CARNOT ALGEBRAS WITH SMALL GRADED AUTOMORPHISM GROUPS

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ABSTRACT. We study a class of nilpotent Lie algebras called Carnot algebras that have graded automorphism groups consisting of homotheties. Pierre Pansu proved several results about Carnot algebras in 1989 ([Pan89]) in a famous work about quasi-isometric rigidity of rank-one Lie groups, following Gromov's program. In particular, under appropriate dimension conditions, step-2 Carnot algebras generically have small automorphism groups. We are exploring the situations not covered in Pansu's results. In this paper, we present some results in low dimensions and explorations into the general step-2 case.

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1. INTRODUCTION

Our central question involves a particular type of nilpotent Lie algebra called a Carnot algebra. We begin with some basic definitions.

Definition 1.1 (Carnot algebra)**.** *Let V be a Lie algebra. We call V a* Carnot algebra *of step-n if it contains nonzero vector subspaces $V^1, \ldots, V^n$ such that*

(1) $V = V^1 \oplus \cdots \oplus V^n$
(2) $[V^1, V^i] = V^{i+1}$ *for each* $1 \leq i < n$
(3) $[V^1, V^n] = 0$.

Notice that a Carnot algebra of step-n is nilpotent of degree n. Recall that there is a natural notion of isomorphism between Lie algebras.

Definition 1.2 (Lie algebra isomorphism)**.** *A* Lie algebra isomorphism *is a vector space isomorphism $\phi : V \to W$ satisfying $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in V$.*

Between two Carnot algebras V and W, we define *graded* isomomorphims to be Lie algebra isomorphisms that preserve the Carnot structures on V and W.

Definition 1.3 (Graded Isomorphism). *Let $V = V^1 \oplus \cdots \oplus V^n$ and $W = W^1 \oplus \cdots \oplus W^n$ be Carnot algebras. A Lie algebra isomorphism $\phi : V \to W$ is called* graded *if $\phi(V^i) = W^i$ for each i.*
A graded automorphism *of V is an isomorphism from V to itself.*

We identify the space of all step-k Carnot algebras $V^1 \oplus \cdots \oplus V^k$ with $\dim V^i = n_i$ with the space of all Lie brackets $[,]$ that make $\mathbb{R}^{n_1} \oplus \cdots \oplus \mathbb{R}^{n_k}$ a Carnot algebra. Let $I_g(V)$ denote the set of Carnot algebras that are isomorphic to V via a graded isomorphism. Moreover, let $\mathrm{Aut}_g(V)$, or sometimes $\mathrm{Aut}_g([,])$, denote the group of graded automorphisms of the Carnot algebra $(V, [,])$. If $V = V^1 \oplus \cdots \oplus V^n$, then every graded automorphism of V is given by a n-tuple of matrices $(A_1, \ldots, A_n)$ where $A_i \in \mathrm{GL}(V^i)$. Using the definition of a Carnot algebra 1.1 and graded automorphisms 1.3, one can verify that a graded automorphism is completely determined by its component in $\mathrm{GL}(V^1)$, A_1. Therefore we can realize $\mathrm{Aut}_g(V)$ as a subgroup of $\mathrm{GL}(V^1)$.

The aim of this research is to study the set of Carnot algebras that have "small" graded automorphism groups. Such algebras are called *asymmetric*, and we define them in the following way. Given any Carnot algebra V, there exists a one-paremeter subgroup $\mathcal{D}$ of $\mathrm{Aut}_g(V)$ consisting of *dilations*.

Definition 1.4 (Dilation). *For each $c \in \mathbb{R}_+$, define a* dilation *$\delta_c : V \to V$ that scales V^i by a factor of c^i. That is, for all $1 \le i \le n$ and $v_i \in V^i$, $\delta_c(v_i) = c^i v_i$. One can check that in fact each δ_c is a graded automorphism of V.*

Notice that $\mathcal{D}$ is contained in the center of $\mathrm{Aut}_g(V)$, so we can always write $\mathrm{Aut}_g(V) = L\mathcal{D}$ for some subgroup L of $L\mathcal{D}$.

Definition 1.5 (Asymmetry, [CNS21]). *We call a Carnot algebra $V = V^1 \oplus \cdots \oplus V^n$* asymmetric *if its graded automorphism group has the form $\mathrm{Aut}_g(V) = L\mathcal{D}$ where L is a compact subgroup of $\mathrm{SL}_{\pm}(V^1)$.*

Ultrarigid algebras are those whose graded automorphsm group consists solely of dilations. They form a sub-class of asymmetric algebras that Le Donne studied in [LDOW14], Theorem 1.2. The following results regarding asymmetry are already known.

Theorem 1.6. [Pan89, Theorem 10.1] *Let $m \ge 2$ and $X = \mathbb{H}H^m$, the quaternionic hyperbolic symmetric space. Let N be the maximal unipotent subgroup of the isometric group of X and $\mathcal{N}$ the associated Lie algebra. Then $\mathcal{N}$ is asymmetric.*

Theorem 1.7. [Pan89, Theorem 13.2] *Suppose n is even, $n \ge 10$, and $3 \le p < 2n-3$. Consider the space of step-2 Carnot algebras with first level dimension n and second level dimension p. Asymmetry is a Zariski open condition on such algebras.*

Theorem 1.8. [CNS21, Corollary 2.10] *Let G be a connected and simply connected Carnot Lie group of dimension n. If G is asymmetric then there is a neighborhood U of G in the variety of n-dimensional Carnot Lie groups such that U consists of asymmetric groups.*

We prove a few results extending the theorems above in low dimensions and progress towards the general step-2 case.

Theorem 1.9. *Let $V = V^1 \oplus V^2$ be a step-2 Carnot algebra and suppose* $\dim\ V^1 = n$ *and* $\dim V^2 = p$*. V is not asymmetric if one of the following is true:*

- $n \le 3$

- $n = 4$ *and* $p \neq 3$
- $p = 1$

One application of Theorem 1.9 is the following result.

Theorem 1.10. *If V is a Carnot algebra such that the dimension of V is ≤ 5, then V is not asymmetric.*

We expect that asymmetry is a common phenomenon. As Pansu proved in Theorem 1.7, barring some dimension conditions we expect that step-2 Carnot algebras are generically (in the Zariski topology) asymmetric in large dimension. However, this is not the case when dimension is small. We prove the following result.

Theorem 1.11. *Asymmetry is not a Zariski open condition on the space of step-2 Carnot algebras with first level dimension 4 and second level dimension 3.*

We also investigate whether asymmetry of step-2 Carnot algebras with higher dimensions is a generic property in the cases not covered by Pansu's Theorem 1.7. In this direction, we have the following partial result. Pansu arrived at an analog of Theorem 1.12 in step (1) of his proof of Theorem 1.7 (see [Pan89], page 50). However, his result was somewhat stronger since it proved the condition on an open and dense set rather than just a dense set.

Theorem 1.12. *Suppose n is odd and $p \geq 3$. For every $[,]$ that makes $(\mathbb{R}^n \oplus \mathbb{R}^p, [,])$ a Carnot algebra, we consider the group homomorphism*

$$\begin{aligned} f : \mathrm{Aut}_g([,]) &\to \mathrm{GL}(p, \mathbb{R}) \\ (A, B) &\mapsto B. \end{aligned}$$

Then on a dense subset of the space of all such $[,]$ (see Section 3), f is injective. In other words, for dense choice of $[,]$, $\mathrm{Aut}_g([,])$ can be realized as a subgroup of $\mathrm{Aut}_g(p, \mathbb{R})$.

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3. Step-2 Carnot algebras

We say that a step-2 Carnot algebra $V = V^1 \oplus V^2$ is type-(n, p) if $\dim V^1 = n$ and $\dim V^2 = p$. Recall that the set of such algebras is equivalent to the set of Lie brackets that make $\mathbb{R}^n \oplus \mathbb{R}^p$ a Carnot algebra. For vector spaces U and V, we will let $S(U, V)$ denote the set of surjective linear maps from U to V. Any bracket $[,]$ that makes $\mathbb{R}^n \oplus \mathbb{R}^p$ a Carnot algebra is completely determined by a surjective linear map $[,]\big|_{\Lambda^2\mathbb{R}^n} : \Lambda^2\mathbb{R}^n \to \mathbb{R}^p$. Therefore the set of type-(n, p) carnot Lie algebras can be identified with $S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$, an open and dense set in $\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$, where $\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ is given the Euclidean topology from its vector space isomorphism with $\mathbb{R}^{\frac{n(n-1)}{2}p}$. In this paper, we will study $S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ with this induced Euclidean topology and refer only to another topology, such as the Zariski topology, when explicitly stated.

Given a matrix $A \in \mathrm{GL}(n, \mathbb{R})$, let $\Lambda^2 A : \Lambda^2\mathbb{R}^n \to \Lambda^2\mathbb{R}^n$ be defined by $\Lambda^2 A(v \wedge w) = Av \wedge Aw$. The group $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(p, \mathbb{R})$ acts on $S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ continuously via $(A, B) \cdot [,] = B \circ [,] \circ \Lambda^2 A^{-1}$. Notice that $\mathrm{Aut}_g([,]) = \mathrm{Stab}_{\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(p,\mathbb{R})}([,])$ and $I_g([,]) = (\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(p, \mathbb{R})) \cdot [,]$. The set of isomorphism classes of type-(n,p) Carnot algebras is therefore $S(\Lambda^2\mathbb{R}^n,\mathbb{R}^p)/(\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(p,\mathbb{R}))$. Moreover, we claim that every graded isomorphism class $I_g([,])$ has the structure of a smooth manifold.

Proposition 3.1. *For all $[,]\in S(\Lambda^2\mathbb{R}^n,\mathbb{R}^p)$, $I_g([,])$ is a smooth manifold.*

Proof. Since the action of $\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(p,\mathbb{R})$ on $S(\Lambda^2\mathbb{R}^n,\mathbb{R}^p)$ is continuous, the map

$$\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(p,\mathbb{R})/\operatorname{Stab}_{\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(p,\mathbb{R})}([,])\to(\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(p,\mathbb{R}))\cdot[,]$$

is a homeomorphism. Recall that $\operatorname{Stab}_{\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(p,\mathbb{R})}([,])=\operatorname{Aut}_g([,])$ and $(\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(p,\mathbb{R}))\cdot[,]=I_g([,])$. So $I_g([,])$ is homeomorphic to $\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(p,\mathbb{R})/\operatorname{Aut}_g([,])$. Now $\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(p,\mathbb{R})$ is a Lie group and $\operatorname{Aut}_g([,])$ is a (topologically) closed subgroup since it is the stabilizer of a continuous action of $\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(p,\mathbb{R})$. Therefore $\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(p,\mathbb{R})/\operatorname{Aut}_g([,])$ is a smooth manifold, so $I_g([,])$ is a smooth manifold. ☐

Corollary 3.2. *For any $[,]\in S(\Lambda^2\mathbb{R}^n,\mathbb{R}^p)$,*

$$\dim I_g([,])=n^2+p^2-\dim\operatorname{Aut}_g([,]).$$

Next, we will explore the relationship between type-(n,p) Carnot algebras and the Grassmannian.

