Ratios for Spanning Tree Polynomials of Covering Graphs

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Abstract

The spanning tree polynomial $T(\Gamma)$ of an undirected edge-weighted multigraph Γ is the sum of the weights of the spanning trees of Γ. For any covering graph $\tilde{\Gamma}$, we give an explicit formula for the ratio $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$, working off of an analogous result from Chepuri et al. [2] for directed graphs. We also prove the coefficients of $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$ are positive when Γ is a tree, contains only one vertex, when Γ is a collection of cycles joined together at one vertex (a *flower*), and when Γ is a spanning tree with some additional edges such that contracting all non-cyclic edges leaves a flower.

1 Introduction

The object of this paper is to further the investigation into the relationship between the arborescences of graphs and the arborescences of their covering graphs. The question of this relationship first arose in Galashin and Pylyavskyy's [3] study of R-systems. An R-system is a discrete dynamical system on an edge-weighted directed graph Δ with a state vector $(X_v)_{v\in\Delta}$ that evolves based on the edges in the graph. In particular, Galashin and Pylyavskyy showed for a directed graph Δ and a k-cover $\tilde{\Delta}$ of Δ that the ratio of arborescence polynomials $\frac{A_{\tilde{v}}(\tilde{\Delta})}{A_v(\Delta)}$ is independent of the choice of vertex v in Δ and corresponding lift $\tilde{v}$ in $\tilde{\Delta}$.

This invariance of choosing the root of the arborescences was the motivating factor for Chepuri et al. [2] to derive an explicit formula for the arborescence polynomial ratio in terms of the determinant of a certain matrix $[\mathcal{L}(\Delta)]_{\mathbb{Z}[E]}$.

Theorem 1.1 (Theorem 1.3 in [2]). *Let Δ be an edge-weighted directed multigraph, and let $\tilde{\Delta}$ be a k-fold cover of Δ. Then, for any vertex v of Δ and lift $\tilde{v}$ of v in $\tilde{\Delta}$, we have*

$$\frac{A_{\tilde{v}}(\tilde{\Delta})}{A_v(\Delta)} = \frac{1}{k}\det[\mathcal{L}(\Delta)]_{\mathbb{Z}[E]}.$$

Further, Chepuri et al. [2] conjectured the following.

Conjecture 1.2 (Conjecture 1.7 in [2]). *The ratio $\frac{A_{\tilde{v}}(\tilde{\Delta})}{A_v(\Delta)}$ has positive integer coefficients.*

Since considerable effort has been spent on this conjecture to no avail, we will study special cases of an analogous conjecture which is narrower in scope.

Conjecture 1.3. *For an undirected edge-weighted multigraph Γ and a k-cover $\tilde{\Gamma}$ of Γ, the ratio of the spanning tree polynomials $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$ has positive integer coefficients.*

While it is clear that Conjecture 1.2 and Conjecture 1.3 are similar, it is not immediately obvious why the latter is simpler. The reason for this is that directed graphs can be viewed as a special class of unidrected graphs, therefore studying undirected graphs not only restricts the type of graph we get, but also the types of covers of that graph. This restriction will become clearer in Section 3 where the relationship between directed and undirected graphs is discussed at length.

To begin our work on Conjecture 1.3 we derive a formula for undirected graphs using Theorem 1.1.

Theorem 1.4. *Let Γ be an undirected edge-weighted multigraph and let $\tilde{\Gamma}$ be a k-cover. Define matrices $n(k-1) \times n(k-1)$ matrices $\mathcal{A}, \mathcal{D}$, with rows and columns indexed by $v_1^2, ..., v_n^2, ..., v_1^k, ..., v_n^k$, such that for $\tilde{\Gamma}$, k-cover of undirected graph $\Gamma = (V, E, wt)$,*

$$\mathcal{A}[v_i^t, v_j^r] = \sum_{\tilde{E} \ni \tilde{e} = \{v_i^t, v_j^r\}} wt(\tilde{e}) - \sum_{E \ni \tilde{e} = \{v_j^r, v_i^1\}} wt(\tilde{e})$$

$$\mathcal{D}[v_i^t, v_j^r] = \delta_{ij}\delta_{tr} \cdot \sum_{\tilde{e} \in \tilde{E} \text{ adjacent to } v_i^t} wt(e)$$

Let $[\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]} = \mathcal{D} - \mathcal{A}$. Then $\frac{T(\tilde{\Gamma})}{T(\Gamma)} = [\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]}$.

This formula implies that the spanning tree ratio for undirected graphs has integer coeficients since we are taking a determinant. The main result of this paper proves Conjecture 1.3 for a broad class of undirected graphs called *extended flowers*, defined in Section 6.1.

Theorem 1.5. *If Γ is an extended flower with a k-cover $\tilde{\Gamma}$, then $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$ is a polynomial with positive integer coefficients.*

This result is meaningful because it sets a foundation for proving the original conjecture posed by Chepuri et al. [2], which in turn makes progress towards finding a combinatorial interpretation of the $[\mathcal{L}(\Delta)]_{\mathbb{Z}[E]}$ matrix.

The rest of this paper will proceed as follows. Section 2 will provide the necessary background and restate Chepuri et al's. [2] formula for $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$. In Section 3 we adapt this result for undirected graphs. The remainder of the paper focuses on proving the positivity conjecture in special cases of undirected graphs. In Section 4 we prove the conjecture for when the spanning tree polynomial of our graph is a monomial. In Section 5 we prove the conjecture for flower graphs. In Section 6 we prove our main result, positivity for extended flowers. Finally, in Section 7, we suggest next steps towards proving Conjecture 1.3. Additionally, we discuss how our results give some insight into certain cases of Conjecture 1.2 and pose related conjectures that we find interesting.

2 Background

2.1 Arborescences and Spanning Trees

Let $\Delta = (V, E, wt)$ be a directed, edge-weighted multigraph. That is, (V, E) is a directed multigraph and there is a weight function $wt : E \to \mathbb{R}_{>0}$ such that each edge is assigned a positive real number. Likewise, let $\Gamma = (V, E, wt)$ be an undiredcted edge-weighted multigraph.

