s_{i+1})}$$

has the same distribution as the GUE maximum eigenvalue. Taken out of the context of polymer free energy, the same result has implications in queueing theory as well[1].

Also, we can use this model to test the accuracy of our calculation. For example, at $n = 1$, this upper limit is simply $D_1 = B_{(0,s)}$, where $0 < s < 1$. We can run a very simple simulation to illustrate this relationship

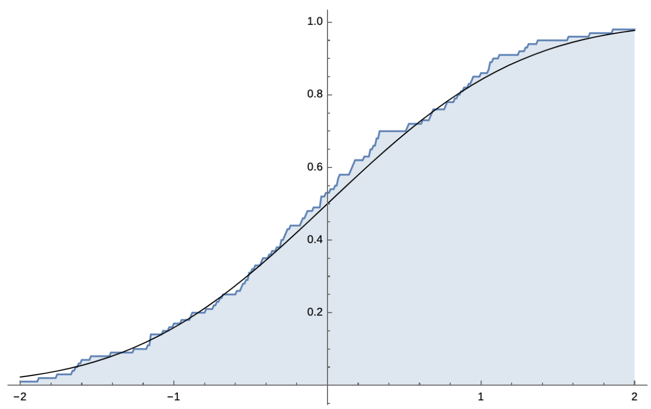


Figure 2: Prob($D_1 < x$), with x on the horizontal axis

We can see that it fits very well with the normal distribution denoted by the thin black line.

The $n = 2$ case can still be simulated with easy by breaking $(0, 1)$ into s small intervals and pick

$$D_2 \sim \max_{i=1,2,\ldots,s} (B^{(1)}_{(0,\frac{i}{s})} + B^{(2)}_{(\frac{i}{s},1)})$$

Again, it fits well with our predicted cumulative distribution function curve denoted by the black line:

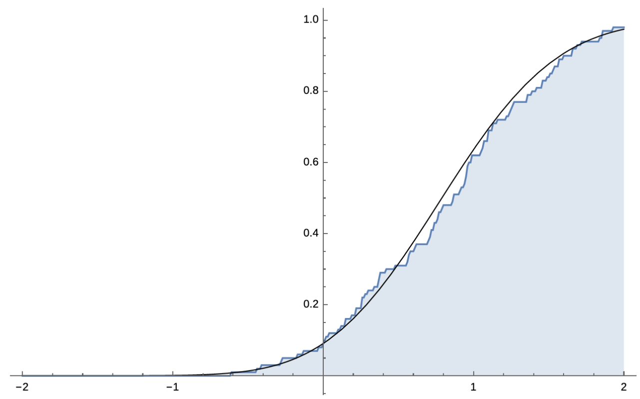


Figure 3: Prob($D_2 < x$), with x on the horizontal axis

7.3 Fermions In Confining Potential

Statistical physicists made the connection to GUE when modeling near-the-edge behavior of fermions trapped in confining potentials[8]. In the center of the trap, the physical properties of the fermions are well established by the Local Density Approximation by Thomas Fermi. These approximation, however, fails at the edge of the trap.

To be more specific, the one-body Hamiltonian for N spinless fermions can be written as

$$\boldsymbol{H} = \frac{\boldsymbol{p}^2}{2} + V(x)$$

In the one-dimensional zero-temperature case ($d = 1, T = 0$), we consider the harmonic oscillator potential

$$V(x) = \frac{1}{2}x^2$$

In the ground state, we fill up the first N energy levels

$$\epsilon_k = k + \frac{1}{2}, \quad k = 0, 1, 2, \ldots, N-1$$

with N fermions. Then we can establish the wave function

$$\Psi_0(x_1, x_2, \ldots, x_N) \sim e^{-\frac{1}{2}\sum_{i=1}^N x_i^2} \det[H_i(x_j)]$$

up to a constant. Here the term $H_i(x_j)$ refers to Hermite polynomial we introduced before. Its determinant can be reduced to a Vandermonde determinant

$\prod_{i<j}(x_i - x_j)$. As a result,

$$|\Psi_0(x_1, x_2, \ldots, x_N)|^2 = \frac{1}{C}\prod_{i<j}(x_i - x_j)^2 e^{-\sum_{i=1}^N x_i^2}$$

This is exactly the eigenvalue probability distribution of GUE. As an analogy, the largest eigenvalue correspond to the right-most fermion in the trap. The distribution asymptotic behavior we calculated translates directly into the likelihood of the rightmost fermion appearing in some point far from the origin.

8 Mathematica Coding Sample

8.1 Calculate And Plot The Distributions

```
n=3; %%choose the matrix size
mat= Table[Integrate[x^(i+j-2)*Exp[-x^2],{x,-Infinity, t}], {i, 1, n}, {j, 1, n}];
dist = Det[mat]
const = Limit[dist,t->Infinity];
Plot[dist/const, {t,-2,2}]
```

8.2 Simulating Brownian Queues

8.2.1 N=1 Case

```
sample = RandomFunction[WienerProcess[], {0, 1, 1}, 100]["States"][[All,2]];
Show[DiscretePlot[CDF[EmpiricalDistribution[sample], x], {x, -2, 2, .01}],
  Plot[CDF[NormalDistribution[0,1], y],{y,-2,2}, PlotStyle->{Black,Thin}]]
```

8.2.2 N=2 Case

```
stepsize = 100;
samplesize = 100;
endpoint = 1/2;
sample1 = RandomFunction[WienerProcess[], {0, endpoint, endpoint/stepsize},
    samplesize]["States"];
sample2 = RandomFunction[WienerProcess[], {0, endpoint, endpoint/stepsize},
    samplesize]["States"];
sampleCombined = Table[Max[Table[sample1[[i,j]] - sample2[[i,j]] +
       sample2[[i,stepsize + 1]], {j, 1, stepsize + 1}]], {i, 1, samplesize}];
Show[DiscretePlot[CDF[EmpiricalDistribution[sampleCombined], x], {x, -2, 2, 0.01}],
  Plot[((-(1/4)/E^(2*t^2) + Pi/8 - ((1/4)*Sqrt[Pi]*t)/E^t^2 +
      (1/4)*Pi*Erf[t] - ((1/4)*Sqrt[Pi]*t*Erf[t])/E^t^2 + (1/8)*Pi*Erf[t]^2)*
     2)/Pi, {t, -2, 2}, PlotStyle -> {Black, Thin}], PlotRange -> All]
```

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