3.1. **The Grassmannian viewpoint.** Given a vector space V with dimension n, let the p-Grassmannian $\mathrm{Gr}(V,p)$ to be the set of p-planes in V. We can define a topology on $\mathrm{Gr}(V,p)$ in the following way. Let $A\subset\mathbb{R}^{np}$ be the set of p-tuples $(v_1,\dots,v_p)$ such that $v_1,\dots,v_p\in\mathbb{R}^n$ are linearly independent. Let π be the map $A\to\mathrm{Gr}(V,p)$ defined by $(v_1,\dots,v_p)\mapsto\operatorname{Span}(v_1,\dots,v_p)$. π surjects onto $\mathrm{Gr}(V,p)$, and so we can give $\mathrm{Gr}(V,p)$ the quotient topology from π. Let τ be this topology. Another topology τ' that one might define on $\mathrm{Gr}(V,p)$ is generated by the basis sets

$$U(v_1,\dots,v_p,\epsilon_1,\dots,\epsilon_p)=\{\operatorname{Span}(w_1,\dots,w_p)\mid|w_i-v_i|<\epsilon_i\}.$$

Proposition 3.3. *The two topologies defined above are equivalent, that is, $\tau=\tau'$.*

Proof. First we show that $\tau\subset\tau'$. Suppose $U\in\tau$. Then $\pi^{-1}(U)\subset\mathbb{R}^{np}$ is open. Let $q\in U$ be given. We want to show there exists $\epsilon_1,\dots,\epsilon_p$ such that $q\in U(v_1,\dots,v_p,\epsilon_1,\dots,\epsilon_p)\subset U$. Let $(v_1,\dots,v_p)\in\pi^{-1}(q)$. Since $\pi^{-1}(U)\subset\mathbb{R}^{np}$ is open, there exists $\epsilon_1,\dots,\epsilon_p$ such that

$$(v_1,\dots,v_p)\in S:=\{(w_1,\dots,w_p)\mid|w_i-v_i|<\epsilon_i\}\subset\pi^{-1}(U).$$

Then $q\in\pi(S)\subset U$, and we are done since $\pi(S)=U(v_1,\dots,v_p,\epsilon_1,\dots,\epsilon_p)$.

Conversely we will show that $\tau'\subset\tau$. We will demonstrate that each basis set $U(v_1,\dots,v_p,\epsilon_1,\dots,\epsilon_p)$ of τ' is in τ. Given $v_1,\dots,v_p\in\mathbb{R}^n$ and $\epsilon_1,\dots,\epsilon_p>0$, we have

$$\pi^{-1}(U(v_1,\dots,v_p,\epsilon_1,\dots,\epsilon_p))=\{(w_1,\dots,w_p)\mid|w_i-v_i|<\epsilon_i\}$$

which is open in $\mathbb{R}^{np}$. Thus $U(v_1,\dots,v_p,\epsilon_1,\dots,\epsilon_p)\in\tau$. ☐

For every $[,]\in S(\Lambda^2\mathbb{R}^n,\mathbb{R}^p)$, there is an associated element of $\mathrm{Gr}(\Lambda^2\mathbb{R}^n,p)$ obtained in the following way. Since $[,]$ is surjective, its dual map $[,]^*:(\mathbb{R}^p)^*\to\Lambda^2(\mathbb{R}^n)^*$ is injective. Therefore $\operatorname{im}([,]^*)$ is a p-plane in $\Lambda^2(\mathbb{R}^n)^*$, meaning $\operatorname{im}([,]^*)\in\mathrm{Gr}(\Lambda^2(\mathbb{R}^n)^*,p)$. We have the following proposition relating graded isomorphisms of Lie algebras and certain isomorphisms of their corresponding p-planes.

Proposition 3.4. *Let $[,],[,]'\in S(\Lambda^2\mathbb{R}^n,\mathbb{R}^p)$. If $(A,B)\in\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(p,\mathbb{R})$ is a graded isomorphism from $[]$ to $[]'$, then $\Lambda^2A^*(\operatorname{im}([,]^*))=\operatorname{im}(([,]')^*)$. Conversely, if $A\in\mathrm{GL}(n,\mathbb{R})$ and $\Lambda^2A^*(\operatorname{im}([,]^*))=\operatorname{im}(([,]')^*)$, then there exists $B\in\mathrm{GL}(p,\mathbb{R})$ such that (A,B) is a graded isomorphism from $[]$ to $[]'$.*

Proof. Suppose that $[,]$ and $[,]'$ are isomorphic via a graded isomorphism. Then there exist $A \in \mathrm{GL}(n,\mathbb{R})$ and $B \in \mathrm{GL}(p,\mathbb{R})$ such that

$$B \circ [,]' = [,] \circ \Lambda^2 A$$

and we obtain

$$([,]')^* \circ B^* = \Lambda^2 A^* \circ [,]^*.$$

Thus $\mathrm{im}(([,]')^*) = \mathrm{im}(([,]')^* \circ B^*) = \mathrm{im}(\Lambda^2 A^* \circ [,]^*)$. Now $\mathrm{im}(\Lambda^2 A^* \circ [,]^*) = \Lambda^2 A^*(\mathrm{im}([,]^*))$, so we get $\Lambda^2 A^*(\mathrm{im}([,]^*)) = \mathrm{im}(([,]')^*)$.

Suppose that $A \in \mathrm{GL}(n,\mathbb{R})$ and $\Lambda^2 A^* : \Lambda^2(\mathbb{R}^n)^* \to \Lambda^2(\mathbb{R}^n)^*$ maps $\mathrm{im}([,]^*)$ to $\mathrm{im}(([,]')^*)$. Then, for each $v \in \mathbb{R}^p$ there exists $f(v) \in \mathbb{R}^p$ such that $\Lambda^2 A^* \circ [,]^*(v) = ([,]')^*(f(v))$. We claim that $f : \mathbb{R}^p \to \mathbb{R}^p$ is linear, and in fact an isomorphism. If this is true, then $\Lambda^2 A^* \circ [,]^* = ([,]')^* \circ f$ so $[,] \circ \Lambda^2 A = f^* \circ [,]'$ and we get that (A, f^*) is a graded isomorphism from $[,]$ to $[,]'$. For the claim, notice that $\Lambda^2 A^* \circ [,]^*$ is linear, so $([,]')^* \circ f$ is linear, and we get $([,]')^* \circ f(v+w) = ([,]')^*(f(v)) + ([,]')^*(f(w))$. $([,]')^*$ is also linear, so in fact $([,]')^* \circ f(v+w) = ([,]')^*(f(v)+f(w))$. Since $([,]')^*$ is injective, this implies $f(v+w) = f(v) + f(w)$. Finally, note that $[,]^*$ and $\Lambda^2 A^*$ are injective, implying that $\Lambda^2 A^* \circ [,]^* = ([,]')^* \circ f$ is injective. Since $([,]')^*$ is injective, we therefore get that f is injective. Thus f is an isomorphism. $\square$

The equivalence described above allows us to relate graded automorphisms of a given $[,]$ and certain automorphisms of its corresponding p-plane.

Definition 3.5 (Graded automorphism class of a p-plane)**.** *Let $Z \subset \Lambda^2(\mathbb{R}^n)^*$ be a p-plane. Define the* graded automorphism class *of Z to be*

$$\mathrm{Aut}_g(Z) := \left\{ A \in \mathrm{GL}(n,\mathbb{R}) \mid \Lambda^2 A^*(Z) = Z \right\}.$$

We know that $\mathrm{Aut}_g(Z)$ is a group with respect to composition.

Corollary 3.6. *Let $[,] \in S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ and $Z = \mathrm{im}([,]^*)$. Then $\mathrm{Aut}_g(Z) = \mathrm{Aut}_g([,])$ when we regard $\mathrm{Aut}_g([,])$ as a subgroup of $\mathrm{GL}(n,\mathbb{R})$.*

Proof. Proposition 3.4 implies that if $A \in \mathrm{Aut}_g([,])$ then $\Lambda^2 A^*(Z) = Z$, so $A \in \mathrm{Aut}_g(Z)$. Moreover, given $A \in \mathrm{Aut}_g(Z)$, by Proposition 3.4 we can find $B \in \mathrm{GL}(p,\mathbb{R})$ such that (A,B) is an automorphism of $[,]$. It follows that $A \in \mathrm{Aut}_g([,])$. $\square$

Now we can restate Corollary 3.2 in the following way.

Corollary 3.7. *Suppose $[,] \in S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ and $Z = \mathrm{im}([,]^*)$. Then*

$$\dim I_g([,]) = n^2 + p^2 - \dim \mathrm{Aut}_g(Z)$$

3.2. **Worked example.** Now consider the Lie algebra $\mathfrak{h} \oplus \mathfrak{h}$ corresponding to the Lie group $H^3 \times H^3$, where H^3 is the Heisenberg group in three dimensions. This is a type-$(4,2)$ Carnot algebra defined by the relations $[X_1, X_2] = X_6$ and $[X_3, X_4] = X_7$, where (X_1, X_2, X_3, X_4) is a basis for V^1 and (X_5, X_6) is a basis for V^2. Let $Y_i = X_i^*$. Then $[,]^*(Y_6) = Y_1 \wedge Y_2$ and $[,]^*(Y_7) = Y_3 \wedge Y_4$. So $Z := \mathrm{im}([,]^*) = \mathrm{Span}(Y_1 \wedge Y_2, Y_3 \wedge Y_4)$.

Proposition 3.8. $\mathrm{Aut}_g(\mathfrak{h} \oplus \mathfrak{h}) \subset \mathrm{GL}(4,\mathbb{R})$ *consists of all matrices*

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \text{ or } \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} \text{ for } X, Y \in \mathrm{GL}(2,\mathbb{R}).$$

Proof. Let $A = \begin{pmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} & \vec{v_4} \end{pmatrix} \in \mathrm{Aut}(Z)$, where each

$$v_i = \begin{pmatrix} v_i^1 \\ v_i^2 \\ v_i^3 \\ v_i^4 \end{pmatrix} \in \mathbb{R}^4.$$

Then $\Lambda^2 A(Y_1 \wedge Y_2) = \sum_{i<j}(v_1^i v_2^j - v_1^j v_2^i) Y_i \wedge Y_j$. But $\Lambda^2(Y_1 \wedge Y_2)A \in \mathrm{Span}(Y_1 \wedge Y_2, Y_3 \wedge Y_4)$, so we get

(1) $v_1^1 v_2^3 - v_1^3 v_2^1 = 0$
(2) $v_1^1 v_2^4 - v_1^4 v_2^1 = 0$
(3) $v_1^2 v_2^3 - v_1^3 v_2^2 = 0$
(4) $v_1^2 v_2^4 - v_1^4 v_2^2 = 0$.