Definition 2.1. An *arborescence* T_v of Δ rooted at v is a subgraph of Δ such that for each $w \in V$, there is a unique directed path from w to v. Equivalently, T_v is a connected subgraph of Δ such that the out-degree of each vertex other than v in T_v is one, the outdegree of v is zero.

Definition 2.2. The *weight of an arborescence* is the product of the weights of the edges in the arborescence. That is, $wt(A_v) = \prod_{e \in A_v} wt(e)$ where E_v is the set of edges in A_v. Further, if $\mathcal{A}_v$ is the set of all arborescences rooted at v of Δ then the *arborescence polynomial* of Δ rooted at v is

$$A_v(\Delta) = \sum_{T_v \in \mathcal{A}_v} wt(A_v).$$

Example 2.1. Let Δ be the directed graph in Figure 1. This graph has three arborescences rooted at v_4 (pictured in Figure 2). We can compute $A_{v_4}(\Delta) = abc + bde + bcd$.

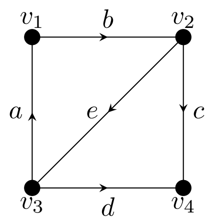

Figure 1: An edge-weighted directed multigraph Δ.

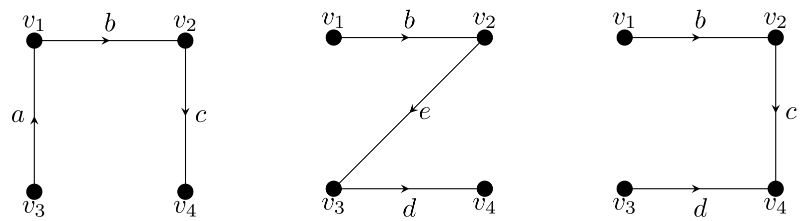

Figure 2: The arborescences of Δ rooted at v_4.

We can define analgous objects for undirected graphs. These will be the primary focus of this paper.

Definition 2.3. A spanning tree S of Γ is a subgraph of Γ where for any two vertices $v, w \in V$ there is a unique path between v and w in S. Equivalently, S is a connected subgraph of Γ with no cycles.

Definition 2.4. The *weight of a spanning tree* is the product of the weights of the edges in the spanning tree. That is, $wt(S) = \prod_{e\in S} wt(e)$. Further, if $\mathcal{S}$ is the set of all spanning trees of Γ then the *spanning tree polynomial* of Γ is

$$T(\Gamma) = \sum_{S\in\mathcal{S}} wt(S).$$

Example 2.2. Let Γ be the undirected graph in Figure 3. There are eight spanning trees of Γ (pictured in Figure 4). Thus, $T(\Gamma) = acd + abd + abc + bcd + ace + bde + ade + bce$.

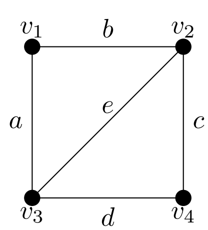

Figure 3: An edge-weighted undirected multigraph Γ

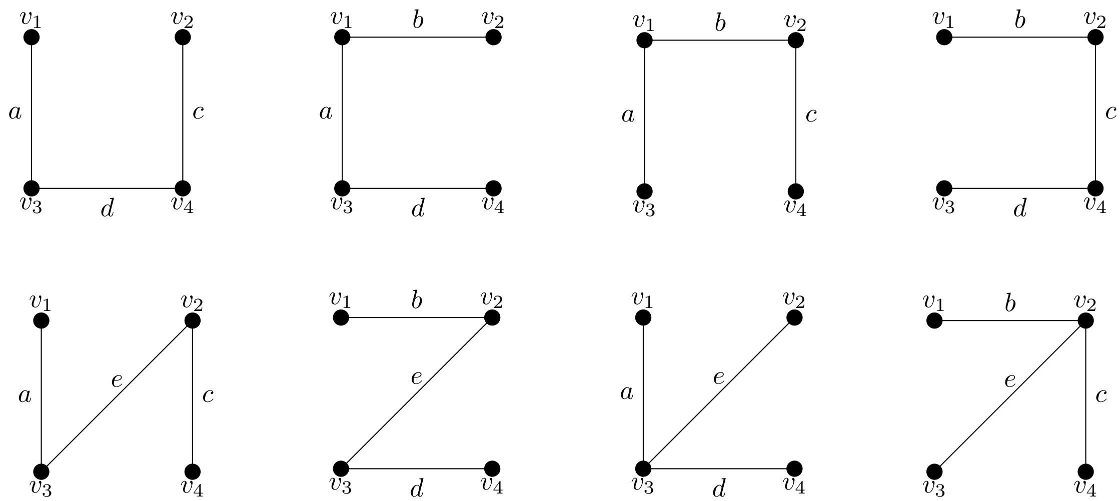

Figure 4: The spanning trees of Γ

While finding the spanning tree polynomial of small graphs by hand may seem straightforward, it is difficult for bigger graphs. As such, we will use the Matrix Tree Theorem as a tool to handle these computations.

Definition 2.5. For a directed graph Δ with vertices $v_1, ..., v_n$, the *Laplacian matrix* $L(\Delta)$ is the difference of the degree matrix and adjacency matrix of Δ. That is, $L(\Delta) = D - A$ where

$$D[v_i, v_i] = \sum_{e \in E_s(v_i)} wt(e)$$

$$A[v_i, v_j] = \sum_{e=(v_i, v_j)} wt(e)$$

Note that this matrix can be constructed similarly for undirected graphs by considering connections rather than directed connections.

