QNM EXPANSION TO SOLUTIONS OF MAXWELL'S EQUATION IN OPEN DOMAINS

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Abstract. In one-spatial dimension, the Maxwell equations governing the propagation of light can sometimes be phrased as a Sturm-Liouville problem. From Sturm-Liouville theory we know that such problems have real eigenvalues and eigenfunctions that are orthogonal and complete. In case one subjects the Maxwell equations to absorbing boundary conditions, the problem no longer fits Sturm-Liouville theory. In certain cases, however, the equations have complex eigenvalues with eigenfunctions "orthogonal" in a (non-positive definite) bilinear form. In the literature such eigenfunctions are known as quasi-normal modes. In this project we try to expand solutions to the Maxwell equations using these quasi-normal modes. More specifically, we try to address the problem of expanding field-responses of bounded scatters measured by a distant source and receiver.

1. Introduction

When one pluck a guitar string with fixed ends, the vibration can be decomposed into a sum of normal modes. However, in the case of plucking an infinite string with energy dissipating to infinity, one can decompose the vibration with quasi-normal modes instead.

Quasi-normal modes were first introduced in the context of characterizing the oscillation of black holes, due to the inevitable dissipation of energy of black holes, there exists a damping term in these modes[7].

In our research, we study light waves and introduce a numerical solver for electromagnetic waves scattering through a medium in open domain subject to an outside signal. Thanks to professor Zimmerling for providing most of the derivations in the report[9].

2. Maxwell's Equation to Wave Equation

In one spatial-domain, assume the field quantities, medium and sources only vary in the x direction. We may set $\partial y = 0$ and $\partial z = 0$ in Maxwell's equations[10]. Assuming $E = E(x)\hat{y}$, then $H = H(x)\hat{z}$, we take the Laplace transform with respect to t and obtain

$$\partial_x H_z + \sigma E_y + \varepsilon_r s E_y = -J_y^{ext}, \qquad \partial_x E_y + \mu_r s H_z = 0, \tag{1}$$

where E_y: electric field strength, H_z: magnetic field strength. J_y^{ext}: volume density of external electric-current source. ε_r: permittivity, μ_r: permeability, and σ: conductivity.

Rename $y = E_y, \widehat{y} = H_z$, assume μ_r is constant, no external magnetic field current, and all variations are in ε_r to arrive at

$$\partial_x \widehat{y} + \sigma y + \varepsilon_r s y = -J_y^{\text{ext}}, \qquad \partial_x y + \mu_r s \widehat{y} = 0, \tag{2}$$

Taking the derivative in the x -direction of the second equation and substitute in $\partial_x \widehat{y}$ from the first equation we get

$$\partial_{xx} y - \mu s(\sigma + \varepsilon s) y = \mu s J_y^{ext} = g(x). \tag{3}$$

3. Problem Statement

When the loss is proportional to the permittivity, we can set $\sigma = 0$. Finally, we arrive at the wave equation form of Maxwell's Equations

$$\partial_{xx} y - \frac{s^2}{c^2} y = g(x). \tag{4}$$

where $\frac{1}{c^2} = \mu_r \varepsilon_r$, and c represents the speed of electromagnetic waves that varies according to ε_r.

Equation(4) we obtained above is called the Helmholtz Equation of the form

$$\Delta u + k^2 u = 0. \tag{5}$$

which is the time-independent wave equation obtained after separation of variables of the standard wave equation.

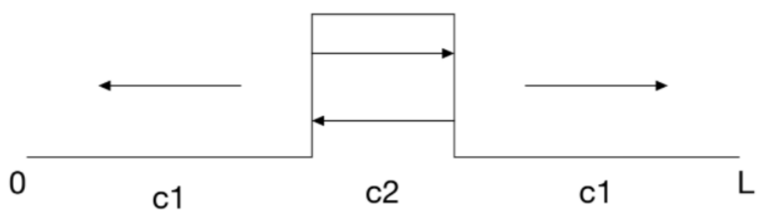
The goal of our research is to expand solutions of equation(4) with a forcing function (light source, signal) $g(x)$ scattering on a Fabry-Perot interferometer (the slab) subject to Sommerfeld radiation boundary condition, which simulates the extension of the domain to infinity, with solutions obtained solving the homogeneuous equation

$$\partial_{xx} y - \frac{s^2}{c^2} y = 0. \tag{6}$$

The system after taking Laplace transform is in frequency domain, since setting $s = i\omega$ in a standard Laplace transform $F(s) = \int_{-\infty}^{\infty} f(t) e^{-st}\, \mathrm{d}t$ results in the Fourier transform $F(i\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, \mathrm{d}t$ which transformed the system to frequency domain, where s is the Laplace frequency variable obtained.

In the appendix we explained the connection of this sort of open domain problem with singular Sturm Liouville problem.

FIGURE 1. The Slab



3.1. **Sommerfeld radiation condition(absorbing boundary condition).**

Arnold Sommerfeld defined the condition of radiation for a scalar field satisfying the Helmholtz equation as "The sources must be sources, not sinks of energy. The energy which is radiated from the sources must scatter to infinity; no energy may be radiated from infinity into the field."[5]

$$\lim_{r\to\infty} r\left(\frac{\partial y}{\partial r} - iky\right) = 0, \quad r = \sqrt{x^2+y^2+z^2}, \quad i = \sqrt{-1}.$$

This adopted condition ensures the uniqueness of solution towards scattering boundary value problems[11].

