THE RELATIVISTIC VLASOV-MAXWELL SYSTEM IN TWO SPACE DIMENSIONS

ADVIKA RAJAPAKSE

Abstract. The global existence and smoothness of the 3-dimensional relativistic Vlasov-Maxwell system, a longstanding problem in analysis, has not yet been proved without invoking further assumptions. Glassey-Schaeffer [2], [4] proved that in two spatial dimensions, there exist unique global in time classical solutions, given regular initial data with compact support in the momentum variable. Luk-Strain [5] proved a more general case, instead requiring that the initial data in the momentum variable has polynomial decay. In this paper, we consider the case where there are two spatial dimensions and the initial data is symmetric under rotations. Our results work towards a bound of the support of particle velocities for large time and large radius.

Contents

1. Introduction

A plasma is an ionized gas consisting of electrons with charge $-e$ and positive ions of charge $+Ze$ ($Z \in \mathbb{N}$). We assume that our plasma is at a high temperature, and collisions among particles do not significantly affect the plasma's behavior. This scenario is not particularly unusual, so our assumption can often be applied to phenomena in the physical world.

We consider the relativistic case, where the distribution of our particles obeys the laws of special relativity. We assume that we have several species of particles with masses m_α and charges e_α, $1 \leq \alpha \leq N$. We define the relativistic velocity

$$\hat{v}_\alpha = \frac{v_\alpha}{\sqrt{(m_\alpha)^2 + v_\alpha^2/c^2}}$$

where c is the speed of light. We immediately see that $|v_\alpha| < c$, which agrees with our understanding of special relativity. Indeed, no particle with mass can reach

Date: DEADLINES: Draft AUGUST 17 and Final version AUGUST 29, 2020.

the speed of light and our plasma is certainly not an exception to this rule. The particle densities $f_\alpha(t,x,v)$ satisfy

$$\begin{gathered}\partial_t f_\alpha + \hat{v}_\alpha \cdot \nabla_x f_\alpha + (E + \frac{\hat{v}_\alpha}{c} \times B) \cdot \nabla_v f_\alpha = 0 \\ \partial_t E = \nabla_x \times B - j, \qquad \partial_t B = -\nabla_x \times E, \\ \nabla_x \cdot E = \rho, \qquad \nabla_x \times B = 0\end{gathered}$$

where the charge density ρ is defined by

$$\rho(t,x) := 4\pi \int_{\mathbb{R}^3} \sum_\alpha e_\alpha f_\alpha(t,x,p)\ dv$$

and the current is defined by

$$j(t,x) := 4\pi \int_{R^3} \hat{v}_\alpha e_\alpha f_\alpha(t,x,p)\ dv.$$

In these equations, E represents the electric field and B represents the magnetic field. We also assume that we have initial data $f_\alpha(0,x,v) = f_{\alpha 0}(x,v) \in C_0^1(\mathbb{R}^3 \times \mathbb{R}^3)$, $E(0,x) = E_0(x) \in C^2(\mathbb{R}^3)$ and $B(0,x) = B_0(x) \in C^2(\mathbb{R}^3)$, satisfying $\nabla \cdot E_0 = \rho_0$, $\nabla \cdot B_0 = 0$, and $\int \rho_0\ dx = 0$. Given this initial-value problem, Glassey [3] proved the following result:

Theorem 1.1. *Let $0 \le f_{\alpha 0} \in C_c^1$ satisfy the above constraints. Suppose as well that for all $f(t,x,v)$ defined for $0 \le t \le T$, $T > 0$, satisfying the above constraints, there exists a continuous function $\beta(t)$ satisfying*

$$f_\alpha(t,x,v) = 0 \text{ for all } x \in \mathbb{R}^3,\ 1 \le \alpha \le N,\ v > \beta(t).$$

Then there exists a unique C^1 solution (f_α, E, B) for all t.