Theorem 2.1 (Matrix Tree Theorem [1]). *For a directed graph Δ, the arborescence polynomial rooted at v_i is equal to the determinant of the i^{th} minor of $L(\Delta)$. Similarly, the spanning tree polynomial of Γ is equal to the determinant of any minor of $L(\Gamma)$.*

$$A_{v_i}(\Delta) = \det L_i^i(\Delta)$$

$$T(\Gamma) = \det L_1^1(\Gamma)$$

Example 2.3. We now use the matrix tree theorem to compute polynomials of both Δ and Γ, starting with the directed graph Δ.

$$D(\Delta) = \begin{bmatrix} b & 0 & 0 & 0 \\ 0 & e+c & 0 & 0 \\ 0 & 0 & a+d & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A(\Delta) = \begin{bmatrix} 0 & b & 0 & 0 \\ 0 & 0 & e & c \\ a & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we need to take the determinant of $L_4^4(\Delta)$ to find $A_{v_4}(\Delta)$, so we can compute $D_4^4(\Delta) - A_4^4(\Delta)$.

$$L_4^4(\Delta) = \begin{bmatrix} b & 0 & 0 \\ 0 & e+c & 0 \\ 0 & 0 & a+d \end{bmatrix} - \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & e \\ a & 0 & 0 \end{bmatrix} = \begin{bmatrix} b & -b & 0 \\ 0 & e+c & -e \\ -a & 0 & a+d \end{bmatrix}$$

Finally, $A_{v_4}(\Delta) = \det L_4^4(\Delta) = abc + bde + bcd$ which confirms our computation in Example 2.1. Similarly, for Γ:

$$D(\Gamma) = \begin{bmatrix} a+b & 0 & 0 & 0 \\ 0 & b+c+e & 0 & 0 \\ 0 & 0 & a+d+e & 0 \\ 0 & 0 & 0 & d+c \end{bmatrix}$$

$$A(\Gamma) = \begin{bmatrix} 0 & b & a & 0 \\ b & 0 & e & c \\ a & e & 0 & d \\ 0 & c & d & 0 \end{bmatrix}$$

We can then compute the determinant of any minor of $L(\Gamma)$. Without loss of generality, we choose $L_1^1(\Gamma)$.

$$L_1^1(\Gamma) = \begin{bmatrix} b+c+e & 0 & 0 \\ 0 & a+d+e & 0 \\ 0 & 0 & d+c \end{bmatrix} - \begin{bmatrix} 0 & e & c \\ e & 0 & d \\ c & d & 0 \end{bmatrix} = \begin{bmatrix} b+c+e & -e & -c \\ -e & a+d+e & -d \\ -c & -d & d+c \end{bmatrix}$$

Then, $T(\Gamma) = \det L_1^1(\Gamma) = acd + abd + abc + bcd + ace + bde + ade + bce$ which confirms the calculation in Example 2.2.

2.2 Covering Graphs

Definition 2.6. $\tilde{\Delta} = (\tilde{V}, \tilde{E}, \tilde{wt})$ is a *covering graph* of a directed graph Δ provided that there exists a function $\pi : \tilde{\Delta} \to \Delta$ such that

1. π maps vertices to vertices and edges to edges.
2. $|\pi^{-1}(v)| = |\pi^{-1}(e)| = k$ for all $v \in V$ and $e \in E$.
3. For all $\tilde{e} \in \tilde{E}$, $wt(\tilde{e}) = wt(\pi(\tilde{e}))$.
4. For all $\tilde{v} \in \tilde{V}$, $|E_s(\tilde{v})| = |E_s(\pi(\tilde{v}))|$ and $|E_t(\tilde{v})| = |E_t(\pi(\tilde{v}))|$

Where, following the notation used by Chepuri et al [2], $E_s(\tilde{v})$ denotes the set of edges direced away from $\tilde{v}$, and $E_t(\tilde{v})$ the set of edges directed towards $\tilde{v}$. Additionally, we can define the covering graph $\tilde{\Gamma}$ of an undirected graph Γ by changing (4) to be $|E(\tilde{v})| = |E(v)|$ where $E(v)$ is the set of edges adjacent to v.

Example 2.4. Let β be the directed graph in Figure 5. Then, $\tilde{\beta}$ in Figure 6 is a 2-cover of β.

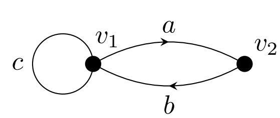

Figure 5: A directed graph β.

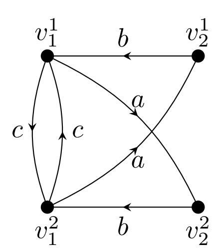

Figure 6: $\tilde{\beta}$, a 2-cover of β.

2.3 Arborescence Ratio

Now that we have defined the relevant terms, we state the formula for the arboresence ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ given by Chepuri et al. [2]; however, we must first construct the following matrix.

Definition 2.7. For some directed graph Γ with vertices $\{v_1, ..., v_n\}$ and a k-cover $\tilde{\Gamma}$ where each vertex v_i of Γ is lifted to $v_i^1, ..., v_i^k$ of $\tilde{\Gamma}$ we define the matrices $\mathcal{A}$ and $\mathcal{D}$ on the basis $v_1^2, ..., v_n^2, v_1^3, ..., v_n^3, ..., v_1^k, ..., v_n^k$ where

$$\mathcal{A}[v_i^t, v_j^r] = \sum_{e=(v_i^t, v_j^r)} wt(e) - \sum_{e=(v_i^t, v_j^1)} wt(e)$$

$$\mathcal{D}[v_i^t, v_i^t] = \sum_{e \in E_s(v_i^t)} wt(e)$$

for $1 < t, r \leq k$. We then define

$$[\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]} = \mathcal{D} - \mathcal{A}.$$

Theorem 1.1 (Theorem 1.3 in [2])**.** *Let Δ be an edge-weighted directed multigraph, and let $\tilde{\Delta}$ be a k-fold cover of Δ. Then, for any vertex v of Δ and lift $\tilde{v}$ of v in $\tilde{\Delta}$, we have*

$$\frac{A_{\tilde{v}}(\tilde{\Delta})}{A_v(\Delta)} = \frac{1}{k} \det[\mathcal{L}(\Delta)]_{\mathbb{Z}[E]}.$$