In one dimension case we adopt the simplest notion of scattering waves travelling in opposite direction and goes to infinity. The first equation encapsulates a right propogating wave $exp(-\frac{s}{c(L)}x)$ at the boundary and the second equation encapsulates a left propogating wave $exp(\frac{s}{c(0)}x)$ at the boundary

$$\begin{aligned} y(L) + \partial_x y(L)\frac{c(L)}{s} &= 0, \\ y(0) - \partial_x y(0)\frac{c(0)}{s} &= 0. \end{aligned} \tag{7}$$

3.2. **Fabry-Perot interferometer(the slab).**

Fabry-Perot interferometer arises when light shines through a cavity bounded by two reflective parallel surfaces. Each time the light encounters one of the surfaces, a portion of it is transmitted out, and the remaining part is reflected back[1].

In one dimension, we simplify the condition as

$$c(x) = c_2 \text{ if } x \in [2/5L, 3/5L], \text{ else } c(x) = c_1. \tag{8}$$

which reflects the change of wavespeed when light scatters across the interfaces.

4. NUMERICAL SOLVER

Setting $\widehat{y} = s^{-1}\partial_x y$, we rewrite the second-order system as

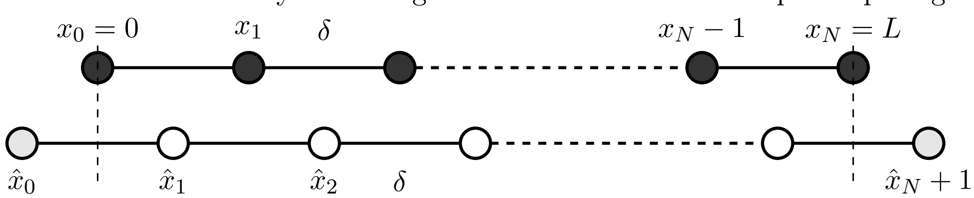
$$\begin{bmatrix} -s & c^2\partial_x \\ \partial_x & -s \end{bmatrix} \begin{bmatrix} y(x,s) \\ \widehat{y}(x,s) \end{bmatrix} = 0. \tag{9}$$

Expanding the first order matrix form

$$-sy + c^2 \partial_x \widehat{y} = 0,$$
$$\partial_x y - s\widehat{y} = 0, \tag{10}$$

we have the second equation encoding information of the newly introduced variable, and substituting that relation in the first equation we have the governing equation(6) of the system.

4.1. **Finite difference method.** We introduce a primary and dual grid descritization for the system along the x direction both with equal δ spacing.



discretization grid

4.1.1. *general discretizing scheme.* We approximate $y(x_i) = y_i$ on the primary grid black nodes, and $\hat{y}(\hat{x}_i) = \hat{y}_i$ on the dual grid white nodes. With the primary grid encoding information for y and dual grid encoding information for $\hat{y}$. We also denote $c(x_i) = c_i$.

For the middle part of the grid, taking the difference of adjacent nodes, and divide by the distance δ, we have an $\mathcal{O}(\delta^2)$ error approximation of the derivative of y at the dual grid mid point. Similarly, we approximate derivative of $\hat{y}$ as

$$-s\hat{y}_{i+1} + \frac{y_{i+1} - y_i}{\delta} = 0, \quad \forall i = 0 \ldots, N,$$
$$\frac{\hat{y}_i - \hat{y_{i-1}}}{\delta} - \frac{s}{c_{i-1}^2} y_{i-1} = 0, \quad \forall i = 1 \ldots, N+1. \tag{11}$$

4.1.2. *discretizing boundary conditions.* The boundary conditions are discretized as

$$c_0^{-2}(-c_0 y_0/\delta - 1/2 s y_0) + \hat{y}_1/\delta = 0,$$
$$c_N^{-2}(-c_N y_N/\delta - 1/2 s y_N) - \hat{y}_N/\delta = 0. \tag{12}$$

We introduce two 'ghost' nodes $\hat{x}_0$, $\hat{x}_{N+1}$ in grey that are outside boundaries $[0, L]$. For the boundary condition at 0, we take the average of $\hat{y}$ at the first two dual grid nodes and due to the fact that $s\hat{y}(x) = \partial_x y(x)$, we can approximate the value of $\partial_x y(0)$ as

$$\partial_x y(x_0 = 0) = s\hat{y}(x_0 = 0) = s\frac{\hat{y}(\hat{x_0}) + \hat{y}(\hat{x}_1)}{2}. \tag{13}$$

The error of approximation is $\mathcal{O}(\delta^2)$, which is consistent with the general finite difference scheme.

Proof. Taylor expanding $\hat{y}$ at $x_0 = 0$ on both sides, we get $\hat{y}(\hat{x}_0)$ and $\hat{y}(\hat{x}_1)$

$$\hat{y}(x_0+\frac{\delta}{2}) = \hat{y}(x_0)+(\frac{\delta}{2})\hat{y}'(x_0)+(\frac{\delta}{2})^2\frac{\hat{y}''(x_0)}{2!}+(\frac{\delta}{2})^3\frac{\hat{y}'''(z_1)}{3!}, z_1 \in (\hat{x}_0, \hat{x}_1), \tag{14}$$

$$\hat{y}(x_0-\frac{\delta}{2}) = \hat{y}(x_0)-(\frac{\delta}{2})\hat{y}'(x_0)+(\frac{\delta}{2})^2\frac{\hat{y}''(x_0)}{2!}-(\frac{\delta}{2})^3\frac{\hat{y}'''(z_2)}{3!}, z_2 \in (\hat{x}_0, \hat{x}_1). \tag{15}$$