Since A is invertible, so is $\Lambda^2 A$. Therefore at least one of $v_1^1 v_2^2 - v_1^2 v_2^1$ or $v_1^3 v_2^4 - v_1^4 v_2^3$ is nonzero. Consider the case of $v_1^1 v_2^2 - v_1^2 v_2^1 \neq 0$. Then one of $v_1^1, v_2^1 \neq 0$. Suppose that both are nonzero. From (1) and (3) we get that $v_2^3 = \frac{v_1^3 v_2^1}{v_1^1} = \frac{v_1^3 v_2^2}{v_1^2}$. If $v_1^3 \neq 0$, then we obtain $\frac{v_2^1}{v_1^1} = \frac{v_2^2}{v_1^2}$ and $v_1^1 v_2^2 - v_1^2 v_2^1 = 0$, a contradiction. So $v_1^3 = 0$, and moreover $v_2^3 = 0$. Similarly, $v_2^4 = \frac{v_1^2 v_2^1}{v_1^1} = \frac{v_1^4 v_2^2}{v_1^2}$ yields $v_1^4 = v_2^4 = 0$. Now suppose that $v_1^1 \neq 0$ and $v_2^1 = 0$. Then (2) implies that $v_2^4 = 0$ and (1) implies $v_2^3 = 0$. Since $v_2^1 = 0$, we must have $v_2^2 \neq 0$ or else $v_1^1 v_2^2 - v_1^2 v_2^1 = 0$. Then (3) and (4) yield $v_1^3 = v_1^4 = 0$. Analogously, in the case the $v_1^1 = 0$ while $v_1^2 \neq 0$ we get that $v_1^3 = v_1^4 = v_2^3 = v_2^4 = 0$. Therefore $v_1^1 v_2^2 - v_1^2 v_2^1 \neq 0$ implies

$$A = \begin{pmatrix} X & * \\ 0 & * \end{pmatrix}.$$

Since $\mathrm{Span}(Y_1 \wedge Y_2, Y_3 \wedge Y_4)$ is invariant under the swap $Y_1 \iff Y_3$, $Y_2 \iff Y_4$, the equations in the entries of $\vec{v_3}, \vec{v_4}$ obtained from requiring $\Lambda^2 A(Y_3 \wedge Y_4) \in \mathrm{Span}(Y_1 \wedge Y_2, Y_3 \wedge Y_4)$ are analogous. Therefore in the case of $v_1^1 v_2^2 - v_1^2 v_2^1 \neq 0$ we get

$$A = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

where $X, Y \in \mathrm{GL}(2, \mathbb{R})$. Similarly, the case of $v_1^1 v_2^2 - v_1^2 v_2^1 = 0$ implies $v_1^3 v_2^4 - v_1^4 v_2^3 \neq 0$ and we get

$$A = \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix}$$

for $X, Y \in \mathrm{GL}(2, \mathbb{R})$. □

By Corollary 3.7, $I_g([,])$ is a submanifold of $S(\Lambda^2\mathbb{R}^4, \mathbb{R}^2)$ with dimension $20 - \dim \mathrm{Aut}_g(Z)$. Here $\dim \mathrm{Aut}_g(Z) = 8$, so $\dim I_g([,]) = 12$. $S(\Lambda^2\mathbb{R}^4, \mathbb{R}^2)$ is a manifold of dimension 12 and $I_g([,]) \subset S(\Lambda^2\mathbb{R}^4, \mathbb{R}^2)$ has no boundary, so $I_g([,])$ is an open subset of $S(\Lambda^2\mathbb{R}^4, \mathbb{R}^2)$.

3.3. **Asymmetry of step-2 Carnot algebras in small dimensions.** We are now ready to prove Theorem 1.9.

Proof of Theorem 1.9. Recall from Corollary 3.2 that given $[,] \in \mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$, $I_g([,])$ is a manifold of dimension $\dim \mathrm{GL}(n, \mathbb{R}) + \dim \mathrm{GL}(p, \mathbb{R}) - \dim \mathrm{Aut}_g([,]) = n^2 + p^2 - \dim \mathrm{Aut}_g([,])$. Since $I_g([,]) \subset S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$, we must always have that $\dim I_g([,]) \leq \dim S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ i.e.

$$n^2 + p^2 - \dim \mathrm{Aut}_g([,]) \leq \frac{n(n-1)}{2} p. \tag{3.1}$$

Suppose that $[,] \in S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ is asymmetric. Recall again the dilation group $\mathcal{D} \subset \mathrm{Aut}_g([,])$ 1.4. Then $\mathrm{Aut}_g([,]) = \mathcal{D}L$ where $L \subset \mathrm{SL}_{\pm}(n, \mathbb{R})$ is compact. Any maximal compact subgroup of $\mathrm{SL}_{\pm}(n, \mathbb{R})$ is conjugate to the orthogonal group $O(n, \mathbb{R})$, thus a compact subgroup of $\mathrm{SL}_{\pm}(n, \mathbb{R})$ is conjugate to a subgroup of $O(n, \mathbb{R})$. Therefore $\dim G \leq \dim O(n, \mathbb{R}) = n(n-1)/2$ and we obtain $\dim \mathrm{Aut}_g([,]) = 1 + \dim G = 1 + n(n-1)/2$. So equation (3.1) implies that

$$n^2 + p^2 - \frac{n(n-1)}{2}p \leq \dim \mathrm{Aut}_g([,]) \leq 1 + \frac{n(n-1)}{2}. \tag{3.2}$$

It can be verified that the situation of $n^2 + p^2 - \frac{n(n-1)}{2}p > 1 + \frac{n(n-1)}{2}$ occurs exactly when $n \geq 3$, $n = 4$ and $p \neq 3$, or $p = 1$. Therefore the proposition follows. □

4. Asymmetry in dimension ≤ 5

In this section, we will prove Theorem 1.10, which states that that no Carnot algebra of dimension ≤ 5 is asymmetric.

Proof of Theorem 1.10. According to [DT20], the Carnot algebras with dimension ≤ 5 up to isomorphism are $N_{3,2}, N_{4,2}, N_{5,2,1}, N_{5,2,3}, N_{5,3,1}, N_{5,3,2}, N_{3,2} \times \mathbb{R}, N_{4,2} \times \mathbb{R}^2, N_{4,2} \times \mathbb{R}$, and $\mathbb{R}^5$. The algebra $\mathbb{R}^5$ is abelian with dimension 5, and thus it is not asymmetric since $\mathrm{Aut}_g(\mathbb{R}^5) = \mathrm{GL}(5, \mathbb{R})$. Moreover $N_{3,2}, N_{5,3,1}, N_{5,3,2}, N_{3,2} \times \mathbb{R}, N_{4,2} \times \mathbb{R}$, and $N_{4,2} \times \mathbb{R}^2$ are step-2 and thus not asymmetric by Theorem 1.9. We prove that $N_{4,2}, N_{5,2,1}$, and $N_{5,2,3}$ are not asymmetric individually.

1. $N_{4,2}$ Filiform type 1

$$[X_1, X_2] = X_3, [X_1, X_3] = X_4$$

$$\begin{aligned} V^1 &= \mathrm{Span}(X_1, X_2) \\ V^2 &= \mathrm{Span}(X_3) \\ V^3 &= \mathrm{Span}(X_4). \end{aligned}$$

One can check that every matrix of the form

$$\begin{pmatrix} a & 0 & 0 & 0 \\ b & b_0 & 0 & 0 \\ 0 & 0 & ab_0 & 0 \\ 0 & 0 & 0 & a^2b_0 \end{pmatrix} \in \mathrm{GL}(4, \mathbb{R}).$$

defines an automorphism of $N_{4,2}$. Therefore $\left\{ \begin{pmatrix} a & 0 \\ b & b_0 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}) \right\} \subset \mathrm{Aut}_g(N_{4,2})$. Even after factoring out dilations, $\left\{ \begin{pmatrix} a & 0 \\ b & b_0 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}) \right\}$ is unbounded so we conclude that $N_{4,2}$ is not asymmetric.

2. $N_{5,2,1}$ Filiform type 1

$$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5$$

$$\begin{aligned} V^1 &= \mathrm{Span}(X_1, X_2) \\ V^2 &= \mathrm{Span}(X_3) \\ V^3 &= \mathrm{Span}(X_4) \\ V^4 &= \mathrm{Span}(X_5). \end{aligned}$$

Every matrix of the form

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ b & b_0 & 0 & 0 & 0 \\ 0 & 0 & ab_0 & 0 & 0 \\ 0 & 0 & 0 & a^2b_0 & 0 \\ 0 & 0 & 0 & 0 & a^3b_0 \end{pmatrix} \in \mathrm{GL}(5, \mathbb{R})$$

defines an automorphism of $N_{5,2,1}$. Therefore $\left\{ \begin{pmatrix} a & 0 \\ b & b_0 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}) \right\} \subset \mathrm{Aut}_g(N_{5,2,1})$ and we conclude that $N_{5,2,1}$ is not asymmetric.

3. $N_{5,2,3}$ Filiform type 2

$$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_3] = X_5$$

$$\begin{aligned} V^1 &= \mathrm{Span}(X_1, X_2) \\ V^2 &= \mathrm{Span}(X_3) \\ V^3 &= \mathrm{Span}(X_4, X_5). \end{aligned}$$

Consider matrices of the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & A \cdot \det A \end{pmatrix}$$

where $A \in \mathrm{GL}(2, \mathbb{R})$. Each such matrix defines an automorphism of $N_{5,2,3}$, so $\mathrm{GL}(2, \mathbb{R}) \subset \mathrm{Aut}_g(N_{5,2,3})$ and we conclude that $N_{5,2,3}$ is not asymmetric.

□

5. Type-(4,3) Carnot algebras

5.1. **Classification.** [DT20] classifies type-$(4, 3)$ Carnot algebras, and finds that there are six such algebras up to isomorphism. The algebras are labeled $37A, 37B, 37B1, 37C, 37D$, and $37D1$. For definitions of each, see the proof of the proposition below. In each case, we will determine whether $\mathrm{Aut}_g([,])$ is asymmetric and whether $I_g([,]) \subset S(\Lambda^2\mathbb{R}^4, \mathbb{R}^3)$ has full dimension, i.e. has dimension $=$ 18 since $S(\Lambda^2\mathbb{R}^4, \mathbb{R}^3)$ is an open subset of $\mathbb{R}^{18}$. Our exploration will prove the following proposition, using the labelings from [DT20].

Proposition 5.1. *For type-$(4, 3)$ algebras, we have the following classification.*

- $37A, 37B, 37B1, 37C$ *are each not asymmetric and have graded isomorphism classes of less-than-full dimension.*
- $37D$ *is not asymmetric and its graded isomorphism class has full dimension.*
- $37D1$ *is asymmetric and its graded isomorphism class has full dimension.*

Proof. Recall that $\dim I_g([,]) = 4^2 + 3^2 - \dim \mathrm{Aut}_g([,])$, so $\dim I_g([,]) = 18$ (< 18) if and only if $\dim \mathrm{Aut}_g([,]) = 7$ (> 7). Now we go over each automorphism class of type-$(4, 3)$ Carnot algebras. We let $(X_1, \ldots, X_4)$ be the standard basis of $\mathbb{R}^4$ and (X_5, X_6, X_7) the standard basis of $\mathbb{R}^3$. For each $[,]$, let $Z = \mathrm{im}([,]^*)$. By Corollary 3.6, it is enough to find matrices $A \in \mathrm{GL}(4, \mathbb{R})$ such that $\Lambda^2 A^*(Z) = Z$. We make use of the natural isomorphism $\Lambda^2(\mathbb{R}^n)^* \to \Lambda^2\mathbb{R}^n$ that maps $X_i^* \wedge X_j^* \mapsto X_i \wedge X_j$ in order to write Z in terms of the X_i's instead of the X_i^*'s.

1. *37A*

$$[X_2, X_1] = X_5, [X_1, X_3] = X_6, [X_1, X_4] = X_7$$

$$\begin{aligned} Z &= \mathrm{Span}(X_2 \wedge X_1, X_1 \wedge X_3, X_1 \wedge X_4) \\ &= \mathrm{Span}(X_1 \wedge X_2, X_1 \wedge X_3, X_1 \wedge X_4) \\ &= X_1 \wedge \mathrm{Span}(X_2, X_3, X_4). \end{aligned}$$

Notice that for every

$$A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}, \qquad B \in \mathrm{GL}(3, \mathbb{R})$$

we have $\Lambda^2 A(Z) = Z$. Therefore $\mathrm{Aut}_g(37A)$ contains an embedding of $\mathrm{GL}(3, \mathbb{R})$, so $\dim \mathrm{Aut}_g(27A) \geq 9 > 7$.