Example 2.5. With Theorem 1.1, we can compute the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ for any directed graph Δ and k-cover $\tilde{\Delta}$. In particular, for β and $\tilde{\beta}$ in Example 2.4:

$$\mathcal{D} = \begin{array}{c c} & \begin{array}{cc} 1^2 & 2^2 \end{array} \\ \begin{array}{c} 1^2 \\ 2^2 \end{array} & \begin{bmatrix} a+c & 0 \\ 0 & b \end{bmatrix} \end{array}$$

$$\mathcal{A} = \begin{array}{c c} & \begin{array}{cc} 1^2 & 2^2 \end{array} \\ \begin{array}{c} 1^2 \\ 2^2 \end{array} & \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \end{array} - \begin{array}{c c} & \begin{array}{cc} 1^2 & 2^2 \end{array} \\ \begin{array}{c} 1^2 \\ 2^2 \end{array} & \begin{bmatrix} c & a \\ 0 & 0 \end{bmatrix} \end{array}$$

Then, $[\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]} = \mathcal{D} - \mathcal{A}$, so:

$$\frac{A_{v_2^2}(\tilde{\beta})}{A_{v_2}(\beta)} = \frac{1}{2} \det \begin{bmatrix} a+2c & a \\ -b & b \end{bmatrix} = ab + bc.$$

3 Explicit Formula for the Ratio of Spanning Tree Polynomials

Let $\Gamma = (V, E, wt)$ be an undirected edge-weighted graph with vertices $v_1, ...v_n$, and $\tilde{\Gamma}$ be a k-cover of Γ where vertex v_i of Γ is lifted to $v_i^1, ..., v_i^k$ in $\tilde{\Gamma}$. For Γ, define the directed graph $\Gamma' = (V', E', wt')$ as follows: V' simply contains a copy of each vertex in Γ, $E' = \{(v_i, v_j), (v_j, v_i) | \{v_i, v_j\} \in E\}$; that is, if $\{v_i, v_j\}$ is in E, then both (v_i, v_j) and (v_j, v_i) are in E'. Let $wt' : E' \to \mathbb{R}$ such that $wt'((v_i, v_j)) = wt'((v_j, v_i)) = wt(\{v_i, v_j\})$. Define $(\tilde{\Gamma})'$ similarly for $\tilde{\Gamma}$.

Lemma 3.1. *$(\tilde{\Gamma})'$ is a k-cover of Γ'.*

Proof. Since $\tilde{\Gamma}$ is a $k-$cover of Γ, there exists a map $\pi : \tilde{\Gamma} \to \Gamma$ satisfying Definition 2.6. Let $\pi' : (\tilde{\Gamma})' \to \Gamma'$ be the map such that for vertices $v \in \tilde{\Gamma}'$, $\pi'(v) = \pi(v)$ and for edges $e \in \tilde{\Gamma}'$, $\pi'(e = (v_a^b, v_c^d)) = (\pi(v_a^b), \pi(v_c^d))$.

Since π' acts on the same vertices as π, condition 2 is satisfied for vertices. Since $\tilde{\Gamma}'$ contains k copies of each $e = (v_a, v_c)$ and only lifts of e get mapped to e, indeed $|\pi'^{-1}(e)| = k$, satisfying condition 2 for edges.

The weights of the edges $(v_i^j, v_k^l), (v_k^l, v_i^j)$ in the cover are the same as the corresponding weights $\{v_i^j, v_k^l\}$ in $\tilde{\Gamma}$, and likewise the weights of edges $\{v, w\}$ in Γ, and so we have satisfied condition 3.

Each vertex v_i^j in $(\tilde{\Gamma})'$ has the same incoming and outgoing degree as the corresponding v_i in Γ': We first note that the incoming and outgoing degree of each vertex in Γ' and $(\tilde{\Gamma})'$ are the same as the degree of the corresponding vertex in Γ and $\tilde{\Gamma}$ respectively. By construction each vertex in $\tilde{\Gamma}$ has the same degree as its corresponding vertex in Γ. Then since each vertex in Γ has the same degree as the corresponding vertex in Γ', each vertex v_i^j in $(\tilde{\Gamma})'$ has the same incoming and outgoing degree as the corresponding v_i in Γ'. Hence π is a local homeomorphism and we have satisfied condition 4.
Hence, we can conclude that $(\tilde{\Gamma})'$ is by definition a k-cover of Γ'. $\square$

Lemma 3.2. *The Laplacian matrices of Γ and $\tilde{\Gamma}$ are the same as those of Γ' and $(\tilde{\Gamma})'$ respectively.*

Proof. The definition of Γ' is equivalent to simply creating the Laplacian matrix for Γ, and then reading it as if it were the Laplacian matrix of an directed graph. This directed graph is exactly Γ'. $\square$

We can now use this result to derive a formula for the ratio $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$.

Theorem 1.4. *Let Γ be an undirected edge-weighted multigraph and let $\tilde{\Gamma}$ be a k-cover. Define matrices $n(k-1) \times n(k-1)$ matrices $\mathcal{A}, \mathcal{D}$, with rows and columns indexed by $v_1^2, ..., v_n^2, ..., v_1^k, ..., v_n^k$, such that for $\tilde{\Gamma}$, k-cover of undirected graph $\Gamma = (V, E, wt)$,*

$$\mathcal{A}[v_i^t, v_j^r] = \sum_{\tilde{E} \ni \tilde{e} = \{v_i^t, v_j^r\}} wt(\tilde{e}) - \sum_{E \ni \tilde{e} = \{v_j^r, v_i^1\}} wt(\tilde{e})$$

$$\mathcal{D}[v_i^t, v_j^r] = \delta_{ij}\delta_{tr} \cdot \sum_{\tilde{e} \in \tilde{E} \text{ adjacent to } v_i^t} wt(e)$$

Let $[\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]} = \mathcal{D} - \mathcal{A}$. Then $\frac{T(\tilde{\Gamma})}{T(\Gamma)} = [\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]}$.