Taking the average results

$$(\hat{y}(\hat{x_0}) + \hat{y}(\hat{x_1}))/2 = (\hat{y}(x_0 - \frac{\delta}{2}) + \hat{y}(x_0 - \frac{\delta}{2}))/2 = \hat{y}(x_0) + \mathcal{O}(\delta^2). \quad \square \tag{16}$$

Substituting equation(12) our approximation of $\partial_x y(0)$ into the boundary condition, we get

$$y(0) - s\frac{\hat{y}(\hat{x_0}) + \hat{y}(\hat{x_1})}{2}\frac{c(0)}{s} = 0. \tag{17}$$

Next, we substitute the newly obtained approximated value at the 'ghost' node

$$\hat{y}(\hat{x_0}) = \frac{2y(0)}{c(0)} - \hat{y}(\hat{x_1}), \tag{18}$$

into the general finite difference scheme

$$\frac{\hat{y}(\hat{x_1}) - \hat{y}(\hat{x_0})}{\delta} - \frac{s}{c(0)^2}y(0) = 0, \tag{19}$$

to obtain

$$c(x_0)^2\hat{y}(x_1) - c(x_0)y(x_0) = 1/2sy_0. \tag{20}$$

encoding information for the boundary condition at 0. We descretize the other boundary similarly.

4.2. **System Setup.** Now we can write the continuous system in a finite difference case storing all finite-difference approximations in the vectors $\mathbf{y}$ and $\hat{\mathbf{y}}$ as

$$\mathbf{y} = [y_1, \dots y_{N+1}]^T, \hat{\mathbf{y}} = [\widehat{y}_1, \dots, \widehat{y}_N]^T, \tag{21}$$

$$\mathbf{B} = \mathrm{diag}(\frac{c(0)}{\delta}, 0, 0 \dots, 0, \frac{c(L)}{\delta})_{(N+1)\times(N+1)},$$

$$\mathbf{C} = \mathrm{diag}(c(x_0), c(x_2), \dots, c(x_N))_{(N+1)\times(N+1)},$$

$$\mathbf{D} = \begin{pmatrix} -\delta^{-1} & \delta^{-1} & & & \\ & -\delta^{-1} & \delta^{-1} & & \\ & & & & \\ & & \cdot & \cdot & \\ & & & -\delta^{-1} & \delta^{-1} \end{pmatrix}_{N\times(N+1)},$$

$$\mathbf{E} = \mathrm{diag}(\frac{1}{2}, 1, 1 \dots, 1, \frac{1}{2})_{(N+1)\times(N+1)}.$$

The system becomes

$$\begin{bmatrix} \mathbf{C}^{-2}(-\mathbf{B} - s\mathbf{E}) & -\mathbf{D}^T \\ \mathbf{D} & -s\mathbf{I}_N \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \widehat{\mathbf{y}} \end{bmatrix} = 0. \tag{22}$$

separating terms associated with s, we get

$$\left(\begin{bmatrix} -\mathbf{C}^{-2}\mathbf{B} & -\mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix} - s \begin{bmatrix} \mathbf{C}^{-2}\mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_N \end{bmatrix} \right) \begin{bmatrix} \mathbf{y} \\ \widehat{\mathbf{y}} \end{bmatrix} = 0. \tag{23}$$

4.2.1. *Transforming to eigenvalue problem.* Let $\mathbf{K} = \begin{bmatrix} -\mathbf{C}^{-2}\mathbf{B} & -\mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix}$ and $\mathbf{Z} = \begin{bmatrix} \mathbf{C}^{-2}\mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_N \end{bmatrix}$. The equation becomes

$$(\mathbf{K} - s\mathbf{Z}) \begin{bmatrix} \mathbf{y} \\ \widehat{\mathbf{y}} \end{bmatrix} = 0. \tag{24}$$

We turn equation (23) into a standard eigenvalue problem by diagonal scaling, multiplying $\mathbf{K}$ with $\mathbf{Z}^{-1/2} = \mathrm{diag}(\mathbf{CE}^{-1/2}, I_N)$ on both sides and obtain

$$(\mathbf{K} - s\mathbf{Z}) \begin{bmatrix} \mathbf{y} \\ \widehat{\mathbf{y}} \end{bmatrix} = \mathbf{Z}^{1/2}(\mathbf{Z}^{-1/2}\mathbf{K}\mathbf{Z}^{-1/2} - s)\mathbf{Z}^{1/2} \begin{bmatrix} \mathbf{y} \\ \widehat{\mathbf{y}} \end{bmatrix} = 0, \tag{25}$$

which is equivalent to solving

$$(\mathbf{Z}^{-1/2}\mathbf{K}\mathbf{Z}^{-1/2} - s)\mathbf{Z}^{1/2} \begin{bmatrix} \mathbf{y} \\ \widehat{\mathbf{y}} \end{bmatrix} = 0. \tag{26}$$

Setting $\mathbf{A} = \mathbf{Z}^{-1/2}\mathbf{K}\mathbf{Z}^{-1/2}$, we arrive at the eigenvalue problem form of the matrix equation

$$(\mathbf{A} - s)\mathbf{Z}^{1/2} \begin{bmatrix} \mathbf{y} \\ \widehat{\mathbf{y}} \end{bmatrix} = \left(\begin{bmatrix} -\mathbf{E}^{-1}\mathbf{B} & -\mathbf{C}\mathbf{E}^{-\frac{1}{2}}\mathbf{D}^T \\ \mathbf{D}\mathbf{E}^{-\frac{1}{2}}\mathbf{C} & \mathbf{0} \end{bmatrix} - s \right) \mathbf{Z}^{1/2} \begin{bmatrix} \mathbf{y} \\ \widehat{\mathbf{y}} \end{bmatrix} = 0. \tag{27}$$

The matrix $\mathbf{A}$ we obtained is symmetric in the $\mathbf{M}$ bilinear form with eigenvectors orthogonal in the $\mathbf{M}$ bilinear form and the eigenvalues obtained are non-positive and bounded. The proofs of these claims are given in the matrix operator analysis section. Note the eigenvectors obtained are scaled by $\mathbf{Z}^{-1/2}$ which needs to be scaled back when considering the solution of the original system.