This theorem motivates the problem of finding functions such as $\beta(t)$ that bound the support of f_α in v. While such a bound has never been proved in three dimensions, researchers have made efforts to tackle lower dimensional analogues to this problem. In particular, Glassey-Schaeffer-Pankavich [1] study the relativistic Vlasov-Maxwell system in one spatial dimension and two velocity dimensions:

$$\begin{aligned}\partial_t f_\alpha + \hat{v}_{\alpha 1}\partial_x f_\alpha + e_\alpha(E_1 + \hat{v}_{\alpha 2} B)\partial_{v_1} f_\alpha& \\ + e_\alpha(E_2 - \hat{v}_{\alpha 1} B)\partial_{v_2} f_\alpha = 0& \\ \rho(t,x) = \int \sum_\alpha e_\alpha f_\alpha(t,x,p)\ dv& \\ j(t,x) = \int \sum_\alpha e_\alpha f_\alpha(t,x,v)\hat{v}_\alpha\ dv& \\ E_1(t,x) = \frac{1}{2}\int_{-\infty}^x \rho(t,y)\ dy - \frac{1}{2}\int_x^\infty \rho(t,y)\ dy& \\ \partial_t E_2 + \partial_x B = -j_2& \\ \partial_t B + \partial_x E_2 = 0&\end{aligned} \tag{1.2}$$

for $1 \le \alpha \le N$. Here, $t \ge 0$ is time, x is the position component, and $v = (v_1, v_2) \in \mathbb{R}^2$ is the velocity. Similar to the definition in three dimensions, the relativistic velocity is given by

$$\hat{v}_\alpha = \frac{v_\alpha}{\sqrt{(m_\alpha)^2 + v_\alpha^2}}.$$

Also similar are the initial data we are given

$$\begin{aligned} f_\alpha(0,x,v) &= f_{\alpha 0}(x,v) \geq 0 \\ E_2(0,x) &= E_0(x) \\ B(0,x) &= B_0(x). \end{aligned}$$

with $f_{\alpha 0} \in C_c^1(\mathbb{R}^3)$, E_2, $B \in C_c^1(\mathbb{R})$. The most glaring difference between (1.2) and the 3-dimensional Vlasov-Maxwell system is that the speed of light c has been normalized to 1 to ease computation. In their paper, Glassey-Pankavich-Schaeffer study two cases. The first is the neutral case, where they assume

$$\int \int e_\alpha f_{\alpha 0} \; dv \; dx = 0.$$

The second is the monocharge case, where they work under the assumption that $N = 1$, so there is only one α value. In this case, we denote $f = f_\alpha = f_1$. The following theorems that bound the components of v are due to the work of Glassey-Schaeffer-Pankevich:

Theorem 1.3 (Glassey-Pankavich-Schaeffer [1])**.** *In the neutral case, there is a positive constant C such that*

$$|v_2| \leq C + C\sqrt{t - |x| + C_0}$$

on the support of f_α for every α. In the monocharge case, there is a positive constant C such that

$$|v_2| \leq C + C\sqrt{(t + C_0)^2 - x^2}$$

on the support of f.

Theorem 1.4 (Glassey-Pankavich-Schaeffer [1])**.** *In the neutral case, there exists a constant C such that*

$$|v_1| \leq C + Ct^{\frac{1}{2}}(t - |x| + 2C_0)^{\frac{1}{4}}$$

on the support of f_α for every α.

Our paper focuses on a similar goal, bounding the velocity of particlies in a low-dimensional analogue of the Vlasov-Maxwell system. We mainly consider the case where there is only one particle species and the plasma is confined within two dimensions in the spatial and momentum variables. We must also note that there exists the case where there are two spatial dimensions, but the momenta are allowed to have values in $\mathbb{R}^3$. The latter scenario can be thought of as identical to the three-dimensional Vlasov-Maxwell system, but with the initial data having translational symmetry along the x_3 axis. We will refer to the first case as the two-dimensional (2D) case and the second case as the two-and-one-half dimensional ($2\frac{1}{2}$D) case. With these two cases in mind, we define the particle density $f : \mathbb{R}_t \times \mathbb{R}_x^2 \times \mathbb{R}_p^{d_p} \to \mathbb{R}^+$ as a function of time $t \in \mathbb{R}$, position $x \in \mathbb{R}^2$, and momentum $p \in \mathbb{R}^{d_p}$