Proof. Since the Laplacian matrices of Γ and $\tilde{\Gamma}$ are the same as those of Γ' and $\tilde{(\Gamma)}'$ respectively by Lemmas 3.1 and 3.2, the ratio $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$ is equal to $\frac{A_v((\tilde{\Gamma})')}{A_v(\Gamma')}$. Rewriting $[\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]}$ in terms of the edges from Γ rather than Γ', we obtain our result. $\square$

4 Positivity for Simple Cases

To begin our investigation of positivity, we first look at a simple case. In particular we study graphs with spanning tree polynomials that have only one term. The purpose of this is twofold. First, it is illustrative of an argument style used throughout the paper; namely, showing that the spanning tree polynomial of Γ can be factored out of the spanning tree polynomial of $\tilde{\Gamma}$. Second, two important special cases follow directly in Corollary 4.2 and Corollary 4.3.

Lemma 4.1. *Let $\Gamma = (V, E, wt)$ be a graph whose spanning tree polynomial is a monomial. Then, for any k-cover $\tilde{\Gamma}$ of Γ, the spanning tree ratio $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$ has positive integer coefficients.*

Proof. We first note that $T(\Gamma)$ divides $T(\tilde{\Gamma})$ by our Theorem 1.4. Further, since $T(\Gamma)$ is a monomial, it divides each term of $T(\tilde{\Gamma})$. In particular, the coefficient of $T(\Gamma)$ divides each coefficient of $T(\tilde{\Gamma})$. Since the coefficients of $T(\Gamma)$ and $T(\tilde{\Gamma})$ are positive integers by construction, we can then conclude the ratio of spanning tree polynomials $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$ has positive integer coefficients. $\square$

In particular, we get the following corollaries.

Corollary 4.2. *If Γ is a graph with exactly one vertex, then for any k-cover $\tilde{\Gamma}$ of Γ, $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$ has positive integer coefficients.*

Corollary 4.3. *If Γ is a spanning tree, then for any k-cover $\tilde{\Gamma}$ of Γ, $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$ has positive integer coefficients.*

5 Positivity For Flowers

Definition 5.1. A *flower* is a graph which is a union of cycles centered at one vertex, and a *petal* is a simple cycle in a flower.

Example 5.1. Consider the graph in Figure 7. This is a flower with two petals adjoined at v_2, both of which are 2-cycles.

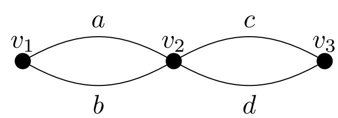

Figure 7: A flower Γ composed of two 2-cycles with central vertex v_2 .

Definition 5.2. For a flower Γ with central vertex v, and a k-cover $\tilde{\Gamma}$ of Γ, a *quasi-cycle* of $\tilde{\Gamma}$ is a subgraph constructed by starting at a lift of v and traversing adjacent vertices until a, possibly different, lift of v is reached. A quasi-cycle Q always projects down to a petal of Γ, and each quasi cycle contains the same number of edges as the petal it is lifted from.

Example 5.2. The graph $\tilde{\Gamma}$ in Figure 8 is a 2-cover of Γ in Figure 7. Observe that $\tilde{\Gamma}$ has four quasi-cycles, two that correspond to each petal of Γ. In particular, note that the edges of one of these quasi-cycles are highlighted in red.

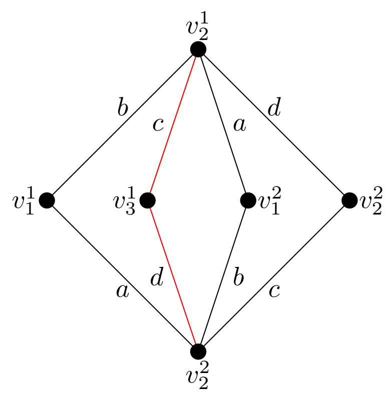

Figure 8: $\tilde{\Gamma}$, a 2-cover of the flower Γ

Lemma 5.1. *Let Γ be a flower. For any petal C of Γ and any spanning tree T of $\tilde{\Gamma}$, there exists a quasi-cycle lifted from C that is missing eaxtly one edge in T. Further, no quasi-cycles can be missing more than one edge in T.*

Proof. Suppose for contradiction that for some spanning tree T of $\tilde{\Gamma}$ and some cycle C of Γ there exists no quasi-cycle lifted from C which is missing an edge. Then, since these quasi-cycles together contain all lifts of the vertices and edges in C, they themselves form a cycle or a collection of cycles, so our spanning tree has T has some cycle, which is a contradiction. Therefore some quasi-cycle lifted from C must be missing at least one edge.

Now suppose for contradiction that a quasi-cycle Q with vertices $v_1, ..., v_n$ and edges $e_1 = \{v_1, v_2\}, ...e_{n-1} = \{v_{n-1}, v_n\}$ was missing at least edges e_i, e_j in a spanning tree of T. Then since $v_1, ...v_{j+1}$ each had degree two in T, this segment is now isolated, and so we do not have a spanning tree, which is a contradiction.

Therefore there must exist a quasi-cycle in T lifted from C which is missing exactly one edge, and no quasi-cycles in T missing more than one edge. ∎

Lemma 5.2. *If we have an edge missing in a spanning tree T of $\tilde{\Gamma}$, we can swap it any edge in the same quasi-cycle and still have spanning tree of $\tilde{\Gamma}$.*

Proof. Let $e = \{u, w\}$ be the missing edge of the quasi cycle. If e is not the only edge in the quasi cycle, then at least one of u, w is not a lift of the central vertex v. Suppose w is not a lift of v, then the degree of w is 1, since it was 2 in $\tilde{\Gamma}$ but we removed an edge adjacent to w in producing T.