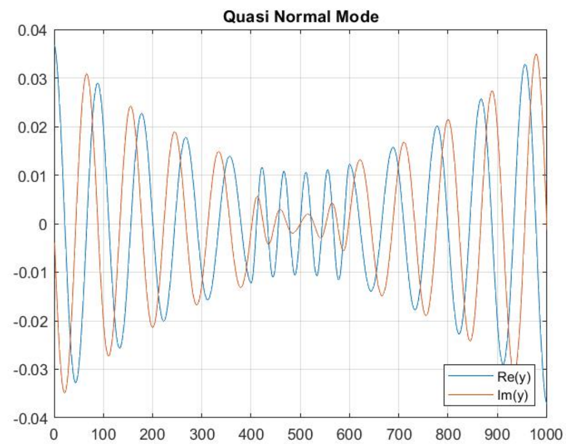
Next, we introduce

$$\mathbf{M} = diag(-\mathbf{I}_{N+1}, \mathbf{I}_N), \tag{28}$$

We can further symmetrize matrix A as a complex symmetrix matrix

$$\mathbf{M}^{1/2} A \mathbf{M}^{-1/2} = \begin{bmatrix} -\mathbf{E}^{-1}\mathbf{B} & -i\mathbf{C}\mathbf{E}^{-\frac{1}{2}}\mathbf{D}^T \\ -\mathbf{D}\mathbf{E}^{-\frac{1}{2}}\mathbf{C}i & \mathbf{0} \end{bmatrix}, \tag{29}$$

FIGURE 2. Quasi-Normal Mode



where $i = \sqrt{-1}$ is the imaginary unit. Then the eigenvalue problem can be rewritten as

$$(\mathbf{M}^{1/2}\mathbf{A}\mathbf{M}^{-1/2} - s)\mathbf{M}^{1/2}\mathbf{Z}^{1/2}\begin{bmatrix}\mathbf{y}\\ \widehat{\mathbf{y}}\end{bmatrix} = 0. \tag{30}$$

4.3. **Quasi Normal Mode Expansion.**

Definition 4.1. The solutions s of our original problem equation(6) are the quasi-normal frequencies of the system, and the corresponding solutions y are the quasi-normal modes of the system.

Equation(30) is the final numerical version of the problem we try to solve. We assume that our complex symmetrized matrix $M^{1/2}AM^{-1/2}$ has distinct eigenvalues, which has been shown true numerically through our simulation so far. Also, since the set of complex $n \times n$ matrices that are not diagonalizable over $\mathbb{C}$ as a subset of $\mathbb{C}_{N\times N}$ has lebesgue measure 0[4]. We made a bold assumption that it is most likely that our complex symmetric matrix is diagonalizable. By [8, Theorem 4.4.27], it can be complex orthogonally diagonalized. One can write, $\mathbf{M}^{1/2}A\mathbf{M}^{-1/2} = \mathbf{V}\Lambda\mathbf{V}^T$, where $\Lambda = \begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{2N+1}\end{bmatrix}, V = \begin{bmatrix} | & | & & | \\ \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \cdots & \tilde{\mathbf{v}}_{2N+1} \\ | & | & & | \end{bmatrix}$ such that $\mathbf{V}^T\mathbf{V} = I$.

Here, Λ contains the eigenvalues which are the quasi normal frequencies of the system, and the $\mathbf{v}_i$s after rescaling with $(\mathbf{M}^{1/2}\mathbf{Z}^{1/2})^{-1}$ corresponds to solutions $\begin{bmatrix}\mathbf{y}\\ \hat{\mathbf{y}}\end{bmatrix}$ which form the quasi-normal modes of the system. Figure 2 illustrates a quasi-normal mode obtained.

4.3.1. *Expandin Forcing Function.* When one subject the equation to an outside source (forcing term) $\mathbf{g}$, we attempted to solve the equation with the quasi normal modes as follows.

Let $\tilde{\mathbf{y}} = \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \end{bmatrix}$, $\tilde{\mathbf{g}} = \begin{bmatrix} \mathbf{g} \\ 0 \end{bmatrix}$ where $\mathbf{g}$ an arbitrary forcing function of the system

$$(\mathbf{A} - s\mathbf{I})\tilde{\mathbf{y}} = \tilde{\mathbf{g}}. \tag{31}$$

Rewrite $\mathbf{A} = (\mathbf{M}^{-1/2}\mathbf{V})\Lambda(\mathbf{M}^{-1/2}\mathbf{V})^{-1}$, since $\mathbf{V}^T = \mathbf{V}^{-1}$ we have

$$((\mathbf{M}^{-1/2}\mathbf{V})\Lambda(\mathbf{M}^{-1/2}\mathbf{V})^{-1} - s\mathbf{I})\tilde{\mathbf{y}} = \tilde{\mathbf{g}}, \tag{32}$$

which is equivalent to

$$((\mathbf{M}^{-1/2}\mathbf{V})(\Lambda - sI)(\mathbf{M}^{-1/2}\mathbf{V})^{-1})\tilde{\mathbf{y}} = \tilde{\mathbf{g}}, \tag{33}$$

therefore

$$\tilde{\mathbf{y}} = ((\mathbf{M}^{-1/2}\mathbf{V})(\Lambda - s\mathbf{I})(\mathbf{M}^{-1/2}\mathbf{V})^{-1})^{-1}\tilde{\mathbf{g}}. \tag{34}$$