2. *37B*

$$[X_1, X_2] = X_5, [X_2, X_3] = X_6, [X_3, X_4] = X_7$$

$$Z = \mathrm{Span}(X_1 \wedge X_2, X_2 \wedge X_3, X_3 \wedge X_4).$$

Consider the set

$$S = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ c & d & 0 & h \\ a' & 0 & e' & f' \\ 0 & 0 & 0 & h' \end{pmatrix} \in \mathrm{GL}(4, \mathbb{R}) \mid a, c, d, h, a', e', f', h' \in \mathbb{R} \right\}.$$

Let $M \in S$ be given, and notice that

$$\begin{aligned} \Lambda^2 M(X_1 \wedge X_2) &= ad X_1 \wedge X_2 - a'd X_2 \wedge X_3 \in Z \\ \Lambda^2 M(X_2 \wedge X_3) &= de' X_2 \wedge X_3 \in Z \\ \Lambda^2 M(X_3 \wedge X_4) &= -he' X_2 \wedge X_3 + e'h' X_3 \wedge X_4 \in Z \end{aligned}$$

so $M \in \mathrm{Aut}_g(37B)$. Since $\dim S = 8$ and $S \subset \mathrm{Aut}_g(37B)$, we conclude $\dim \mathrm{Aut}_g(37B) \geq 8$.

3. *37B1*

$$[X_1, X_2] = X_5, [X_1, X_3] = X_6, [X_1, X_4] = X_7, [X_2, X_4] = X_6, [X_3, X_4] = -X_5$$

$$Z = \mathrm{Span}(X_1 \wedge X_2 - X_3 \wedge X_4, X_1 \wedge X_3 + X_2 \wedge X_4, X_1 \wedge X_4).$$

Consider the set

$$S = \left\{ \begin{pmatrix} a & e & i & c \\ 0 & f & j & 0 \\ 0 & g & k & 0 \\ b & h & \ell & d \end{pmatrix} \in \mathrm{GL}(4, \mathbb{R}) \mid a, e, i, c, b, h, \ell, d, f, j, g, k \in \mathbb{R} \right\}.$$

Let $M \in S$ be given, and notice that

$$\begin{aligned}
\Lambda^2 M(X_1 \wedge X_2 - X_3 \wedge X_4) &= (af + jc)X_1 \wedge X_2 - (bg + kd)X_3 \wedge X_4 \\
&\quad + (ag + kc)X_1 \wedge X_3 - (bf + jd)X_2 \wedge X_4 \\
&\quad + (ah - eb + id - \ell c)X_1 \wedge X_4 \\
\Lambda^2 M(X_1 \wedge X_3 + X_2 \wedge X_4) &= (aj - fc)X_1 \wedge X_2 - (bk - dg)X_3 \wedge X_4 \\
&\quad + (ak - gc)X_1 \wedge X_3 + (ad - bj)X_2 \wedge X_4 \\
&\quad + (a\ell - bi + ed - ch)X_1 \wedge X_4 \\
\Lambda^2 M(X_1 \wedge X_4) &= (ad - bc)X_1 \wedge X_4
\end{aligned}$$

so $\Lambda^2 M(Z) = Z$ if and only if the following system of equations holds.

$$\begin{cases} af + jc = bg + kd \\ ag + kc = -(bf + jd) \\ aj - fc = bk - gd \\ ak - gc = ad - bj \end{cases} \tag{5.1}$$

Suppose we are given any $a, e, i, c, b, h, \ell, d \in \mathbb{R}$. Let

$$Q = \begin{pmatrix} a & c & -b & -d \\ b & d & a & c \\ -c & a & d & -b \\ 0 & b & -c & a \end{pmatrix}.$$

Then to find f, j, g, k such that system 5.1 holds we require

$$Q \begin{pmatrix} f \\ j \\ g \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ ad \end{pmatrix}.$$

If M is invertible, we simply let

$$\begin{pmatrix} f \\ j \\ g \\ k \end{pmatrix} = Q^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ ad \end{pmatrix}.$$

Let S' be the subset of S containing matrices that satisfy 5.1 and for which Q is invertible. That is, S' contains all matrices

$$\begin{pmatrix} a & e & i & c \\ 0 & f & j & 0 \\ 0 & g & k & 0 \\ b & h & \ell & d \end{pmatrix}$$

such that

$$\det Q = a^4 + 2a^2b^2 + b^4 + 2a^2c^2 - 2b^2c^2 + c^4 + 6abcd - a^2d^2 + b^2d^2 + c^2d^2 \neq 0$$

and

$$\begin{pmatrix} f \\ j \\ g \\ k \end{pmatrix} = Q^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ ad \end{pmatrix}.$$

Now S' is a manifold of dimension 8. Moreover, $S' \subset \mathrm{Aut}_g(37B)$ and we get $\dim \mathrm{Aut}_g(37B) \geq 8$.

4. *37C*

$$[X_1, X_2] = X_5, [X_2, X_3] = X_6, [X_2, X_4] = X_7, [X_3, X_4] = X_5$$
$$Z = \mathrm{Span}(X_1 \wedge X_2 + X_3 \wedge X_4, X_2 \wedge X_3, X_2 \wedge X_4).$$

Consider the set

$$S = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & e & 0 & 0 \\ c & 0 & f & h \\ d & 0 & g & i \end{pmatrix} \in \mathrm{GL}(4, \mathbb{R}) \mid a, b, c, d, e, f, h, g, i \in \mathbb{R} \right\}.$$

Let $M \in S$ be given, and notice that

$$\begin{aligned} \Lambda^2 M(X_2 \wedge X_3) &= ef X_2 \wedge X_3 + eg X_2 \wedge X_4 \in Z \\ \Lambda^2 M(X_2 \wedge X_4) &= eh X_2 \wedge X_3 + ei X_2 \wedge X_4 \in Z \\ \Lambda^2 M(X_1 \wedge X_2 + X_3 \wedge X_4) &= ae X_1 \wedge X_2 + (fi - hg) X_3 \wedge X_4 \\ &\quad - ec X_2 \wedge X_3 - ed X_2 \wedge X_4. \end{aligned}$$

Introducing the requirement that $ae = fi - hg$, we get that $\Lambda^2 M(X_1 \wedge X_2 + X_3 \wedge X_4) \in Z$ so $M \in \mathrm{Aut}_g(37C)$. Now

$$S' = \{M \in S \mid ae = fi - hg\}$$

is an algebraic variety with dimension 8. Moreover $S' \subset \mathrm{Aut}_g(37C)$, so $\dim \mathrm{Aut}_g(37C) \geq 8$.

5. *37D*

$$[X_1, X_2] = X_5, [X_1, X_3] = X_6, [X_2, X_4] = X_7, [X_3, X_4] = X_5$$
$$Z = \mathrm{Span}(X_1 \wedge X_2 + X_3 \wedge X_4, X_1 \wedge X_3, X_2 \wedge X_4).$$

We will show that $\dim \mathrm{Aut}_g(37D) = 7$. The idea is to express $\mathrm{Aut}_g(37D)$ as a finite union of manifolds or algebraic varieties, each having dimension ≤ 7. It follows that $\dim \mathrm{Aut}_g(37D) \leq 7$, and since $\dim \mathrm{Aut}_g(37D) \geq 7$ always, we can conclude $\dim \mathrm{Aut}_g(37D) = 7$. Suppose that

$$M = \begin{pmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & \mathcal{O} \\ d & h & \ell & p \end{pmatrix} \in \mathrm{Aut}_g(37D).$$

Let

$$A = \begin{pmatrix} 0 & -c & b & 0 \\ -d & 0 & 0 & a \\ -b & a & d & -c \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & \mathcal{O} & -n & 0 \\ p & 0 & 0 & -m \\ n & -m & -p & \mathcal{O} \end{pmatrix}.$$

Then the requirement that $\Lambda^2 M(Z) = Z$ is equivalent to the coefficients of M satisfying the conditions

$$\begin{pmatrix} i \\ j \\ k \\ \ell \end{pmatrix} \in \ker A, \qquad \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \in \ker B,$$

and

$$\begin{cases} bg - fc + j\mathcal{O} - nk = 0 \\ ah - ed + ip - m\ell = 0 \\ af - eb + in - mj = ch - gd + kp - \mathcal{O}\ell \end{cases} \tag{5.2}$$

Notice that $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \ker A$, so if A has a 1-dimensional kernel then $\begin{pmatrix} i \\ j \\ k \\ \ell \end{pmatrix} \in \operatorname{Span} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ and we get that M is not invertible. So A must not be surjective. This is the case exactly when one of the following holds.

(A) $b = c = 0$
(B) $d = a = 0$
(C) $a = -tc$, $d = tb$ for some nonzero $t \in \mathbb{R}$
(D) $a = c = 0$
(E) $d = b = 0$.

Similarly, B must not be surjective, which occurs exactly when once of the following holds.

(A') $n = \mathcal{O} = 0$
(B') $p = m = 0$
(C') $m = -z\mathcal{O}$, $p = zn$ for some nonzero $z \in \mathbb{R}$
(D') $\mathcal{O} = m = 0$
(E') $p = n = 0$.

We let $\mathrm{Aut}_{(A,A')}$ denote the set of $M \in \mathrm{Aut}_g(37D)$ for which case A and A' hold, which is an algebraic variety contained in $\mathrm{Aut}_g(37D)$. Define $\mathrm{Aut}_{(A,B')}, \mathrm{Aut}_{(A,C')}$, and so forth similarly. Naturally, we have to investigate the possibilities for M in each of the 25 case. We can simplify the problem slightly using a symmetry of Z. First, there is a natural correspondence between (X, Y') and (Y, X') that reduces the problem to 15 cases. Now consider the transformation $S \in \mathrm{Aut}_g(37D)$ which swaps $X_1 \leftrightarrow X_2$ and $X_3 \leftrightarrow X_4$. Moreover, consider $V \in \mathrm{Aut}_g(37D)$ which swaps $X_1 \leftrightarrow X_4$ and $X_2 \leftrightarrow X_3$. The following lemmas use S and V to obtain symmetries in the remaining cases.

Lemma 5.2. $S \cdot \mathrm{Aut}_{(A,A')} = \mathrm{Aut}_{(B,B')}$

Proof. $\Lambda^2 S$ fixes Z, so $S \in \mathrm{Aut}_g(37D)$ and we get $M \in \mathrm{Aut}_g(37D)$ if and only if $SM \in \mathrm{Aut}_g(37D)$. Now $M \in \mathrm{Aut}_{(A,A')}$ if and only if $SM \in \mathrm{Aut}_{(B,B')}$, so $S \cdot \mathrm{Aut}_{(A,A'} = \mathrm{Aut}_{(B,B')}$. □

Lemma 5.3. $S \cdot \mathrm{Aut}_{(C,A')} = \mathrm{Aut}_{(C,B')}$

Proof. Suppose that $M \in \mathrm{Aut}_{(C,A')}$, that is, $M \in \mathrm{Aut}_g(37D), n = \mathcal{O} = 0$, and $a = -tc, d = tb$ for some nonzero $t \in \mathbb{R}$. Let

$$SM = \begin{pmatrix} \tilde{a} & \tilde{e} & \tilde{i} & \tilde{m} \\ \tilde{b} & \tilde{f} & \tilde{j} & \tilde{n} \\ \tilde{c} & \tilde{g} & \tilde{k} & \widetilde{\mathcal{O}} \\ \tilde{d} & \tilde{h} & \tilde{\ell} & \tilde{p} \end{pmatrix}.$$