Now consider the other edge out of w, $e' = \{w, x\}$ (it may be that $x = v$, but clearly $w \neq v$). Since the degree of w is 1, the degree of x in T must be greater than 1, else e' would be isolated and we would not have a spanning tree. Therefore, we can remove e' without leaving x isolated, and replace with e' with e such that w is not isolated. Since we have no isolated vertices and the appropriate number of edges, we must still have a spanning tree. Repeating this process, we may swap any two edges in the quasi cycle and still have a spanning tree. □

Theorem 5.3. *Let Γ be a flower and $\tilde{\Gamma}$ be a k-cover, then the ratio $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$ has positive coeffecients.*

Proof. Let Γ be a flower with P petals denoted $C_1, ..., C_i, ...C_P$ each of length n_i with edges $\{e_{i,1}, ..., e_{i,n_i}\}$ where $e_{i,1}$ and e_{i,n_i} adjoin to the central vertex v. Let $\tilde{\Gamma}$ be a k-cover of Γ. Let $\mathcal{Q}_i$ be the set of quasi-cycles which are lifts of C_i and let $Q_j \in \mathcal{Q}_i$ be the quasi cycle containing the j^{th} lift of $e_{i,1}$.

For every spanning tree of $\tilde{\Gamma}$, there exists a $Q_j \in \mathcal{Q}_i$ which is missing an edge by Lemma 5.1. Thus for each i we can partition the spanning trees of of $\tilde{\Gamma}$ into sets $\tilde{\Gamma}^i_j$ where j is minimal for which $Q_j \in \mathcal{Q}_i$ is missing an edge

We define a function $g_{j,\{e_{i,a},e_{i,b}\}} : \tilde{\Gamma}^i_j \to \tilde{\Gamma}^i_j$. such that $g_{j,\{e_{i,a},e_{i,b}\}}(T \in \tilde{\Gamma}^i_j)$ acts on T by replacing $e_{i,a}$ and removing $e_{i,b}$ or visa versa when T is missing a lift of $e_{i,a}$ or $e_{i,b}$, and $g_{j,\{e_{i,a},e_{i,b}\}}(T) = T$ otherwise. By Lemma 5.2, $g_{j,\{e_{i,a},e_{i,b}\}}(T)$ will always be a spanning tree, and clearly if $T \in \tilde{\Gamma}^i_j$, then $g_{j,\{e_{i,a},e_{i,b}\}}(T) \in \tilde{\Gamma}^i_j$.

Assume T is not missing a lift of $e_{i,a}$ or $e_{i,b}$, then $g_{j,(e_{i,a},e_{i,b})}(T) = g^2_{j,(e_{i,a},e_{i,b})}(T) = T$. Now, without loss of generality, assume T is missing a lift of $e_{i,a}$ in Q_j, and let $g_{j,\{e_{i,a},e_{i,b}\}}(T) = T'$. Then T and T' differ only in that the lift $e_{i,b}$ is missing from T' but the lift of $e_{i,a}$ is not. We see then that $g_{j,\{e_{i,a},e_{i,b}\}}(T') = T$, since $g_{j,\{e_{i,a},e_{i,b}\}}$ replaces the lift of $e_{j,b}$ and removes $e_{i,a}$. Therefore $g_{i,\{e_{i,a},e_{i,b}\}}$ is an involution. It follows that $g_{i,\{e_{i,a},e_{i,b}\}}$ is bijective.

For each $\tilde{\Gamma}^i_j$ of $\tilde{\Gamma}$, we have

$$\sum_{T \in \tilde{\Gamma}^i_j} wt(T) = \sum_{T \in \tilde{\Gamma}^i_j} wt(g_{j,e_{i,a},e_{i,b}}(T)) \tag{1}$$

since $g_{j,\{e_{i,a},e_{i,b}\}}$ is a bijection. For ease of reading we will refer to $g_{j,\{e_{i,a},e_{i,b}\}}$ simply as g. We can then say $\tilde{\Gamma}^i_j = \tilde{\Gamma}^i_j/e_{i,a} \cup \tilde{\Gamma}^i_j/e_{i,b} \cup \tilde{\Gamma}^i_j*$, where $\tilde{\Gamma}^i_j/e_{i,a}$ (resp. $\tilde{\Gamma}^i_j/e_{i,b}$) contains spanning trees in $\tilde{\Gamma}^i_j$ for which Q_j does not contain a lift of $e_{i,a}$ (resp. $e_{i,b}$) and $\tilde{\Gamma}^i_j*$ contains spanning trees in $\tilde{\Gamma}^i_j$ for which Q_j contains lifts of both $e_{i,a}$ and $e_{i,b}$. Therefore we may rewrite Equation 1 as

$$\sum_{T \in \tilde{\Gamma}^i_j/e_{i,a}} wt(T) + \sum_{T \in \tilde{\Gamma}^i_j/e_{i,b}} wt(T) + \sum_{T \in \tilde{\Gamma}^i_j*} wt(T) = \sum_{T \in \tilde{\Gamma}^i_j/e_{i,a}} wt(g(T)) + \sum_{T \in \tilde{\Gamma}^i_j/e_{i,b}} wt(g(T)) + \sum_{T \in \tilde{\Gamma}^i_j*} wt(g(T)) \tag{2}$$

But since $g(T) = T$ for $T \in \tilde{\Gamma}^i_j*$, we may cancel out our third terms. Looking at the left hand side, we may write $\sum_{T \in \tilde{\Gamma}^i_j/e_{i,a}} wt(T)$ as $\prod_{l \in [n_i] \setminus \{a\}} wt(e_{i,l}) \sum_{T \in \tilde{\Gamma}^i_j/e_{i,a}} wt(T - Q_j)$, where $T - Q_j$ denotes deleting all edges in Q_j from the spanning tree T. Each tree in $\tilde{\Gamma}^i_j/e_{i,a}$ contains the same copy of $Q_j - e_{i,a}$, so we are simply factoring this out.