The electromagnetic fields used in the Relativistic Vlasov-Maxwell system are denoted E and B. When $d_p = 2$, we define $E : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, $B : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, where the first index is in the time variable and the second index is in the space variable. The images of these functions are written in the form

$$E = (E^1(t, x_1, x_2), E^2(t, x_1, x_2), 0), \qquad B = (0, 0, B(t, x_1, x_2)) \tag{1.5}$$

so that we can properly compute the curl and divergence of these fields as in the Maxwell equations (1.9), (1.10). When $d_p = 3$, we define $E : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^3$, $B : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^3$. The images of these functions are written in the form

$$E = (E^1(t, x_1, x_2), E^2(t, x_1, x_2), E^3(t, x_1, x_2)), \tag{1.6}$$

$$B = (B^1(t, x_1, x_2), B^2(t, x_1, x_2), B^3(t, x_1, x_2)). \tag{1.7}$$

In 2D and $2\frac{1}{2}D$, the relativistic Vlasov-Maxwell System can be written in the form

$$\partial_t f + \hat{p} \cdot \nabla_x f + (E + \hat{p} \times B) \cdot \nabla_p f = 0 \tag{1.8}$$

$$\partial_t E = \nabla_x \times B - j, \qquad \partial_t B = -\nabla_x \times E, \tag{1.9}$$

$$\nabla_x \cdot E = \rho, \qquad \nabla_x \times B = 0 \tag{1.10}$$

where the charge density ρ is defined by

$$\rho(t, x) := 4\pi \int_{\mathbb{R}^{d_p}} f(t, x, p)\ dp$$

and the current j is defined by

$$j_i(t, x) := 4\pi \int_{d_p} \hat{p}_i f(t, x, p)\ dp, \qquad i = 1, ..., d_p.$$

Here, we have defined our relativistic velocity

$$\hat{p} = \frac{p}{p_0}, \qquad \sqrt{1 + |p|^2}. \tag{1.11}$$

Notice how similar this system of equations shown above looks similar to the three dimensional case we initially introduced, minus the fact that we have normalized our constants $c = m_\alpha = e_\alpha = 1$ for simplicity. We also observe that we only account one species of plasma, hence the identity $f_\alpha = f_1 = f$. We are given initial data of the form

$$(f, E, B)|_{t=0} = (f_0, E_0, B_0)$$

for some (f_0, E_0, B_0) satisfying (1.10).

Because of the Vlasov equation (1.8), we see that given a solution (f, E, B), the particle density is constant along certain trajectories, which we often call characteristics. We formalize this useful statement in a theorem.

Theorem 1.12. *For all $(t, x, p) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{d_p}$, there exist trajectories of the form $(X(s), P(s)) = (X(s; t, x, p), P(s; t, x, p))$, which verify the differential equations:*

$$\frac{dX}{ds}(s) = \hat{P}(s), \qquad \frac{dP}{ds}(s) = E(s, X(s)) + \hat{P} \times B(s, X(s)) \tag{1.13}$$

$$P(t) = p, \qquad X(t) = x, \tag{1.14}$$

where we have defined $\hat{P} := \frac{P}{\sqrt{1+|P|^2}}$.

We saw from Theorem 1.4 and 1.3 that in one space dimension and two velocity dimensions, there exists a bound for the velocity of particles. We state a version of a theorem given by Luk-Strain that not only gives a bound for v in the 2D and 2.5D case, but proves its global existence and uniqueness. We do not state their theorem in its full generality, since we mainly consider the case where our data f_0 has compact support.

Theorem 1.15 (Luk-Strain [5]). *Suppose that we are given some initial data $(f_0(x,p), E_0(x), B_0(x))$ that not only satisfies the constraints (1.10), but obeys either (1.5) in the $2D$ case or (1.6) in the $2.5D$ case. Suppose also that $f_0 \in C_c^1(R)$ and the electromagnetic fields E_0, $B_0 \in H^3(\mathbb{R}_x^2)$ obey the bounds*

$$\sum_{0\le k\le 3} (\|\nabla_x^k E_0\|_{L_x^2} + \|\nabla_x^k B_0\|_{L_x^2}) < \infty. \tag{1.16}$$

Then there exists a unique global in time solution (f, E_0, B_0) to the relativvstic Vlasov-Maxwell system obeying the constraints (1.8)-(1.10). Furthermore, there exist positive constants C and k such that

$$\|E(t)\|_{L_x^\infty} + \|B(t)\|_{L_x^\infty} \le Ce^{Ct^k}.$$

Throughout our study of the $2D$ and 2.5D cases, we will always assume (1.16) for our data E_0, B_0.