Then $\tilde{m} = n$, $\tilde{p} = \mathcal{O}$, $\tilde{a} = b$, $\tilde{d} = c$, so $\tilde{m} = \tilde{p} = 0, \tilde{b} = -t\tilde{d}, \tilde{c} = t\tilde{a}$. Letting $t_0 = -\frac{1}{t}$ we get $\tilde{a} = -t_0\tilde{c}$ and $\tilde{d} = t_0\tilde{b}$, so $SM \in \mathrm{Aut}_{(C,B')}$. Given $M' \in \mathrm{Aut}_{(C,B')}$, it is analogous to show that $SM' \in \mathrm{Aut}_{(C,B')}$. We conclude that $S \cdot \mathrm{Aut}_{(C,A')} = \mathrm{Aut}_{(C,B')}$. □

Lemma 5.4. $V \cdot \mathrm{Aut}_{(C,D')} = \mathrm{Aut}_{(C,E')}$

Proof. Suppose that $M \in \mathrm{Aut}_{(C,D')}$, that is, $M \in \mathrm{Aut}_g(37D)$, $\mathcal{O} = m = 0$, and $d = -tb, a = tc$ for some nonzero $t \in \mathbb{R}$. Let $\tilde{a}, \tilde{b}$, etc denote the coefficients of VM, as in the preceding lemma. By the definition of V, we have that $\tilde{a} = d$, $\tilde{d} = a$, $\tilde{b} = c$, $\tilde{c} = b$, $\tilde{m} = p$, $\tilde{p} = m$, $\tilde{n} = \mathcal{O}$, and $\widetilde{\mathcal{O}} = n$. Therefore $\tilde{p} = \tilde{n} = 0$ and $\tilde{d} = -t\tilde{b}, \tilde{a} = t\tilde{c}$ and we see that $VM \in \mathrm{Aut}_{(C,E')}$. Thus $V \cdot \mathrm{Aut}_{(C,D')} \subset \mathrm{Aut}_{(C,E')}$, Since $V : \mathrm{Aut}_{(C,D')} \to \mathrm{Aut}_{(C,E')}$ is its own inverse, we conclude $V \cdot \mathrm{Aut}_{(C,D')} = \mathrm{Aut}_{(C,E')}$. □

Lemma 5.5. $V \cdot \mathrm{Aut}_{(A,D')} = \mathrm{Aut}_{(A,E')}$

Proof. Suppose that $M \in \mathrm{Aut}_{(A,D')}$, that is, $M \in \mathrm{Aut}_g(37D)$, $\mathcal{O} = m = 0$, and $b = c = 0$. Let $\tilde{a}, \tilde{b}$, etc denote the coefficients of VM. By the definition of V, we have that $\tilde{a} = d$, $\tilde{d} = a$, $\tilde{b} = c$, $\tilde{c} = b$, $\tilde{m} = p$, $\tilde{p} = m$, $\tilde{n} = \mathcal{O}$, and $\widetilde{\mathcal{O}} = n$. Therefore $\tilde{p} = \tilde{n} = 0$ and $\tilde{b}\tilde{c} = 0$ and we see that $VM \in \mathrm{Aut}_{(A,E')}$. Thus $V \cdot \mathrm{Aut}_{(A,D')} \subset \mathrm{Aut}_{(A,E')}$. Since $V : \mathrm{Aut}_{(A,D')} \to \mathrm{Aut}_{(A,E')}$ is its own inverse, we conclude $V \cdot \mathrm{Aut}_{(A,D')} = \mathrm{Aut}_{(A,E')}$. □

Lemma 5.6. $V \cdot \mathrm{Aut}_{(B,D')} = \mathrm{Aut}_{(B,E')}$

Proof. Suppose that $M \in \mathrm{Aut}_{(B,D')}$, that is, $M \in \mathrm{Aut}_g(37D)$, $\mathcal{O} = m = 0$, and $a = d = 0$. Let $\tilde{a}, \tilde{b}$, etc denote the coefficients of VM. By the definition of V, we have that $\tilde{a} = d$, $\tilde{d} = a$, $\tilde{b} = c$, $\tilde{c} = b$, $\tilde{m} = p$, $\tilde{p} = m$, $\tilde{n} = \mathcal{O}$, and $\widetilde{\mathcal{O}} = n$. Therefore $\tilde{p} = \tilde{n} = 0$ and $\tilde{a}\tilde{d} = 0$ and we see that $VM \in \mathrm{Aut}_{(B,E')}$. Thus $V \cdot \mathrm{Aut}_{(B,D')} \subset \mathrm{Aut}_{(B,E')}$. Since $V : \mathrm{Aut}_{(B,D')} \to \mathrm{Aut}_{(B,E')}$ is its own inverse, we conclude $V \cdot \mathrm{Aut}_{(A,D')} = \mathrm{Aut}_{(A,E')}$. □

Lemmas 5.2, 5.3, 5.4, 5.5, and 5.6 prove that

$$\begin{aligned} \dim \mathrm{Aut}_{(A,A')} &= \dim \mathrm{Aut}_{(B,B')} \\ \dim \mathrm{Aut}_{(C,A')} &= \dim \mathrm{Aut}_{(C,B')} \\ \dim \mathrm{Aut}_{(C,D')} &= \dim \mathrm{Aut}_{(C,E')} \\ \dim \mathrm{Aut}_{(A,D')} &= \dim \mathrm{Aut}_{(A,E')} \end{aligned}$$

and

$$\dim \mathrm{Aut}_{(B,D')} = \dim \mathrm{Aut}_{(B,E')}.$$

The remaining cases are

$$\begin{gathered} (B,B'), (A,B'), (A,D'), (B,D'), (D,D') \\ (E,E'), (D,E'), (C,B'), (C,D'), (E,E'). \end{gathered}$$

We treat each case separately.

$\mathrm{Aut}_{(A,B')}$:

Let $M \in \mathrm{Aut}_{(A,B')}$ be given. Then

$$\ker A = \operatorname{Span}\left\{\begin{pmatrix} a \\ 0 \\ 0 \\ d \end{pmatrix}, \begin{pmatrix} 0 \\ -d \\ a \\ 0 \end{pmatrix}\right\},$$

$$\ker B = \operatorname{Span}\left\{\begin{pmatrix} -\mathcal{O} \\ 0 \\ 0 \\ n \end{pmatrix}, \begin{pmatrix} 0 \\ n \\ \mathcal{O} \\ 0 \end{pmatrix}\right\},$$

and, for some $\gamma, \iota, \alpha, \beta \in \mathbb{R}$,

$$M = \begin{pmatrix} a & -\gamma\mathcal{O} & \alpha a & 0 \\ 0 & \iota n & -\beta d & n \\ 0 & \iota\mathcal{O} & \beta a & \mathcal{O} \\ d & \gamma n & \alpha d & 0 \end{pmatrix}.$$

System (5.2) restricts $(a, d, n, \mathcal{O}, \gamma, \beta, \alpha, \iota)$ to some variety of dimension ≤ 7, so $\dim \mathrm{Aut}_{(A,B')}(V) \leq 7$.

$\mathrm{Aut}_{(B,B')}$:

Let $M \in \mathrm{Aut}_{(B,B')}$ be given. Then

$$\ker A = \operatorname{Span}\left\{\begin{pmatrix} -c \\ 0 \\ 0 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ b \\ c \\ 0 \end{pmatrix}\right\}$$

$$\ker B = \operatorname{Span}\left\{\begin{pmatrix} -\mathcal{O} \\ 0 \\ 0 \\ n \end{pmatrix}, \begin{pmatrix} 0 \\ n \\ \mathcal{O} \\ 0 \end{pmatrix}\right\}$$

and, for some $\gamma, \iota, \alpha, \beta \in \mathbb{R}$,

$$M = \begin{pmatrix} 0 & -\gamma\mathcal{O} & -\alpha c & 0 \\ b & \iota n & \beta b & n \\ c & \iota\mathcal{O} & \beta c & \mathcal{O} \\ 0 & \gamma n & \alpha b & 0 \end{pmatrix}$$

System (5.2) restricts $(b, c, n, \mathcal{O}, \gamma, \beta, \alpha, \iota)$ to some variety of dimension ≤ 7, so $\dim \mathrm{Aut}_{(B,B')} \leq 7$.

$\mathrm{Aut}_{(B,D')}$:

Let $M \in \mathrm{Aut}_{(B,D')}$ be given. Then

$$\ker A = \mathrm{Span}\left\{ \begin{pmatrix} -c \\ 0 \\ 0 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ b \\ c \\ 0 \end{pmatrix} \right\}$$

$$\ker B = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

and, for some $\gamma, \iota, \alpha, \beta \in \mathbb{R}$,

$$M = \begin{pmatrix} 0 & 0 & -\alpha c & 0 \\ b & \gamma & \beta b & n \\ c & 0 & \beta c & 0 \\ 0 & \iota & \alpha b & p \end{pmatrix}$$

System (5.2) restricts $(b, c, n, p, \gamma, \beta, \alpha, \iota)$ to some variety of dimension ≤ 7, so $\dim \mathrm{Aut}_{(B,D')} \leq 7$.

$\mathrm{Aut}_{(A,D')}$:

Let $M \in \mathrm{Aut}_{(A,D')}$ be given. Then

$$\ker A = \mathrm{Span}\left\{ \begin{pmatrix} a \\ 0 \\ 0 \\ d \end{pmatrix}, \begin{pmatrix} 0 \\ -d \\ a \\ 0 \end{pmatrix} \right\}$$

$$\ker B = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

and, for some $\gamma, \iota, \alpha, \beta \in \mathbb{R}$,

$$M = \begin{pmatrix} a & 0 & \alpha a & 0 \\ 0 & \gamma & -\beta d & n \\ 0 & 0 & \beta a & 0 \\ d & \iota & \alpha d & p \end{pmatrix}$$

System (5.2) restricts $(a, b, n, p, \gamma, \beta, \alpha, \iota)$ to some variety of dimension ≤ 7, so $\dim \mathrm{Aut}_{(A,D')} \leq 7$.

$\mathrm{Aut}_{(D,E')}$:

Let $M \in \mathrm{Aut}_{(D,E')}$ be given. Then

$$\ker A = \operatorname{Span}\left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\},$$

$$\ker B = \operatorname{Span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\},$$

and, for some $\gamma, \iota, \alpha, \beta \in \mathbb{R}$, we have.

$$M = \begin{pmatrix}0 & \gamma & 0 & m\\ b & 0 & \alpha & 0\\ 0 & \iota & 0 & \mathcal{O}\\ d & 0 & \beta & 0\end{pmatrix},$$

The space of such M is dimension 8, and System (5.2) restricts $(b, d, m, \mathcal{O}, \gamma, \iota, \alpha, \beta \in \mathbb{R})$ to a nontrivial algebraic variety. So $\dim \mathrm{Aut}_{(D,E')} \leq 7$.

$\mathrm{Aut}_{(D,D')}$ and $\mathrm{Aut}_{(E,E')}$:

Let $M \in \mathrm{Aut}_{(D,D')}$ be given. Then

$$\ker A = \operatorname{Span}\left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$$

and

$$\ker B = \operatorname{Span}\left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}.$$

The first row of M contains all zeroes, so M is not invertible and we get that $\mathrm{Aut}_{(D,D')}$ is empty. Analogously, one finds that that $\mathrm{Aut}_{(E,E')}$ is empty.