To expand the solution in a transpose manner, we can factor out a $\mathbf{M}$ term and achieve

$$\tilde{\mathbf{y}} = ((\mathbf{M}^{-1/2}\mathbf{V})(\Lambda - s\mathbf{I})^{-1}(\mathbf{M}^{-1/2}\mathbf{V})^T)\mathbf{M}\tilde{\mathbf{g}}. \tag{35}$$

Notice that $(\Lambda - s\mathbf{I})$ may not be invertible when the frequency s is at one of the eigenvalues, the solution would be infinite in these cases, which indicates resonance of the system.

Setting q_i to be the i_{th} column vector in $\mathbf{M}^{-1/2}\mathbf{V}$, with $\mathbf{V}$ containing the quasi-normal modes. we obtain the following quasi-normal mode expansion of the solution

$$\tilde{\mathbf{y}}(s) = \sum_{i=1}^{2N+1} \mathbf{q}_i(1/(\lambda_k - s))\mathbf{q}_i^T\mathbf{M}\tilde{\mathbf{g}}, \tag{36}$$

Taking the inverse Laplace transform of $\tilde{\mathbf{y}}(s)$, we obtain the solution back in time domain

$$\tilde{\mathbf{y}}(t) = -\sum_{i=1}^{2N+1} \mathbf{q}_i exp(\lambda_k t)\mathbf{q}_i^T\mathbf{M}\tilde{\mathbf{g}}. \tag{37}$$

The first $N+1$ entries of $\tilde{\mathbf{y}}(t)$ corresponds to our approximated solution of y.

5. Testing numerical solver

We tested our numerical solver with a specific configuration of the slab

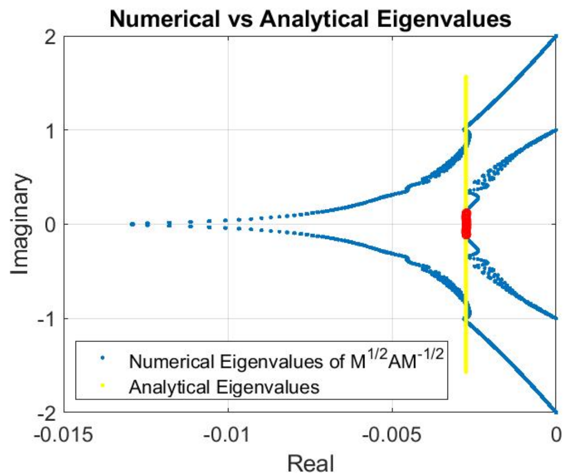
$$L = 1000, \delta = 1, \tag{38}$$

We set the Fabry-Perot interferometer as

$$c(x) = 0.5. \text{ If } x \in [2/5L, 3/5L], \text{ else } c(x) = 1 \tag{39}$$

and obtained the following results.

FIGURE 3. Eigenvalues



Due to the piecewise constant nature of the slab, we could solve the problem analytically in the time domain with eigenvalues

$$\lambda_n = -\frac{1}{\triangle T}\ln\frac{c_1+c_2}{c_2-c_1}+\frac{i\pi n}{\triangle T}. \tag{40}$$

with

$$\triangle T = (x_2-x_1)/c_2, \tag{41}$$

and plotted part of the analytical eigenvalues (colored yellow). The numerical eigenvalues obtained are complex and comes in conjugate pairs as shown in Figure 3.

To make our numerical solver more efficient, we took the 30 closest numerical eigenvalues to the analytical eigenvalues with relatively small real and imaginary parts (colored red), and expanded the solution via these eigenvalues corresponding quasi-normal modes.

Next, we animated our full expansion and truncated expansion of a Gaussian forcing function $g = \mathcal{N}(500, 25)$ in the time domain and obtained the following result.

The approximation works well when the Gaussian forcing term lives within the slab as Figure 4 shows. The left column showcases the full numerical solved solution. The right column showcases the solution with truncated expansion of only 30 quasi-normal modes that have numerical eigenvalues closer to the analytical eigenvalues. .

The truncated expansion becomes inaccurate when the Gaussian slab is placed outside the slab,i.e. $g = \mathcal{N}(200, 25)$, as Figure 5 shows. One can see the decomposition of reflection and transmitted parts as the wave scatters the slab at 400.