In this paper, we often assume that the data is radially symmetric, meaning that given any matrix $\mathcal{O} \in SO(2,\mathbb{R})$, our data satisfies the condition:

$$f_0(x,p) = f_0(\mathcal{O}x, \mathcal{O}p), \quad \mathcal{O}E_0(x) = E_0(\mathcal{O}x), \quad B_0(x) = B_0(\mathcal{O}x) \tag{1.17}$$

when $d_p = 2$ and

$$f_0(x,p) = f_0(\mathcal{O}x, \mathcal{O}_3 p), \quad \mathcal{O}_3 E_0(x) = E_0(\mathcal{O}x), \quad \mathcal{O}_3 B_0(x) = B_0(\mathcal{O}x) \tag{1.18}$$

when $d_p = 3$. Here we define $\mathcal{O}_3 \in SO(3,\mathbb{R})$ by

$$\mathcal{O}_3 := \begin{pmatrix} \mathcal{O} & 0 \\ 0 & 1 \end{pmatrix}.$$

We often use the fact that solutions of the Relativistic Vlasov-Maxwell system with data symmetric under rotation stay symmetric under rotation. We summarize this result in a lemma.

Lemma 1.19. *Suppose that we are given the following data set (f_0, E_0, B_0) satisfying the constraints (1.17) in the 2-dimensional case and (1.17) in the 2.5-dimensional case. Then for all $\mathcal{O} \in SO(2,\mathbb{R})$, the unique solution (f, E, B) satisfying $f(0,x,p) = f_0(x,p)$, $E(0,x) = E_0(x)$, $B(0,x) = B_0(x)$ has the property*

$$f(t,x,p) = f(t,\mathcal{O}x,\mathcal{O}p), \quad \mathcal{O}E(t,x) = E(t,\mathcal{O}x), \quad B(t,x) = B(t,\mathcal{O}x) \tag{1.20}$$

in the 2-dimensional case and

$$f(t,x,p) = f(t,\mathcal{O}x,\mathcal{O}_3 p), \quad \mathcal{O}_3 E(t,x) = E(t,\mathcal{O}x), \quad \mathcal{O}_3 B(t,x) = B(t,\mathcal{O}x) \tag{1.21}$$

in the 2.5-dimensional case.

Proof. Suppose not, and in the 2D case, there exists an $\mathcal{O} \in SO(2,\mathbb{R})$ such that (1.20) does not hold. Then we see that

$$(f(t,x,p), E(t,x), B(t,x)) \text{ and } (f(t,\mathcal{O}x,\mathcal{O}_3 p), \mathcal{O}^{-1}E(t,\mathcal{O}x), B(t,\mathcal{O}x))$$

are two different solutions that share the same initial data. This contradicts Luk-Strain's result on the uniqueness of solutions. Similarly in 2.5D, there would exist two different solutions

$$(f(t,x,p), E(t,x), B(t,x)) \text{ and } (f(t,\mathcal{O}x,\mathcal{O}_3 p), \mathcal{O}_3^{-1}E(t,\mathcal{O}x), \mathcal{O}_3^{-1}B(t,\mathcal{O}x)).$$

□

We see from this lemma that if we are dealing with a radially symmetric solution, it might be easier to deal with cylindrical coordinates rather than Cartesian coordinates. With this idea in mind, we can pick characteristics $X(s) = X(s,t,x,v)$, $P(s) = P(s,t,x,v)$ and define

$$\hat{\phi}(s) := \frac{-X_2(s)\hat{x_1} + X_1(s)\hat{x_2}}{|X(s)|} = \frac{-X_2(s)\hat{x_1} + X_1(s)\hat{x_2}}{R(s)} \tag{1.22}$$

$$\hat{r}(s) := \frac{X_1(s)\hat{x_1} + X_2(s)\hat{x_2}}{R(s)} \tag{1.23}$$

for $R(s) := |X(s)| \neq 0$. We see here that $P(s) \cdot \hat{\phi}(s)$ is the angular component of $P(s)$ and $P(s) \cdot \hat{r}(s)$ is the radial component. We have a theorem bounding the angular component:

Theorem 1.24. *Suppose that we are in the $2D$ case with radial symmetry, where f_0 has compact support. Also assume that the initial data has finite energy*

$$\frac{1}{2}\int_{\mathbb{R}^2} (|E_0|^2 + |B_0|^2)\ dx + 4\pi \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} p_0 f_0\ dp\ dx < \infty$$

If $f(t,x,p) \neq 0$, then we have

$$P(t) \cdot \hat{\phi}(t) = \frac{-A(t,X(t)) + P(0) \cdot R(0)\hat{\phi}(0) + A(0,X(0))}{R(t)},$$

which implies

$$\Big| P(t) \cdot \hat{\phi}(t) \Big| \leq C + \frac{C}{R(t)}.$$

This theorem leads to a very interesting corollary, which in an intuitive sense, bounds how fast particles can approach the center of the plane.

Corollary 1.25. *Assume again that we are in the $2D$ case with radial symmetry and compactly supported data f_0. There exists a constant C such that for any characteristic $(X(s;t,x,p), P(s;t,x,p))$ with $X(t;t,x,p) \neq 0$, we have*

$$\frac{P(0;t,x,p) \cdot R(0;t,x,p)\hat{\phi}(0;t,x,p) + A(0, X(0;t,x,p))}{1 + te^{Ct^k}} \leq CR(t;t,x,p)$$

The above theorem and corollary will be proved in section 3.

2. Conservation Laws

We introduce and energy density term e and show that its integral over x is constant over time. This result is analagous to the law of energy conservation in classical electrodynamics. Our proof is similar to the proof in [5].

Proposition 2.1. *If we define our energy density term*

$$e(t,x) := \frac{1}{2}(|E(t,x)|^2 + |B(t,x)|^2) + 4\pi \int_{\mathbb{R}^{d_p}} p_0 f(t,x,p)\ dp$$

and suppose that the integral

$$\int_{\{0\}\times\mathbb{R}^2} e(t,x)\ dx < \infty,$$

we see that solutions to the relativistic Vlasov-Maxwell system satisfy

$$\int_{\{t\}\times\mathbb{R}^2} e(t,x)\ dx = constant. \tag{2.2}$$

Proof. We observe that in 2, 2.5, or 3 dimensions, solutions to (1.8)-(1.10) obey the property

$$\partial_t e = \sum_{k=1}^{d_x} \partial_{x_k}\left(-(B\times E)_k + 4\pi \int p_k f\ dp\right).$$

This equality is proved through integration by parts in p. Using the above identity, we see that the integral of $\partial_t e$ over x is 0. Therefore,

$$\partial_t\left(\int_{\{t\}\times\mathbb{R}^2} e(t,x)\ dx\right) = 0.$$

□

3. Bound on $v_{\hat{\phi}}$

In this section, we will only consider the 2D case of the Relativistic Vlasov-Maxwell system, with our data having finite energy

$$\frac{1}{2}\int_{\mathbb{R}^2}(|E_0|^2 + |B_0|^2)\ dx + 4\pi\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} p_0 f_0\ dp\ dx < \infty.$$

We also assume that our data (f_0, E_0, B_0) satisfies the radial symmetry condition (1.17). We will consider the characteristic equation

$$(X(s), P(s)) = (X(s,\bar{t},\overline{x},\overline{p}), V(s,\bar{t},\overline{x},\overline{p}))$$

of f, along which $f(s, X(s), P(s)) \neq 0$. For $R(s) := |X(s)| \neq 0$, define

$$\hat{\phi}(s) := \frac{-X_2(s)\hat{x_1} + X_1(s)\hat{x_2}}{|X(s)|} = \frac{-X_2(s)\hat{x_1} + X_1(s)\hat{x_2}}{R(s)} \tag{3.1}$$

$$\hat{r}(s) := \frac{X_1(s)\hat{x_1} + X_2(s)\hat{x_2}}{R(s)} \tag{3.2}$$

$$A(t,x) := \int_0^{|x|} B_3(t,r)\ dr = \frac{1}{2\pi}\int_{B(0,|x|)} B_3(t,y)\ dy \tag{3.3}$$

We often abuse notation and write $A(t,x) = A(t,|x|)$, because B_3 only depends on radius. We furthermore denote $(V(s))_{\hat{\phi}} = (V(s))\cdot\hat{\phi}(s)$ and $(V(s))_{\hat{r}} = (V(s))\cdot\hat{r}(s)$ for any $V:\mathbb{R}\to\mathbb{R}^2$.