$\mathrm{Aut}_{(C,C')}$:

Let $M \in \mathrm{Aut}_{(C,C')}$ be given. Then

$$\ker A = \operatorname{Span}\left\{\begin{pmatrix}c\\b\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-b\\c\end{pmatrix}\right\},$$

$$\ker B = \operatorname{Span}\left\{\begin{pmatrix}-\mathcal{O}\\0\\0\\n\end{pmatrix}, \begin{pmatrix}0\\n\\\mathcal{O}\\0\end{pmatrix}\right\},$$

and, for some $u, v, r, s \in \mathbb{R}$,

$$M = \begin{pmatrix} -tc & -u\mathcal{O} & rc & -z\mathcal{O} \\ b & vn & rb & n \\ c & v\mathcal{O} & -sb & \mathcal{O} \\ tb & un & sc & zn \end{pmatrix}.$$

Let

$$F = \begin{pmatrix} 0 & b\mathcal{O} - nc & b\mathcal{O} & nb \\ -tcn + \mathcal{O}tb & 0 & czn & cz\mathcal{O} \\ \mathcal{O}b - cn & \mathcal{O}tb - tcn & z\mathcal{O}b + cn & bzn + \mathcal{O}c \end{pmatrix}.$$

System (5.2) is equivalent to

$$\begin{pmatrix} u \\ v \\ r \\ s \end{pmatrix} \in \ker F.$$

Let $f : \mathbb{R}^6 \to \mathbb{R}^4$ send $(b, c, t, n, \mathcal{O}, z)$ to the vector containing the determinants of the four 3-by-3 minors of $F(b, c, t, n, \mathcal{O}, z)$. f is continuous, so $f^{-1}(\mathbb{R}^4 \setminus \{0\})$ is an open subset of $\mathbb{R}^6$. Let

$$S = \left\{(x, y) \in \mathbb{R}^6 \times \mathbb{R}^4 \mid y \in \ker F(x)\right\}$$

and

$$S_1 = \left\{(x, y) \in \mathbb{R}^6 \times \mathbb{R}^4 \mid x \in f^{-1}(\mathbb{R}^4 \setminus \{0\}) \ \& \ y \in \ker F(x)\right\}.$$

S is the zero set of a polynomial and therefore an algebraic variety. Notice that S_1 is open in S and is a manifold with dimension $6 + 1 = 7$. Now the coordinates of f are polynomials in the entries of $\mathbb{R}^6$, so its zero set has measure zero and we get that S_1 is dense in S. Thus the dimension of S equals the dimension of S_1 which is 7. Consider the map

$$T : \mathbb{R}^{10} \to \mathbb{R}^{16}$$
$$(b, c, t, n, \mathcal{O}, z, u, v, r, s) \mapsto \begin{pmatrix} -tc & -u\mathcal{O} & rc & -z\mathcal{O} \\ b & vn & rb & n \\ c & v\mathcal{O} & -sb & \mathcal{O} \\ tb & un & sc & zn \end{pmatrix}.$$

We showed earlier that $\mathrm{Aut}_{(C,C')}$ consists of all invertible matrices in $T(S)$. Letting $S' = T^{-1}(\mathrm{Aut}_{(C,C')})$, note that the restriction of T to S' is a diffeomorphism onto $\mathrm{Aut}_{(C,C')}$. S' is an open and dense subset of S, so it is an algebraic variety with dimension equal to that of S. Since the dimension of S is 7, we conclude that the dimension of $\mathrm{Aut}_{(C,C')}$ is 7.

$\mathbf{Aut}_{(\boldsymbol{C},\boldsymbol{B'})}$ **:**

Let $M \in \mathrm{Aut}_{(C,B')}$ be given. Then

$$\ker A = \mathrm{Span}\left\{\begin{pmatrix} c \\ b \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -b \\ c \end{pmatrix}\right\},$$

$$\ker B = \mathrm{Span}\left\{\begin{pmatrix} -\mathcal{O} \\ 0 \\ 0 \\ n \end{pmatrix}, \begin{pmatrix} 0 \\ n \\ \mathcal{O} \\ 0 \end{pmatrix}\right\},$$

and, for some $\gamma, \iota, r, s \in \mathbb{R}$,

$$M = \begin{pmatrix} -tc & -\gamma\mathcal{O} & rc & 0 \\ b & \iota n & rb & n \\ c & \iota\mathcal{O} & -sb & \mathcal{O} \\ tb & \gamma n & sc & 0 \end{pmatrix}.$$

Let

$$K = \begin{pmatrix} b\mathcal{O} & nb & 0 & b\mathcal{O} - cn \\ 0 & 0 & t(\mathcal{O}b - cn) & 0 \\ cn & \mathcal{O}b & \mathcal{O}b - cn & t(cn - \mathcal{O}b) \end{pmatrix}.$$

System (5.2) is equivalent to

$$\begin{pmatrix} r \\ s \\ \gamma \\ \iota \end{pmatrix} \in \ker K.$$

By definition of $\mathrm{Aut}_{(C,C')}$, $b, c \neq 0$. One can see that the first and second columns of K are linearly independent unless

(1) $n = \mathcal{O} = 0$ or

(2) $n, \mathcal{O} \neq 0$ but $n = \mathcal{O}\sqrt{\frac{b}{c}}$.

The first case is not possible since M would not be invertible. In the second case, $\dim\ker K \leq 3$ but the requirement $n = \mathcal{O}\sqrt{\frac{b}{c}}$ reduces dimension by 1. So the restriction of $\mathrm{Aut}_{(C,B')}$ to case (2) has dimension $\leq 5 + 3 - 1 = 7$. If the first and second columns of K are linearly independent, $\dim\ker K \leq 2$. So the restriction of $\mathrm{Aut}_{(C,B')}(V)$ to this case has dimension $\leq 5 + 2 = 7$. We conclude that $\dim \mathrm{Aut}_{(C,B')}(V) \leq 7$.

$\mathbf{Aut}_{(C,D')}$ **:**

Let $M \in \mathrm{Aut}_{(C,D')}$ be given. Then

$$\ker A = \mathrm{Span}\left\{ \begin{pmatrix} c \\ b \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -b \\ c \end{pmatrix} \right\},$$

$$\ker B = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

and, for some $\gamma, \iota, r, s \in \mathbb{R}$,

$$M = \begin{pmatrix} -tc & 0 & rc & 0 \\ b & \gamma & rc & n \\ c & 0 & -sb & 0 \\ tb & \iota & sc & p \end{pmatrix}.$$

Let

$$K = \begin{pmatrix} 0 & nn & -c & 0 \\ cp & 0 & 0 & -tc \\ cn & bp & -tc & c \end{pmatrix}.$$

System (5.2) is equivalent to

$$\begin{pmatrix} r \\ s \\ \gamma \\ \iota \end{pmatrix} \in \ker K.$$

We are requiring at $c \neq 0$, since the $c = 0$ case is equivalent to (D, D'). But $c \neq 0$ implies that the third and fourth columns of K are linearly independent, so $\dim \ker K \leq 2$. It follows that $\dim \mathrm{Aut}_{(C,D')} \leq 5 + 2 = 7$.

6. *37D1* This algebra is isomorphic to the quaternionic Heisenberg algebra of dimension 7, $\mathbb{H}H^2$, in Pansu's Theorem 1.6. Pansu proved that $\mathrm{Aut}_g([,]) \cong \mathbb{R}_+ \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)$, so $37D1$ is asymmetric and has a graded isomorphism class of dimension $25 - 7 = 18$ which is full.

This completes our proof of the Proposition. ∎

5.2. **Asymmetry is not Zariski open.** In this subsection, we will prove Theorem 1.11 which states that asymmetry is not a Zariski open condition on the space of type-$(4,3)$ Carnot algebras. Proposition 5.1 demonstrates that the only asymmetric type-$(4,3)$ Carnot algebra is the quaterionic Heisenberg algebra, 37D1. Therefore it is enough to show that $I_g(37D1)$ is not Zariski open in $S(\Lambda^2\mathbb{R}^4, \mathbb{R}^3)$. If $I_g(37D1)$ were Zariski open and $S(\Lambda^2\mathbb{R}^4, \mathbb{R}^3)$ were connected in the Euclidean topology, then $S(\Lambda^2\mathbb{R}^4, \mathbb{R}^3) \setminus I_g(37D1)$ would have dimension strictly less than $I_g(37D1)$. But $I_g(37D) \subset S(\Lambda^2\mathbb{R}^4, \mathbb{R}^3) \setminus I_g(37D1)$ has full dimension, a contradiction. Therefore to conclude that $I_g(37D1)$ is not Zariski open, it is enough to show that $S(\Lambda^2\mathbb{R}^4, \mathbb{R}^3)$ is connected. We prove this in Lemma 5.8 below.

Lemma 5.7. *Let* $\mathrm{Mat}(m, \mathbb{R})$ *denote the set of all* m*-by-*m *matrices. Suppose* $M \in \mathrm{Mat}(m, \mathbb{R})$ *such that* $\det M = 0$*. Let* U_+ *denote the connected component of* $\mathrm{GL}(m, \mathbb{R})$ *consisting of matrices with positive determinant, and* U_- *the component consisting of matrices with negative determinant. There exists a path* $\gamma : [0,1] \to \mathrm{Mat}(m, \mathbb{R})$ *such that* $\gamma(0) = M$ *and* $\gamma\big|_{(0,1]}$ *maps into* U_+ *(or* U_-*).*

Proof. Write $M = \begin{pmatrix} w_1 & \dots & w_m \end{pmatrix}$. $\mathrm{GL}(m, \mathbb{R})$ is dense in $\mathrm{Mat}(m, \mathbb{R})$, so $\begin{pmatrix} w_1 & \dots & w_m \end{pmatrix}$ is a limit point of $\mathrm{GL}(m, \mathbb{R})$, say $(z_n) \to \begin{pmatrix} w_1 & \dots & w_m \end{pmatrix}$. Without loss of generality, $(z_n) \subset U_+$. Since U_+ is path-connected, we can find a sequence of paths $\gamma_i : [\sum_{j=1}^{i-1}(\frac{1}{2})^j, \sum_{j=1}^{i}(\frac{1}{2})^j] \to U_+$ such that $\gamma_i(\sum_{j=1}^{i-1}(\frac{1}{2})^j) = z_i$ and $\gamma_i(\sum_{j=1}^{i}(\frac{1}{2})^j) = z_{i+1}$. Where $\cdot$ denotes concatenation, let $\gamma^k = \gamma_k \cdot \gamma_{k-1} \cdots \gamma_1$ and $\gamma = \lim_{k\to\infty} \gamma^k$. Then $\gamma(0) = z_1$, $\gamma(1) = \begin{pmatrix} w_1 & \dots & w_m \end{pmatrix}$ and $\gamma\big|_{[0,1)} \subset U_+$. Taking the reverse parameterization of γ gives us the desired path. $\square$

Lemma 5.8. *If $n \neq m$, then $S(\mathbb{R}^n, \mathbb{R}^m)$ is path-connected with respect to the induced topology from $\mathbb{R}^{nm}$ (or equivalently connected, since $S(\mathbb{R}^n, \mathbb{R}^m) \subset \mathbb{R}^{nm}$ is open).*

Proof. Let $A, B \in S(\mathbb{R}^n, \mathbb{R}^m)$ and suppose

$$A = \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix}$$

and

$$B = \begin{pmatrix} w_1 & \dots & w_n \end{pmatrix}$$

where v_i, w_j are column vectors in $\mathbb{R}^m$. We will find a path from B to A. There are 3 cases:

(1) There exist $1 \leq y_1 < \cdots < y_m \leq n$ such that

$$\begin{pmatrix} w_{y_1} & \dots & w_{y_m} \end{pmatrix}$$

and

$$\begin{pmatrix} v_{y_1} & \dots & v_{y_m} \end{pmatrix}$$

are not invertible

(2) There exist $1 \leq y_1 < \cdots < y_m \leq n$ such that

$$\begin{pmatrix} w_{y_1} & \dots & w_{y_m} \end{pmatrix}$$

is not invertible but

$$\begin{pmatrix} v_{y_1} & \dots & v_{y_m} \end{pmatrix}$$

is invertible, or vice versa.