FIGURE 4. Gaussian Forcing within Slab Animation

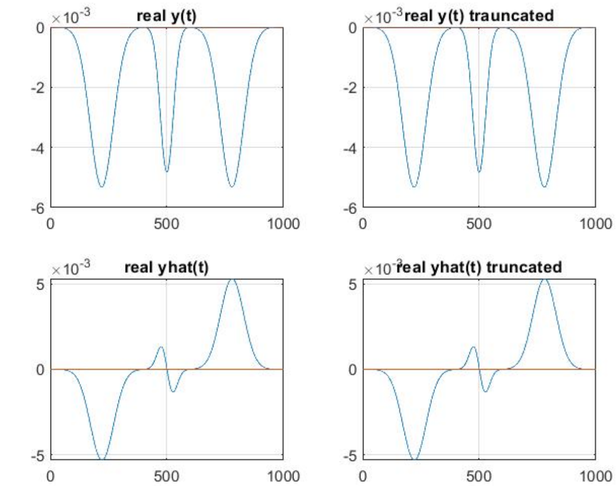
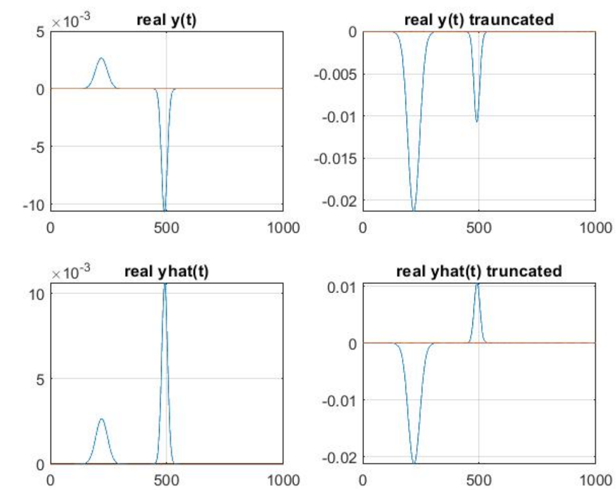


FIGURE 5. Gaussian Forcing outside Slab Animation



6. Relating to Sturm Liouville Problem

6.1. **Sturm Liouville Theory.** [9] regular Sturm Liouville problems are second-order, linear, ordinary differential equations on bounded domains of the form

$$\frac{d}{dx}[p(x)\frac{d}{dx}y(x)] + q(x)y(x) = -\lambda w(x)y(x), \tag{42}$$

with positive $p(x), w(x) > 0$, and $p(x), p'(x), q(x), w(x)$ continuous on $[a, b]$ with non vanishing, homogeneous Robin boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \text{ and } \beta_1 y(b) + \beta_2 y'(b) = 0. \tag{43}$$

The Sturm Liouville problem is called singular if one of the following occurs

(1)$p(a) = 0$ or $p(b) = 0$ or both (2)The interval (a, b) is infinite.

Taking

$$\mathcal{L}y = -\left(\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + q(x)y\right), \tag{44}$$

as a linear operator on function y. The regular Sturm Liouville problem becomes an eigenvalue problem

$$\mathcal{L}y - \lambda w(x)y = 0. \tag{45}$$

$\mathcal{L}$ is a hermitian operator on the bounded domain. For real valued functions f, g we have $\langle \mathcal{L}f, g\rangle = \langle f, \mathcal{L}g\rangle$ under the $w(x)$ weighted inner product $< f(x), g(x) >_{w(x)}= \int_a^b f(x)g(x)w(x)\mathrm{d}x$

Theorem 6.1. [9] *Sturm Lioville problems have an infinite set of eigenvectors y_i and eigenvalues λ_i as solution with the following properties:*

1. *The functions $y_i(x)$ are orthonormal in the $w(x)$ weighted inner product. Thus $< y_i(x), y_j(x) >_{w(x)}= \delta_{ij}$*
2. *The eigenvalues are unique, real, and have no accumulation point $\lambda_1, \lambda_2, \dots, \infty$*
3. *The eigenfunctions form a basis (they are complete) in the Hilbert space $L^2([a, b], w(x))$. This means you can expand arbitrary functions from L^2 in this space.*

Utilizing these properties, we can expand a forcing function f to the equation. The derivation follows worksheet[3].

Consider $f \in L^2([a, b])$

$$\mathcal{L}y - \lambda w(x)y = f. \tag{46}$$

By completeness of the eigenfunctions of Sturm Liouville problem, we can suppose solution y has the form

$$y = \sum_{i=1}^{\infty} a_i v_i(x), \tag{47}$$

where $v_i(x)$s are the eigenfunctions of the homogeneuous equation of the regular Sturm-Liouville problem equation(42), with corresponding eigenvalues λ_is, and a_i being undetermined coefficients.

Then

$$\mathcal{L}\sum_{i=1}^{\infty} a_i v_i(x) - \lambda w(x)\sum_{i=1}^{\infty} a_i v_i(x) = f. \tag{48}$$

By linearity of the operator $\mathcal{L}$ we have

$$\sum_{i=1}^{\infty}(\lambda_i - \lambda)a_i w(x)v_i(x) = f. \tag{49}$$

To obtain desired coefficient a_j we multiply eigenfunction $v_j(x)$ on both sides and integrate

$$\int_a^b v_j(x)\sum_{i=1}^{\infty}(\lambda_i - \lambda)a_i w(x)v_i(x)dx = \int_a^b v_j(x)f dx. \tag{50}$$

By orthogonality of the eigenfunctions in the $w(x)$ weighted inner product

$$(\lambda_j - \lambda)a_j = \int_a^b v_j(x)f dx. \tag{51}$$

We can determine the coefficients as

$$a_j = \frac{\int_a^b v_j(x)f dx}{\lambda_j - \lambda}. \tag{52}$$

Therefore

$$y = \sum_{j=1}^{\infty}\frac{\int_a^b v_j(x)f dx}{\lambda_j - \lambda}v_j(x). \qquad \square \tag{53}$$

6.1.1. *Comparisons.* Setting $p(x) = 1, q(x) = 0, w(x) = 1$, the Sturm Liouville problem becomes

$$\partial_{xx}y(x) + \lambda w(x)y(x) = 0, \tag{54}$$

which is our governing homogeneous equation in disguise if we set $\lambda = -s^2$ and $w(x) = 1/c^2$. The problem is set upon an open domain which matches the second criterion of a singular Sturm Liouville problem.