Looking at the right hand side of the equation, we note that for $T \in \tilde{\Gamma}^i_j/e_{i,a}$, g just swaps in $e_{i,a}$ and removed $e_{i,b}$ and does not affect edges outside of Q_j, so for the same reasoning we may write $\sum_{T \in \tilde{\Gamma}^i_j/e_{i,a}} wt(g(T))$

as $\prod_{l\in[n_i]\setminus\{b\}} wt(e_{i,l}) \sum_{T\in\tilde{\Gamma}^i_j/e_{i,a}} wt(T-Q_j)$. Doing similarly for the second terms on each side of the equal sign and cancelling out the third term, we obtain the following equation.

$$\begin{aligned}
&\prod_{l\in[n_i]\setminus\{a\}} wt(e_{i,l}) \sum_{T\in\tilde{\Gamma}^i_j/e_{i,a}} wt(T-Q_j) + \prod_{l\in[n_i]\setminus\{b\}} wt(e_{i,l}) \sum_{T\in\tilde{\Gamma}^i_j/e_{i,b}} wt(T-Q_j) \\
=&\prod_{l\in[n_i]\setminus\{b\}} wt(e_{i,l}) \sum_{T\in\tilde{\Gamma}^i_j/e_{i,a}} wt(T-Q_j) + \prod_{l\in[n_i]\setminus\{a\}} wt(e_{i,l}) \sum_{T\in\tilde{\Gamma}^i_j/e_{i,b}} wt(T-Q_j)
\end{aligned} \tag{3}$$

From Equation 3 we may conclude that

$$\sum_{T\in\tilde{\Gamma}^i_j/e_{i,a}} wt(T-Q_j) = \sum_{T\in\tilde{\Gamma}^i_j/e_{i,b}} wt(T-Q_j) \tag{4}$$

Equation 4 is saying that spanning trees of $\tilde{\Gamma}$ in $\tilde{\Gamma}^i_j$ missing a lift of $e_{i,a}$ are in bijection with those missing a lift of $e_{i,b}$. Since a and b were arbitrary, the spanning trees of $\tilde{\Gamma}$ in $\tilde{\Gamma}^i_j$ missing a lift of $e_{i,x}$ are in bijection with those missing a lift of $e_{i,y}$ for any x and y. We will denote the terms of Equation 4 as a constant α^i_j. Now we see

$$\begin{aligned}
\sum_{T\in\tilde{\Gamma}^i_j} wt(T) &= \sum_{l\in[n_i]} \sum_{T\in\tilde{\Gamma}^i_j/e_{i,l}} wt(T) \\
&= \sum_{l\in[n_i]} \prod_{l\in[n_i]\setminus\{l\}} wt(e_{i,l}) \sum_{T\in\tilde{\Gamma}^i_j/e_{i,l}} wt(T-Q_j) \\
&= \alpha^i_j \sum_{l\in[n_i]} \prod_{l\in[n_i]\setminus\{l\}} wt(e_{i,l})
\end{aligned} \tag{5}$$

Now, fixing i, we have from Equation 5 that

$$\begin{aligned}
\sum_{j\in[k]} \sum_{T\in\tilde{\Gamma}^i_j} wt(T) &= \sum_{j\in[k]} \alpha^i_j \sum_{l\in[n_i]} \prod_{l\in[n_i]\setminus\{l\}} wt(e_{i,l}) \\
&= \Big(\sum_{l\in[n_i]} \prod_{l\in[n_i]\setminus\{l\}} wt(e_{i,l}) \Big) \Big(\sum_{j\in[k]} \alpha^i_j \Big)
\end{aligned} \tag{6}$$

Since every spanning tree of $\tilde{\Gamma}$ must be missing some edge from some $Q_j \in \mathcal{Q}_i$ for each $i \in [P]$, all spanning trees exist in some $\tilde{\Gamma}^i_j$ for fixed i, so we may factor out $\sum_{l\in[n_i]} \prod_{l\in[n_i]\setminus\{l\}} wt(e_{i,l})$ from $T(\tilde{\Gamma})$, the spanning tree polynomial of $\tilde{\Gamma}$. Doing so for each $i \in [P]$ results in factoring out

$$\prod_{i\in[P]} \sum_{l\in[n_i]} \prod_{l\in[n_i]\setminus\{l\}} wt(e_{i,l}) \tag{7}$$

which is precisely the spanning tree polynomial of Γ, $T(\Gamma)$. Since we are essentially factoring out positive monomials from individual terms of $T(\tilde{\Gamma})$ which themselves are positive monomials, we will be left with positive coefficients. Then in the ratio $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$, we will cancel out $T(\Gamma)$ and be left with positive coefficients. ☐

6 Extended Flowers

While the case of flower graphs may seem narrow, we can expand this result significantly with only slight modification to our argument. This modification allows us to prove the main theorem of the paper in Theorem 1.5.

Definition 6.1. An *extended flower* is an undirected graph that is a flower when all edges that are not in any cycles are contracted.

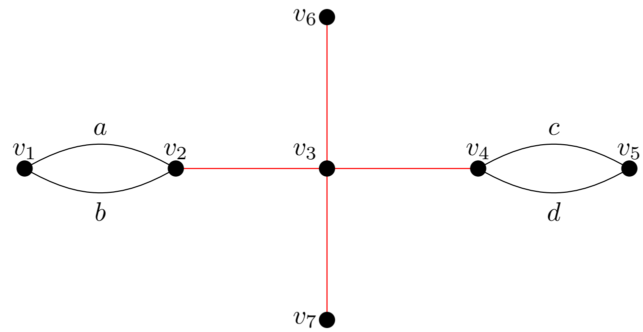

Figure 9: An extended flower Γ with trees highlighted in red.

Example 6.1. Notice that Γ in Figure 9 becomes the flower in Figure 7 when you contract the trees highlighted in red.

Lemma 6.1. *Let Γ be an undirected graph. Then all edges in Γ that are not in any cycles must be in every spanning tree of Γ.*

Proof. Let $e = \{v, w\}$ be an edge not in any cycle of Γ and let S be a spanning tree of Γ not including e. However, this would imply that v and w are disconnected in S which is a contradiction. $\square$

Corollary 6.2. *If e is an edge in Γ that is not in any cycles, then in a cover $\tilde{\Gamma}$ of Γ, all lifts of e are in each spanning tree of $\tilde{\Gamma}$.*

Proof. Let $\tilde{e}$ be a lift of e. Note that if $\tilde{e}$ is in a cycle of $\tilde{\Gamma}$, it would project to a cycle in Γ containing e under π. Thus $\tilde{e}$ is not in any cycle of $\tilde{\Gamma}$. By Lemma 6.1, $\tilde{e}$ is in every spanning tree of $\tilde{\Gamma}$. $\square$

Corollary 6.2 almost immediately lets us apply Theorem 5.3 to conclude our desired result.