We see by the Vlasov-Maxwell system that

$$(\partial_s P)_{\hat{\phi}} = E_{\hat{\phi}} + (\hat{P}\times B)_{\hat{\phi}} = E_{\hat{\phi}} + (\hat{P}_{\hat{r}})B_3 \tag{3.4}$$

for $R \neq 0$. Furthermore,

$$\begin{aligned}
&R(\partial_s P(s))_{\hat{\phi}} + \partial_s(A(s, X(s))) \\
&= R(\partial_s P(s))_{\hat{\phi}} + ([\partial_t A](s, X(s)) + \partial_s X(s) \cdot [\partial_x A](s, X(s))) \\
&= R(\partial_s P(s))_{\hat{\phi}} + \int_0^{R(s)} r[\partial_t B_3](s, r)\ dr + (\partial_s X(s))_{\hat{r}}[\partial_r A](s, R(s))) \\
&= R(\partial_s P(s))_{\hat{\phi}} + \int_0^{R(s)} (-r(\nabla_x \times E)(s, r))_3\ dr + (\hat{P}(s))_{\hat{r}} R(s) B_3(s, X(s)) \\
&= R(\partial_s P(s))_{\hat{\phi}} - \frac{1}{2\pi} \int_{B(0,R(s))} ((\nabla_x \times E)(s, r))_3\ dr + R(s)(\hat{P}(s))_{\hat{r}} B_3(s, X(s)) \\
&= R(\partial_s P(s))_{\hat{\phi}} - \frac{1}{2\pi} \int_{\partial B(0,R(s)} E \cdot\ dl + R(s)\hat{P}(s))_{\hat{r}} B_3(s, X(s)),
\end{aligned}$$

and since

$$\begin{aligned}
R(\partial_s P(s))_{\hat{\phi}} &= R(s) \left(E_{\hat{\phi}}(s, X(s) - (\hat{P(s)})_{\hat{r}} B_3(s, X(s)) \right) \\
&= R(s) \left(\frac{1}{2\pi R(s)} \int_{\partial B(0,R)} E \cdot dl - (\hat{P})_{\hat{r}} B_3 \right) \\
&= \frac{1}{2\pi} \int_{\partial B(0,R)} E(s, x)) \cdot dl - R(s) \left(\hat{P(s)})_{\hat{r}} B_3(s, X(s) \right)
\end{aligned}$$

for $R(s) \neq 0$, we have

$$R(\partial_s P(s))_{\hat{\phi}} + \partial_s(A(s, X(s))) = 0$$

We also observe that for $R(s) \neq 0$,

$$\begin{aligned}
\partial_s \left(P \cdot R\hat{\phi} \right) &= R\partial_s P \cdot \hat{\phi} + P \cdot \partial_s(R\hat{\phi}) \\
&= R\partial_s P \cdot \hat{\phi} + P \cdot \partial_s(-X_2\hat{x}_1 + X_1\hat{x}_2) \\
&= R\partial_s P \cdot \hat{\phi} + P \cdot (-P_2\hat{x}_1 + P_1\hat{x}_2) \\
&= R\partial_s P \cdot \hat{\phi}
\end{aligned}$$

when $R(s) \neq 0$, implying that

$$\begin{aligned}
0 &= \partial_s \left(P(s) \cdot R(s)\hat{\phi}(s) \right) + \partial_s(A(s, X(s))) \\
&= \partial_s \left(P(s) \cdot (-X_2(s)\hat{x}_1 + X_1(s)\hat{x}_2) \right) + \partial_s(A(s, X(s))).
\end{aligned}$$