We chose to chop off the domain and implement absorbing boundary condition to better study the local solutions, which also made computationally solving the problem possible since all computers have finite computing power.

Comparing to the regular Sturm Liouville problem, adopting the Sommerfeld radiation boundary condition with the extra frequency term s in our specified problem's boundary conditions broke the original setup for the regular Sturm-Liouville Theory. The eigenfunctions of the homogeneous equation are no longer orthogonal as we've shown they're orthogonal in a non-positive definite bilinear form instead, which made expanding solutions of the forcing term with the obtained modes difficult.

6.2. **Matrix Operator Analysis.**

Proposition 6.2 ([9])**.** *Matrix* $\mathbf{A}$ *is symmetric in a non-positive definite* $\mathbf{M}$ *bilinear form.*

Proof. Let $\mathbf{v}, \mathbf{w} \in \mathbb{C}^{2N+1}$, since

$$(\mathbf{MA})^T = \mathbf{MA}, \quad \mathbf{M}^T = \mathbf{M}. \tag{55}$$

we have

$$\langle \mathbf{Av}, \mathbf{w} \rangle_{\mathbf{M}} = (\mathbf{Av})^T \mathbf{Mw} = \mathbf{v}^T \mathbf{A}^T \mathbf{Mw} = \mathbf{v}^T (\mathbf{MA})^T \mathbf{w} = \mathbf{v}^T \mathbf{MAw} = \langle \mathbf{v}, \mathbf{Aw} \rangle_{\mathbf{M}}. \tag{56}$$ □

Proposition 6.3. *Suppose matrix* $\mathbf{A}$ *has distinct eigenvalues, then the eigenvectors of matrix* $\mathbf{A}$ *are orthogonal in the* $\mathbf{M}$ *bilinear form.*

Proof. Based on our assumption that matrix $\mathbf{A}$ has distinct eigenvalues, consider two distinct eigenvalues $\lambda_1 \neq \lambda_2$ of matrix $\mathbf{A}$, with corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$. By proposition 6.2

$$\langle \mathbf{Ax}_1, \mathbf{x}_2 \rangle_{\mathbf{M}} - \langle \mathbf{x}_1, \mathbf{Ax}_2 \rangle_{\mathbf{M}} = (\mathbf{Ax}_1)^T \mathbf{Mx}_2 - \mathbf{x}_1^T \mathbf{MAx}_2 = (\lambda_1 \mathbf{x}_1)^T \mathbf{Mx}_2 - \mathbf{x}_1^T \mathbf{M} \lambda_2 \mathbf{x}_2 = 0. \tag{57}$$

Then

$$(\lambda_1 - \lambda_2) \mathbf{x}_1^T \mathbf{Mx}_2 = 0. \tag{58}$$

Since $\lambda_1 \neq \lambda_2$, $(\lambda_1 - \lambda_2) \neq 0$, we have

$$\langle \mathbf{x}_1, \mathbf{x}_2 \rangle_{\mathbf{M}} = \mathbf{x}_1^T \mathbf{Mx}_2 = 0. \tag{59}$$ □

Proposition 6.4. *The solutions* $\begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \end{bmatrix}$ *of equation(23) are orthogonal in a non-positive definite bilinear form, which form the quasi normal modes of the system.*

Proof. By proposition 6.3, we have $\mathbf{x}_1^T \mathbf{Mx}_2 = 0$, recall the eigenvectors of A are scaled by $\mathbf{Z}^{1/2}$ of the original solutions $\begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \end{bmatrix}$ to the system. Let $\begin{bmatrix} \mathbf{y}_1 \\ \hat{\mathbf{y}}_1 \end{bmatrix}$, $\begin{bmatrix} \mathbf{y}_2 \\ \hat{\mathbf{y}}_2 \end{bmatrix}$ be two solutions. substituting in, we have

$$(\mathbf{Z}^{1/2} \begin{bmatrix} \mathbf{y}_1 \\ \hat{\mathbf{y}}_1 \end{bmatrix})^T \mathbf{M} \mathbf{Z}^{1/2} \begin{bmatrix} \mathbf{y}_2 \\ \hat{\mathbf{y}}_2 \end{bmatrix} = 0. \tag{60}$$

Therefore

$$\begin{bmatrix} \mathbf{y}_1 \\ \hat{\mathbf{y}}_1 \end{bmatrix}^T \mathbf{Z}^{1/2} \mathbf{M} \mathbf{Z}^{1/2} \begin{bmatrix} \mathbf{y}_2 \\ \hat{\mathbf{y}}_2 \end{bmatrix} = 0. \tag{61}$$

where $\mathbf{Z}^{1/2}\mathbf{M}\mathbf{Z}^{1/2}$ is a non-positive bilinear form.

Proposition 6.5. *Real part of eigenvalues of matrix* $\mathbf{A}$ *are all non-positive.*

Proof. The proof follows the structure of a proof given in [6, appendix C] Let $\mathbf{X} = \begin{bmatrix} \mathbf{0} & -\mathbf{C}\mathbf{E}^{-\frac{1}{2}}\mathbf{D}^T \\ \mathbf{D}\mathbf{E}^{-\frac{1}{2}}\mathbf{C} & \mathbf{0} \end{bmatrix}$, $\mathbf{R} = \begin{bmatrix} \mathbf{E}^{-1}\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $\mathbf{I}$ be the identity matrix of same size of matrix $\mathbf{A}$.