Theorem 1.5. *If Γ is an extended flower with a k-cover $\tilde{\Gamma}$, then $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$ is a polynomial with positive integer coefficients.*

Proof. Let $M = \prod_{e \in E_m} wt(e)$ where E_m is the set of all edges not in cycles in Γ. If $\tilde{E}_m$ is the set of all lifts of edges in E_m, then $\prod_{e \in \tilde{E}_m} wt(e) = M^k$. Let Γ_f be Γ with all edges in E_m contracted. Likewise, let $\tilde{\Gamma}_f$ be $\tilde{\Gamma}$ with all edges in $\tilde{E}_m$ contracted. By definition, we note that Γ_f is a flower graph, and further that $\tilde{\Gamma}_f$ is a k-cover of Γ_f. We can then appeal to Theorem 5.3 to say $\frac{T(\tilde{\Gamma}_f)}{T(\Gamma_f)}$ is a polynomial with positive integer coefficients. Finally, since $T(\Gamma) = MT(\Gamma_f)$ and $T(\tilde{\Gamma}) = M^k T(\tilde{\Gamma}_f)$ we can conclude that $\frac{T(\tilde{\Gamma})}{T(\Gamma)}$ has positive integer coefficients, as desired. $\square$

7 Future Directions

We conclude the paper by presenting next steps for generalizing the positivity conjecture, as well as some alternative lines of inquiry that could yield interesting results.

7.1 Generalizing the Undirected Case

The natural next step is to prove the positivity conjecture for *chains*.

Definition 7.1. A *chain* is a graph which are a collection of simple cycles adjoined at possibly different vertices.

We did not make this step because our proof technique for flowers did not generalize; without a central vertex, we would need to redefine quasi-cycles to proceed. That being said, we expect similar techniques to work if the definition of a quasi-cycle was tweaked to not be defined in terms of the central vertex, but rather as a minimum set of edges that project to a certain cycle in Γ.

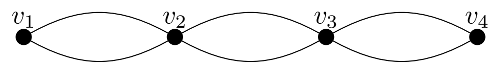

Figure 10: An example of a chain

After that, the final step is to consider graphs which contain edges that are in multiple cycles. We found investigations on this final step to be difficult as our approach was to find an explicit combinatorial formula for the spanning tree polynomial of such a graph. It is possible that a weaker approach, such as an argument similar to that of Section 5 will be more fruitful.

7.2 Implications for Directed Graphs

Note that by Lemma 3.2, we can view undirected graphs as a special type of directed graphs. Namely, an undirected graph Γ is equivalent to a directed graph Γ' with mutually connected vertices; that is, the existence of directed edge $\{v, w\}$ in Γ implies the existence of edges $(v, w), (w, v)$ in Γ' (see Section 3 for more detail). Since the $[\mathcal{L}]_{\mathbb{Z}[e]}$ matrices of Γ and Γ' are the same, the positivity conjecture will hold true for the directed analogues of the cases we have proved. Additionally, the proof technique used in Section 5 should suffice to show the positivity conjecture holds for directed graphs which are simple cycles sharing one common vertex.

7.3 Alternate Approach to the Directed Case

In addition to the more direct approaches for proving this conjecture, we present an alternative approach based on a conjecture posed by Chepuri et al. in [2].

Conjecture 7.1 (Based on Conjecture 5.6. in [2])**.** *Let $\Gamma = (V, E)$ be an undirected graph. Let $\tilde{\Gamma}$ be a random k-cover of Γ, assuming uniform distribution. Then the expected value of the spanning tree polynomial ratio is*

$$\mathbb{E}\left[\frac{T(\tilde{\Gamma})}{T(\Gamma)}\right] = \frac{1}{k}\prod_{v \in V}\left(\sum_{\alpha \in E(v)} wt(\alpha)\right)^{k-1}.$$

Chepuri et al. [2] note that finding an expected value for this ratio as in Corollary 7.1 will possibly give way to a ‘pigeon-hole’ like argument for positivity. That is, assuming for contradiction that the spanning tree ratio has negative coefficients may cause the expected value to be smaller than what would match Conjecture 7.1.

7.4 Other Statistics on Graphs

Another natural direction is to ask whether there are other statistics on graphs for which there is a nice relationship between a graphs and their covers, such as the *Tutte polynomial.*

Definition 7.2. Define to be abridge of graph G if the graph $G - e$ has strictly more connected components than G. Define the *Tutte polynomial* $T_G(x, y)$ for a graph G recursively by $T_G(x, y) = 1$ if G has no edges, and otherwise if e is an edge of G, then

$$T_G(x,y) = \begin{cases} xT_{G-e}(x,y) \text{ if } e \text{ is a bridge} \\ yT_{G-e}(x,y) \text{ if } e \text{ is a loop} \\ T_{G-e}(x,y) + T_{G/e}(x,y) \text{ if } e \text{ is neither a loop nor a bridge} \end{cases}$$

Here $T_{G-e}(x, y)$ and $T_{G/e}(x, y)$ are graphs obtained by deleting edge e and contracting along e respectively.

When $x = y = 1$, the Tutte polynomial counts spanning trees, and so for graph G and cover $\tilde{G}$, $T_G(1,1)$ divides $T_{\tilde{G}}(1,1)$. Further, Verma [4] shows that if G is a flower and $\tilde{G}$ is a 2-cover of G, then $T_G(x,1)$ divides $T_{\tilde{G}}(x,1)$. However, it is still an open question to find exactly what conditions x and y result in $T_G(x,y)$ dividing $T_{\tilde{G}}(x,y)$ in general.

Another line of investigation is to ask whether the *permanent*, the unsigned determinant, of the Laplacian matrices of a graph G divides that of the cover $\tilde{G}$, and if so what the quotient is. Rather than counting spanning trees, the determinant counts the *vertex covers* of a graph.

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