We must also consider the case where $R(s) = 0$. This scenario is much easier to compute, since we can observe by our symmetry condition that $R(s')$ will stay 0 for $s' \geq s$. We make the observation that

$$P(s) \cdot (-X_2(s)\hat{x}_1 + X_1(s)\hat{x}_2) + A(s, X(s)) = 0$$

when $R(s) = 0$, implying that

$$\partial_s \left(P(s) \cdot (-X_2(s)\hat{x}_1 + X_1(s)\hat{x}_2) \right) + \partial_s(A(s, X(s))) = 0.$$

Our above equality now holds for all values of $R(s)$. We can therefore integrate and observe that

$$P(t) \cdot (-X_2(t)\hat{x}_1 + X_1(t)\hat{x}_2) + A(t, X(t)) = P(0) \cdot (-X_2(0)\hat{x}_1 + X_1(0)\hat{x}_2) + A(0, X(0)).$$

Therefore, if $R(t) \neq 0$,

$$(P(t))_{\hat{\phi}} = P(t) \cdot \hat{\phi}(t) = \frac{-A(t, X(t)) + P(0) \cdot R(0)\hat{\phi}(0) + A(0, X(0))}{R(t)}. \tag{3.5}$$

Now we use the above equality to bound $P(t) \cdot \hat{\phi}(t)$. We first use the Hölder inequality and (2.2) to deduce

$$|A(t, X(t))| \leq \frac{1}{2\pi} \int_{B(0,R(t))} |B(t, y)| \, dy \leq CR(t) \|B\|_{L^2(\mathbb{R}^2)} \leq CR(t).$$

Furthermore, by our assumption that f_0 is compactly supported, we finally have our estimate

$$\left| P(t) \cdot \hat{\phi}(t) \right| \leq C + \frac{C}{R(t)}.$$

Our equality (3.5) has another interesting application, bounding from below the radii of characteristic paths. In physical terms, this means that particles cannot move to the center of our plane too quickly. We use Theorem 1.15 to deduce that since $|\partial_s V(t)| \leq \|E(t)\|_{L^\infty_x} + \|E(t)\|_{L^\infty_x} \leq Ce^{Ct^k}$, we have $|V(t)| \leq C + Ct e^{Ct^k}$ by our compact support assumption on f_0. We can now use (3.5) to deduce that for $R(t) \neq 0$, there exists a positive constant C such that

$$\begin{aligned} \frac{P(0) \cdot R(0)\hat{\phi}(0) + A(0, X(0))}{R(t)} - C &\leq \left| \frac{-A(t, X(t)) + P(0) \cdot R(0)\hat{\phi} + A(0, X(0))}{R(t)} \right| \\ &= |V(s)| \\ &\leq C + Cte^{Ct^k}. \end{aligned}$$

Thus we have for any characteristic $(X(s; t, x, p), P(s; t, x, p))$,

$$\frac{P(0; t, x, p) \cdot R(0; t, x, p)\hat{\phi}(0; t, x, p) + A(0, X(0; t, x, p))}{2C + Cte^{Ct^k}} \leq R(t; t, x, p).$$

Acknowledgments

I would like to thank my mentor, Dr. Neel Patel, for his unstinting support throughout this REU and his helpful suggestions that guided my research. I am furthermore grateful to Professor Speyer for organizing this REU during the stress and uncertainty of a global pandemic. I would also like to thank Dr. Jacob Bernstein and Dr. Jonas Lührmann for being outstanding mentors in analysis.

4. Bibliography

References

[1] Robert Glassey, Stephen Pankavich, and Jack Schaeffer. Large time behavior of the relativistic Vlasov Maxwell system in low space dimension. *Differential Integral Equations*, 23(1-2):61–77, 2010.

[2] Robert Glassey and Jack Schaeffer. The "two and one-half-dimensional" relativistic Vlasov Maxwell system. *Comm. Math. Phys.*, 185(2):257–284, 1997.

[3] Robert T. Glassey. *The Cauchy problem in kinetic theory*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.

[4] Robert T. Glassey and Jack Schaeffer. The relativistic Vlasov-Maxwell system in two space dimensions. I, II. *Arch. Rational Mech. Anal.*, 141(4):331–354, 355–374, 1998.

[5] Jonathan Luk and Robert M. Strain. Strichartz estimates and moment bounds for the relativistic Vlasov-Maxwell system. *Arch. Ration. Mech. Anal.*, 219(1):445–552, 2016.