(3) For all $1 \leq y_1 < \cdots < y_m \leq n$ we have

$$\begin{pmatrix} v_{y_1} & \dots & v_{y_m} \end{pmatrix}$$

and

$$\begin{pmatrix} w_{y_1} & \dots & w_{y_m} \end{pmatrix}$$

are invertible.

Case 1: By Lemma 5.7, there exists a path γ from $\begin{pmatrix} w_{y_1} & \dots & w_{y_m} \end{pmatrix}$ to some $M \in U_+$ such that $\gamma\big|_{(0,1]}$ maps into U_+. Similarly, there exists γ' from $\begin{pmatrix} v_{y_1} & \dots & v_{y_m} \end{pmatrix}$ to some $M' \in U_+$ such that $\gamma'\big|_{(0,1]}$ maps into U_+. Apply γ to $\begin{pmatrix} w_{y_1} & \dots & w_{y_m} \end{pmatrix}$ and fix the rest of B to get a matrix $B' \in S$. For each $i \notin \{y_1, \dots, y_n\}$, we connect w_i to v_i using a straight line. Since $M \in U_+$, doing so gives us a path in S. Let B'' denote the new matrix, which now equals M on the columns $y_1, \dots, y_n$ and agrees with A in all other columns. Recall $M, M' \in U_+$, so we can connect M, M' via a path ϕ in U_+. Applying ϕ and then $(\gamma')^{-1}$ to columns $y_1, \dots, y_n$ of B'' and fixing all other columns connects B'' to A yields a path that is in S. Therefore B is path-connected to A in S.

Case 2: Without loss of generality, we assume that $\begin{pmatrix} w_{y_1} & \dots & w_{y_m} \end{pmatrix} \notin \mathrm{GL}(m, \mathbb{R})$ and $\begin{pmatrix} v_{y_1} & \dots & v_{y_m} \end{pmatrix} \in U_+$. Once more, we can find a path γ from $\begin{pmatrix} w_{y_1} & \dots & w_{y_m} \end{pmatrix}$ to some $M \in U_+$ such that $\gamma\big|_{(0,1]}$ maps into U_+. Moreover, there exists ϕ in U_+ connecting M to $\begin{pmatrix} v_{y_1} & \dots & v_{y_m} \end{pmatrix}$. We apply γ and then ϕ to $\begin{pmatrix} w_{y_1} & \dots & w_{y_m} \end{pmatrix}$ while fixing the rest of B to get a matrix B'' that agrees with A on $y_1, \dots, y_m$ and B elsewhere. Since $\begin{pmatrix} v_{y_1} & \dots & v_{y_m} \end{pmatrix} \in U_+$, connecting B'' to A via a straight line on all columns not labeled $y_1, \dots, y_m$ yields a path in S. Therefore B is path-connected to A in S.

Case 3: In this case, $\begin{pmatrix} w_1 & \dots & w_m \end{pmatrix}, \begin{pmatrix} v_1 & \dots & v_m \end{pmatrix} \in \mathrm{GL}(n, \mathbb{R})$. Therefore the straight line path connecting $(w_{m+1}, \dots, w_n)$ to $(v_{m+1}, \dots, v_n)$ and fixing the first m columns of B remains in S. Moreover, notice that the straight line path connecting A to $(-v_1, \dots, v_n)$ is in S since $(v_2, \dots, v_{m+1}) \in \mathrm{GL}(m, \mathbb{R})$. Therefore without loss of generality we assume $\begin{pmatrix} w_1 & \dots & w_m \end{pmatrix}, \begin{pmatrix} v_1 & \dots & v_m \end{pmatrix} \in U_+$. So we can find a path ϕ in U_+ connecting $\begin{pmatrix} w_1 & \dots & w_m \end{pmatrix}$ to $\begin{pmatrix} v_1 & \dots & v_m \end{pmatrix}$. Applying this along with the straight-line path, we connect B to A via a path in S. $\square$

6. Type-(n,p) Carnot algebras for n odd

In this section, we prove some intermediate results in understanding the general type-(n, p) Carnot algebra, including a proof of Theorem 1.12. Throughout, we suppose that $p \geq 3$ and n is odd, noting that the even case was partially addressed by Pansu in Theorem 1.7. Moreover, we introduce the following notations.

Suppose $[,] \in S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$. Let $\omega_{[,]}, \omega'_{[,]}, \omega''_{[,]}$ denote the first, second, and third components of $[,]$, respectively. Each $\omega_{[,]}, \omega'_{[,]}, \omega''_{[,]}$ is an alternating bilinear form on $\mathbb{R}^n$, and we let $c(\omega_{[,]}), c(\omega'_{[,]}), c(\omega''_{[,]})$ denote their centers. For a bilinear form ω on $\mathbb{R}^n$, we define the center of ω to be $c(\omega_{[,]}) = \{v \in \mathbb{R}^n \mid \omega(v, w) = 0 \text{ for all } w \in \mathbb{R}^n\}$. Since n is odd, $c(\omega_{[,]}), c(\omega'_{[,]})$, and $c(\omega''_{[,]})$ are nontrivial, so we can always find nonzero $e_1([,]), e_2([,]), e_3([,]) \in \mathbb{R}^n$ such that $e_1([,]) \in c(\omega_{[,]}), e_2([,]) \in c(\omega'_{[,]})$, and $e_3([,]) \in c(\omega''_{[,]})$. Given such $e_1([,]), e_2([,])$ and $e_3([,])$, we let $W_1([,]) = \ker \omega_{[,]}(e_2([,]), \cdot)$, $W_2([,]) = \ker \omega'_{[,]}(e_3([,]), \cdot)$, and $W_3([,]) = \ker \omega''_{[,]}(e_1([,]), \cdot)$. Notice that in the case that $c(\omega_{[,]}), c(\omega'_{[,]}), c(\omega''_{[,]})$ are one-dimensional, the $W_i([,])$ are well-defined.

We are particularly interested in a few subspaces of $S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$. We will define them here and prove some of their properties in the coming lemmas. Let $K \subset S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ denote the set of $[,] \in S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ such that $\dim c(\omega_{[,]}) = \dim c(\omega'_{[,]}) = \dim c(\omega''_{[,]}) = 1$. Let $P \subset K$ denote the set of $[,] \in K$ such that $\mathrm{Span}(e_1([,]), e_2([,]), e_3([,])) \cap (W_1([,]) \cap W_2([,]) \cap W_3([,])) = \varnothing$. Let $Q \subset K$ denote the set of $[,] \in K$ such that $\dim(W_1([,]) \cap W_2([,]) \cap W_3([,])) = n - 3$.

Lemma 6.1. *Let $S \subset \mathrm{Mat}(n, \mathbb{R})$ denote the set of skew-symmetric matrices. Let $S' \subset S$ be the set of all $M \in S$ such that $\dim \ker M = 1$. Then $S' \subset S$ is open and dense in S.*

Proof. Let $f : S \to \mathbb{R}^{n^2}$ map M to the vector containing the determinants of its $(n-1)$-by-$(n-1)$ minors. An n-by-n matrix M has rank at least $n - 1$ if and only if $f(M)$ is not the zero vector. But when M is skew-symmetric of odd-dimension, the rank of M is $\leq n - 1$ always. Therefore $f^{-1}(\mathbb{R}^{n^2} \setminus \{0\}) = \{M \in S \mid \dim \mathrm{rank}\, M = n - 1\} = S'$. Therefore, since f is continuous, S' is open in S. Moreover, if $M \notin S'$ then one can perturb any of its $(n-1)$-by-$(n-1)$ minors to obtain a matrix in S'. So S' is dense in S. $\square$

Lemma 6.2. *K is open and dense in $S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$.*

Proof. For every $[,] \in S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$, $\omega_{[,]}, \omega'_{[,]}$, and $\omega''_{[,]}$ are skew-symmetric, bilinear forms, and so there exist unique skew-symmetric matrices $B(\omega_{[,]}), B(\omega'_{[,]})$, and $B(\omega''_{[,]})$ such that for all $u, v \in \mathbb{R}^n$, $\omega_{[,]}(u, v) = u^t B(\omega_{[,]}) v$, $\omega'_{[,]}(u, v) = u^t B(\omega'_{[,]}) v$, and $\omega''_{[,]}(u, v) = u^t B(\omega''_{[,]}) v$. Notice that $\ker B(\omega_{[,]}) = c(\omega_{[,]})$, and the same for $\omega'_{[,]}, \omega''_{[,]}$. As in the previous lemma, let $S \subset \mathrm{Mat}(n, \mathbb{R})$ denote the set of skew-symmetric matrices and $S' \subset S$ be the collection of all $M \in S$ such that $\dim \ker M = 1$. By breaking each map in $\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ into its coordinates, we get that $\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ is isomorphic as a vector space to $\underbrace{\mathrm{Hom}(\Lambda^2\mathbb{R}^n,\mathbb{R})\oplus\cdots\oplus\mathrm{Hom}(\Lambda^2\mathbb{R}^n,\mathbb{R})}_{p\ times} = \underbrace{S\oplus\cdots\oplus S}_{p\ times}$. Under this equivalence, K is mapped to $S'\oplus S'\oplus S'\oplus\underbrace{S\oplus\cdots\oplus S}_{p-3\ times}$, so K is homeomorphic to $S'\oplus S'\oplus S'\oplus\underbrace{S\oplus\cdots\oplus S}_{p-3\ times}$. Therefore Lemma 6.1 implies that K is open and dense in $\mathrm{Hom}(\Lambda^2\mathbb{R}^n,\mathbb{R}^p)$ and thus open and dense in $S(\Lambda^2\mathbb{R}^n,\mathbb{R}^p)$. ☐

Lemma 6.3. *P is open and dense in K.*

Proof. Suppose $[,]\in K$ and let $v = ae_1([,])+be_2([,])+ce_3([,])\in\mathrm{Span}(e_1([,]),e_2([,]),e_3([,]))$. Then

$$\begin{aligned}\omega_{[,]}(e_2([,]),v) &= c\omega_{[,]}(e_2([,]),e_3([,]))\\ \omega'_{[,]}(e_3([,]),v) &= -a\omega'_{[,]}(e_1([,]),e_3([,]))\\ \omega''_{[,]}(e_1([,]),v) &= b\omega''_{[,]}(e_2([,]),e_3([,])).\end{aligned}$$

So $[,]\in K$ if and only if $\omega_{[,]}(e_2([,]),e_3([,]))$, $\omega'_{[,]}(e_1([,]),e_3([,]))$, and $\omega''_{[,]}(e_1([,]),e_2([,]))\neq 0$. If $\omega_{[,]}(e_2([,]),e_3([,]))=0$, then we perturb $\omega_{[,]}(e_2([,]),e_3([,]))$ to get a new bracket $[,]'$ for which $\omega_{[,]'}(e_2([,]),e_3([,]))\neq 0$. Since $[,]'$ agrees with $[,]$ except in the first component of $[e_2([,]),e_3([,])]$, $e_1([,])$ is in the center of $\omega_{[,]'}$. Moreover, K is an open set, so perturbing $[,]$ a sufficiently small amount guarantees that $[,]'$ remains in K, and so we have $c(\omega_{[,]'})=\mathrm{Span}(e_1([,]))$ and $e_1([,]')=ce_1([,])$ for some nonzero c. Similarly, we perturb $\omega'_{[,]'},\omega''_{[,]'}$ to obtain $[,]''\in K$ such that
$\omega'_{[,]''}(e_1([,]),e_3([,]))$ and $\omega''_{[,]}(e_1([,]),e_2([,]))\neq 0$ and $c(\omega'_{[,]''})=\mathrm{Span}(e_2([,])), c(\omega''_{[,]''})=\mathrm{Span}(e_3([,]))$. $[,]''$ is therefore in Q, so $Q\subset K$ is dense. To see that Q is open, notice that if $[,]\in Q$, then there is a ball B about $[,]$ such that $[,]'\in B$ implies $\omega_{[,]'}(e_2([,]'),e_3([,]'))$, $\omega'_{[,]'}(e_1([,]'),e_3([,]'))$, and $\omega''_{[,]'}(e_1([,]'),e_2([,]'))\neq 0$. ☐

Lemma 6.4. *Q is open and dense in K.*

Proof. Suppose $[,]\in K$. Each $W_i([,])$ is an $(n-1)$-dimensional vector subspace of $\mathbb{R}^n$, so $\dim(W_1([,])\cap W_2([,])\cap W_3([,]))\geq n-3$. Suppose that $\dim(W_1([,])\cap W_2([,])\cap W_3([,]))>n-3$. Without loss of generality, this means that either $W_1([,])=W_2([,])$ or $W_1([,])\cap W_2([,])\subset W_3([,])$. Note that perturbing $\omega_{[,]}$ will result in a perturbation of $W_1([,])$ alone, and similarly perturbing $\omega'_{[,]},\omega''_{[,]}$will result in perturbations of $W_2([,]),W_3([,])$, respectively. Therefore in the case $W_1([,])=W_2([,])$, we simply perturb $\omega_{[,]}$. The case $W_1([,])\cap W_2([,])\subset W_3([,])$, we can simply perturb $\omega''_{[,]}$.