We can separate matrix as $\mathbf{A} = \mathbf{X} - \mathbf{R}$. Suppose λ_i is an eigenvalue of matrix $\mathbf{A}$ with corresponding eigenvector $\mathbf{v}_i$, then

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v}_i = 0. \tag{62}$$

Multiply the conjugate transpose vector $\mathbf{v}_i^*$ on both sides of the equation then take the real parts we obtain,

$$Re(\mathbf{v}_i^*(\mathbf{X} - \mathbf{R} - \lambda_i \mathbf{I})\mathbf{v}_i) = 0. \tag{63}$$

Due to the skew symmetric nature of matrix $\mathbf{X}$

$$(\mathbf{v}_i^* \mathbf{X} \mathbf{v}_i)^* = -\mathbf{v}_i^* \mathbf{X} \mathbf{v}_i, \tag{64}$$

we must have

$$Re(\mathbf{v}_i^* \mathbf{X} \mathbf{v}_i) = 0. \tag{65}$$

Then the equation simplifies to

$$Re(\lambda_i)\|\mathbf{v}_i\|^2 + Re(\mathbf{v}_i^* \mathbf{R} \mathbf{v}_i) = 0. \tag{66}$$

Since matrix $\mathbf{R}$ is positive semi-definite

$$Re(\mathbf{v}_i^* \mathbf{R} \mathbf{v}_i) \geq 0. \tag{67}$$

Therefore

$$Re(\lambda_i)\|\mathbf{v}_i\|^2 = -Re(\mathbf{v}_i^* \mathbf{R} \mathbf{v}_i) \leq 0. \tag{68}$$

Since eigenvector $\mathbf{v}_i$ is non-zero

$$Re(\lambda_i) \leq 0. \quad \square \tag{69}$$

Proposition 6.6. *The numerical eigenvalues λ obtained for matrix $\mathbf{A}$ are bounded by union of Gershgorin disks, $\lambda \in D(0, 2\sqrt{2}c_1/\delta) \cup D(-2c_1/\delta, \sqrt{2}c_1/\delta)$.*

Proof. Matrix $\mathbf{A}$ has the structure of

$$\left(\begin{array}{cccc|cccc}
 & -2c(0)/\delta & & & -\sqrt{2}c(0)/\delta & & & \\
 & & \ddots & & -c(x_1)/\delta & c(x_1)/\delta & & \\
 & & & & \cdots & & & \\
 & & & -2c(L)/\delta & & & & \sqrt{2}c(L)/\delta \\
\hline
-\sqrt{2}c(0)/\delta & \sqrt{2}c(x_1)/\delta & & & & & & \\
 & -\sqrt{2}c(x_1)/\delta & \sqrt{2}c(x_2)/\delta & & & & & \\
 & & \cdots & & & & & \\
 & & -\sqrt{2}c(x_{N-1})/\delta & \sqrt{2}c(x_N)/\delta & & & & \\
 & & & & & \mathbf{0} & &
\end{array}\right)$$

We bounded the numerical eigenvalues obtained via Gershgorin circle theorem.

For the Upper half of the matrix, the eigenvalues are bounded by $D(-2c_1/\delta, \sqrt{2}c_1/\delta)$ and $D(0, 2c_1/\delta)$. For the bottom half of the matrix, the eigenvalues are bounded by $D(0, 2\sqrt{2}c_1/\delta)$. Taking the union of the three disks, we obtain a bound for the eigenvalues such that $\lambda \in D(0, 2\sqrt{2}c_1/\delta) \cup D(-2c_1/\delta, \sqrt{2}c_1/\delta)$ which showcases the numerical simulation limitations. $\square$

7. Appendix 1

The derivatives in general finite difference scheme are approximated of $\mathcal{O}(\delta^2)$ error since

$$f(x+\delta/2) = f(x)+\delta/2f'(x)+(\delta/2)^2\frac{f''(x)}{2!}+(\delta/2)^3\frac{f'''(\xi)}{3!}, \quad \xi \in (x, x+(\delta/2)),$$

$$f(x-\delta/2) = f(x)-\delta/2f'(x)+(\delta/2)^2\frac{f''(x)}{2!}-(\delta/2)^3\frac{f'''(\xi')}{3!} \quad \xi' \in (x, x+(\delta/2)),$$

$$f'(x) = (f(x+\delta/2) - f(x-\delta/2))/\delta + \mathcal{O}(\delta^2).$$

8. Appendix 2

Definition 8.1 (matrix bilinear form[2]). If we take $V = \mathbb{F}^n$, then every $n \times n$ matrix A gives rise to a bilinear form by the formula

$$\langle v, w\rangle_A = v^t A w$$

Definition 8.2 (hermitian). A Square matrix $A \in M_n$ is Hermitian if $A^* = A$, where $A^* = \bar{A}^T$ is the conjugate tranpose of A, i.e., A Real symmetric matrix $A \in M_n$ is Hermitian.

Definition 8.3 (positive definite). A Hermitian matrix $A \in M_n$ is positive definite if $x^*Ax > 0$ for all nonzero $x \in \mathbf{C}^n$, positive semi-definite if $x^*Ax \geq 0$ for all nonzero $x \in \mathbf{C}^n$

Theorem 8.4 (Gershgorin circle theorem). *If A is an $n \times n$ matrix with complex entries $a_{i,j}$, then every eigenvalue of a matrix lies within at least one Gershgorin disc, where a Gershgorin disc is the disc $D\left(a_{i,i}, r_i(A)\right)$ centered at diagonal entries a_{ii} on the complex plane with radius $r_i(A) = \sum_{i\neq j} |a_{i,j}|$.*

Theorem 8.5 (Horn[8]). *Let $A \in M_n$ be symmetric. Then A is diagonalizable if and only if it is complex orthogonally diagonalizable.*

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