To see that Q is open in K, notice that if $\dim(W_1([,])\cap W_2([,])\cap W_3([,]))=n-3$, then the same is true after sufficiently small perturbations of $W_1([,]),W_2([,]),W_3([,])$, and hence a sufficiently small perturbation of $[,]$. ☐

Lemma 6.5. *Suppose $[,]\in S(\Lambda^2\mathbb{R}^n,\mathbb{R}^p)$ and $(A,I_p)\in\mathrm{Aut}_g([,])$. Then $A(c(\omega_{[,]}))=c(\omega_{[,]})$, $A(c(\omega'_{[,]}))=c(\omega'_{[,]})$, $A(c(\omega''_{[,]}))=c(\omega''_{[,]})$*

Proof. We will show that $A(c(\omega_{[,]}))=c(\omega_{[,]})$ since the other cases are identical. Let $v\in c(\omega_{[,]})$ be given. For any $w\in\mathbb{R}^n$, $0=\omega_{[,]}(v,w)=\omega_{[,]}(Av,Aw)$. Since A is invertible, it follows that $Av\in c(\omega_{[,]})$. ☐

Lemma 6.6. *Suppose $[,]\in K$ and $(A,I_p)\in\mathrm{Aut}_g([,])$. Then $AW_1([,])=W_1([,])$, $AW_2([,])=W_2([,])$, and $AW_3([,])=W_3([,])$*

Proof. We will show that $AW_1([,])=W_1([,])$, and the other cases are identical. Let $v\in W_1([,])$ be given. Then, $0=\omega_{[,]}(e_2([,]),v)=\omega_{[,]}(Ae_2([,]),Av)$. Since A preserves $c(\omega_{[,]})$ and $c(\omega_{[,]})$ has only one dimension, $Ae_1([,])=c_1e_1([,])$ for some nonzero $c_1\in\mathbb{R}$. So $0=\omega_{[,]}(Ae_2([,]),Av)=$

$c_1\omega_{[,]}(e_2([,]), Av)$ and we get $Av \in W_1([,])$. So $AW_1([,]) \subset W_1([,])$. Since A is invertible, it follows that $AW_1([,]) = W_1([,])$. $\square$

Lemma 6.7. [Pan89, Theorem 13.2, Step 1] *There exists open and dense $H \subset \mathrm{Hom}(\Lambda^2\mathbb{R}^{n-3}, \mathbb{R}^p)$ such that for any $[,] \in H$, if $(A, I_p) \in \mathrm{Aut}_g([,])$ then $A = I_{n-3}$.*

Lemma 6.8. *Let $R \subset K$ denote the set of $[,] \in K$ such that $[,]\big|_{W_1([,])\cap W_2([,])\cap W_3([,])} \in H$. R is dense in K.*

Proof. Let $[,] \in K$ be given. Then any perturbation of $[,]\big|_{W_1([,])\cap W_2([,])\cap W_3([,])}$ in K preserves $\omega_{[,]}(e_1([,]), \cdot)$, $\omega'_{[,]}(e_2([,]), \cdot)$, and $\omega''_{[,]}(e_3([,]), \cdot)$, so the centers of $\omega_{[,]}, \omega'_{[,]}, \omega''_{[,]}$ are unchanged. Moreover, such a perturbation preserves $\omega_{[,]}(e_2([,]), \cdot)$, $\omega'_{[,]}(e_3([,]), \cdot)$, and $\omega''_{[,]}(e_1([,]), \cdot)$, so $W_1([,])$, $W_2([,])$, and $W_3([,])$ are unchanged. Now H is dense in $\mathrm{Hom}(\Lambda^2\mathbb{R}^{n-3}, \mathbb{R}^p)$ and $K \subset \mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ is open, so we can perturb $[,]\big|_{W_1([,])\cap W_2([,])\cap W_3([,])}$ to obtain $[,] \in R$. Therefore R is dense in K. $\square$

Proof of Theorem 1.12. It is enough to show that for all $[,]$ in some dense subset of $S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$, $(A, I_p) \in \mathrm{Aut}_g([,])$ implies that $A = I_n$. Lemmas 6.3, 6.4, and 6.8 show that $Q \cap P \cap R$ is dense in K. Since K is open and dense in $S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$, $Q \cap P \cap R$ is in fact dense in $S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$. Let $[,] \in Q \cap P \cap R$ be given and suppose $(A, I_p) \in \mathrm{Aut}_g([,])$. Moreover, by construction of Q and P we have that $\mathbb{R}^n = \mathrm{Span}(e_1([,]), e_2([,]), e_3([,])) \oplus (W_1([,]) \cap W_2([,]) \cap W_3([,]))$. By Lemma 6.5, $Ae_1([,]) = c_1e_1([,])$, $Ae_2([,]) = c_2e_2([,])$, and $Ae_3([,]) = c_3e_3([,])$ for nonzero $c_1, c_2, c_3 \in \mathbb{R}$. But since $[,] \in Q \cap P$, $\omega_{[,]}(e_2([,]), e_3([,])) = \omega_{[,]}(Ae_2([,]), Ae_3([,])) = c_2c_3\omega_{[,]}(e_2([,]), e_3([,]))$ implies that $c_2c_3 = 1$. Similarly, we get that $c_1c_3 = c_1c_2 = 1$ and thus necessarily $c_1 = c_2 = c_3 = 1$. Now by lemma 6.6, $A(W_1([,]) \cap W_2([,]) \cap W_3([,])) = W_1([,]) \cap W_2([,]) \cap W_3([,])$. Let $v_1, \dots, v_{n-3}$ be a basis for $W_1([,]) \cap W_2([,]) \cap W_3([,])$. Then $e_1([,]), e_2([,]), e_3([,]), v_1, \dots, v_{n-3}$ is a basis for $\mathbb{R}^n$ and with respect to it we get

$$A = \begin{pmatrix} I_3 & 0 \\ 0 & M \end{pmatrix}$$

for some $M \in \mathrm{GL}(n-3, \mathbb{R})$. Notice that $(M, I_p) \in \mathrm{Aut}_g([,]\big|_{W_1([,])\cap W_2([,])\cap W_3([,])})$. But since $[,] \in R$, $[,]\big|_{W_1([,])\cap W_2([,])\cap W_3([,])} \in H$ and we must have that $M = I_{n-3}$. Thus $A = I_n$, as desired. $\square$

Proposition 6.9. *In addition to supposing that $p \geq 3$ and n is odd, we further require that $p < 2n - 5$. For dense choice of $[,] \in S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$, $\dim I_g([,]) \geq n^2 + p^2 - n + 1$.*

Proof. It will be enough to show that on some dense subset of $S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$, $\dim \mathrm{Aut}_g([,]) \leq n - 1$. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$ and let $V = \mathrm{Span}(e_1, \dots, e_{n-1})$. Let $S \subset S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ denote the set of $[,] \in S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$ such that $[,]\big|_{\Lambda^2 V}$ is surjective, in other words $[,]\big|_{\Lambda^2 V} \in S(\Lambda^2\mathbb{R}^{n-1}, \mathbb{R}^p)$. Notice that S is open and dense in $S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$. Moreover, it follows from steps (1) and (2) in the proof of Theorem 1.7 that on an open and dense set $K \subset S(\Lambda^2\mathbb{R}^{n-1}, \mathbb{R}^p)$, $[,]' \in K$ implies $\mathrm{Aut}_g([,]') = F\mathcal{D}$ where F is finite (see [Pan89], pages 50-51). So the set of $[,] \in S$ such that $\mathrm{Aut}_g([,]\big|_{\Lambda^2 V}) = F\mathcal{D}$ for some finite F is open and dense in $S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$. Combining these facts, we obtain that the set of $[,]$ such that

- $[,] \in S$
- $\mathrm{Aut}_g([,]\big|_{\Lambda^2 V}) = F\mathcal{D}$ for some finite F
- $[,]$ has the property of Theorem 1.12

is dense in $S(\Lambda^2\mathbb{R}^n, \mathbb{R}^p)$. Let $[,]$ be in this set. We claim that for any $n-1$-dimensional subspace $W \subset \mathbb{R}^n$, the set of $(A, B) \in \mathrm{Aut}_g([,])$ such that $AV = W$ is finite. If this is true then the result follows, since $\dim \mathrm{Aut}_g([,]) \leq \dim \mathrm{Gr}(n,1) = n-1$. First, suppose that $A\big|_V = A'\big|_V$. Then

$$(A\big|_V{}^{-1} \circ A'\big|_V, B^{-1} \circ B') = (I_{n-1}, B \circ B^{-1}) \in \mathrm{Aut}([,]\big|_{\Lambda^2 V})$$

and we get $B \circ B^{-1} = I_p$. So $(A^{-1} \circ A', I_p) \in \mathrm{Aut}_g([,])$ and by Theorem 1.12, $A = A'$. So (A,B) is determined by $A\big|_V$. Now suppose $(A,B),(A',B') \in \mathrm{Aut}_g([,])$ have the property that $AV = W$ and $A'V = W$. Then $(A\big|_V{}^{-1} \circ A'\big|_V, B^{-1} \circ B') \in \mathrm{Aut}_g([,]\big|_{\Lambda^2 V})$. So the collection of $(A,B) \in \mathrm{Aut}_g([,])$ such that $AV = W$ is contained in $\mathrm{Aut}_g([,]\big|_{\Lambda^2 V})$, which is finite. □

References

[CNS21] Chris Connell, Thang Nguyen, and Ralf Spatzier. Carnot metrics, dynamics and local rigidity, 2021. ArXiv.

[DT20] Enrico Le Donne and Francesca Tripaldi. A cornucopia of carnot groups in low dimensions, 2020. ArXiv.

[LDOW14] Enrico Le Donne, Alessandro Ottazzi, and Ben Warhurst. Ultrarigid tangents of sub-Riemannian nilpotent groups. *Ann. Inst. Fourier (Grenoble)*, 64(6):2265–2282, 2014.

[Pan89] Pierre Pansu. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)*, 129(1):1–60, 1989.