Weak formulation of discrete–time Principal–Agent problem

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Abstract

The Principal-Agent problem studies the optimal way to build contracts between economic agents facing asymmetries of information. We can consider for example the case of an employer wanting to remunerate an employee, while having only partial knowledge of the actions that the latter could perform. From the mathematical point of view, the theory is extremely rich, in the sense that it mixes many topics such as stochastic optimal control, optimization on spaces of infinite dimension, variational calculus, partial differential equations, stochastic calculus or the theory of backwards stochastic differential equations. This work focuses on a contractual relationship that is maintained over (discrete) time. We want to discuss how we built weak formulation of Principal-Agent Problem step by step.

1 Introduction

A Principal-Agent problem is to find the optimal contract between two parties, the Principal (she) and the Agent (he). Through his actions, the Agent can control the value of the outcome of the project, which benefits the Principal. Essentially, the Principal hires the Agent to work on her behalf, but both of them are selfish, meaning that they only care about themselves. A typical example is when the Principal acts as an investor and the Agent as a portfolio manager who manages investor's money on her/his behalf. In a more relaxed setting, the relationship between a doctor and a patient can be regarded as a Principal-Agent case.

This theory emerged in the 1970s, when economists realized that the theory of general equilibrium was not able to reproduce and solve situations where it is necessary for one of the agents to create a contract prompting the other agent to behave appropriately. It was pointed out that the unobservable behavior by insured people could reduce economic efficiency. Then Zeckhauser[1] and Spence and Zeckhauser[2] provided early models for such situation and Mirrlees[3, 4] introduced a more general model for moral hazard problem.

The basic model is the single period one. The interaction between the Agent and the Principal happens only once. We consider the case where the Principal leads by making different decisions first and then the Agent follows; this type of Principal–Agent problem corresponds to a Stackelberg game in game theory. There are many important works which discuss single period models including Shavell[5], Grossman and Hart[6], Rogerson[7] and Jewitt[8]. When the single period model was well understood, the natural extension was a dynamic system which involves multi-period models. In the dynamic model, we can consider questions as role memory, savings and commitment of the Agent under contract. There are famous works that discussed the repeated Principal-Agent problem, including Fundenberg, Holmstrom and Milgrom[9], Malcomson and Spinnewyn[10], Holmstrom and Milgrom[11], Lambert[12] amongst others.

The key feature in the problem is the asymmetric information between parties. Based on how much information the Principal is able to access, there are three main cases that have been studied in the literature: first-best case, second-best case, and third-best case. In the first-best case, or risk sharing, it is assumed that the Principal has all the information she needs. Both parties agree on how to share the risk, so she does not lose any utility. In the second-best case, the action of the Agent is not observable or non-contractible and the Principal can only observe the outcome controlled by the Agent, which lead to moral hazard. In the third-best case, the Principal cannot observe the Agent's type. In this case, the Principal faces both moral hazard and adverse selection.

In our paper, we only discuss the second-best case, where the moral hazard is present. Intuitively, the moral hazard describes the situation where the Agent might be willing to take unnecessary risk when he does not suffer the consequences of his actions. For example, when protected by insurance, people (the Agent) tend to take unnecessary risks, which could cause losses for insurance companies (the Principal). The Principal's goal is thus, to provide incentives to the Agent in order to align the interests of both sides. The question can be approached under continuous-time or discrete-time setting, but we will only examine the discrete-time (multi-period) case here.

We consider a discrete-time model with N periods. For each period n, where $n \leq N$, the agent works at the beginning of the period to control the outcome distribution of x_n and receives the payment c_n from the Principal at the end of the period (or at the beginning of the next period), when the outcome x_n is revealed. The effort that the Agent chooses to exert at this period is denoted as a_n, and the payment value c_n will depend on the work outcome (x_n), meaning that $c_n = c(x_1, \ldots, x_n)$. The agent's problem is to find the best sequence of efforts, given the contract offered by the principal. That is

$$\mathrm{V}^A(c) = \sup_a \mathbb{E}^{\mathbb{P}^a}\left[\sum_{n=1}^{N} e^{-rn} u_A(c_n) - \sum_{n=1}^{N} e^{-r(n-1)} g(a_n)\right]$$

The agent gains from the value of the contract $u_A(c_n)$ and loses utility $g(a_n)$ by exerting efforts. Since we choose effort at the beginning the period, we discount the effort by $e^{-r(n-1)}$. We want to find the maximized value of summation of discounted utility the agent gains from period 0 to arbitrary period N.

On the other side, the Principal decides by anticipating the optimal effort $a^\star$ exerted by the agent when he is offered the contract. The principal chooses the best contract, by solving agent's problem for every possible contract. In general, the principal possesses a utility function which benefits from the value of the project and loses from paying the agent the contract c.

Thus, the principal's problem is described by

$$\mathrm{V}^P = \sup_c \mathbb{E}^{\mathbb{P}^{a^\star(c)}}\left[u^P(x, c)\right].$$

such that

$$V^A(c) \geq R_0,$$

where for every c, $a^\star(c)$ is the optimal response of the agent to the contract and R_0 is the minimum utility provided by the contract c such that the agent decides to accept it. The principal has to satisfy the constraint of minimum utility required by the agent and then achieve the optimal expected utility through manipulating the contract value c.

2 Model

We choose a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a family of independent identically distributed random variables $(Y_i)_{i\in\mathbb{N}}$, which have mean 0 and variance 1. Denote by $\mathbb{F}^Y$ the filtration generated by the family $(Y_i)_{i\in\mathbb{N}}$ and define the process S by $S_0 := 0$ and $S_n := \sum_{i=1}^{n} Y_i$ for $n \geq 1$. Moreover, for a fixed $\sigma > 0$, define the process X by $X_0 := 0$ and for $n \geq 1$ define recursively

$$X_{n+1} := X_n + \sigma(S_{n+1} - S_n), \quad \mathbb{P} - \text{a.s.}$$

The process X will be referred as the outcome process and the agent is in charge of controlling its distribution. The random process S represents the noise.

We start by proving that, given our choice of the random variables Y_i, the process S defined above is a $(\mathbb{P}, \mathbb{F}^Y)-$martingale. We compute also the quadratic variation of this process.

Proposition 1. *The process S is a $(\mathbb{P}, \mathbb{F}^Y)-$martingale.*

Proof. By the definition of a martingale, it is sufficient to prove $\mathbb{E}_{\mathbb{P}}(S_{n+1}|\mathcal{F}_n) = S_n$ for every $n \in \mathbb{N}$. Let us fix an arbitrary $n \in \mathbb{N}$. Then

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(S_{n+1}|\mathcal{F}_n) &= \mathbb{E}_{\mathbb{P}}(S_{n+1} - S_n + S_n|\mathcal{F}_n) \\
&= \mathbb{E}_{\mathbb{P}}(S_{n+1} - S_n|\mathcal{F}_n) + \mathbb{E}_{\mathbb{P}}(S_n|\mathcal{F}_n) \\
&= \mathbb{E}_{\mathbb{P}}(Y_{n+1}|\mathcal{F}_n) + S_n \\
&= \mathbb{E}_{\mathbb{P}}(Y_{n+1}) + S_n \\
&= S_n
\end{aligned}$$

□

Proposition 2. *There exists a deterministic process $\langle S \rangle$ such that $M := S^2 - \langle S \rangle$ is a $(\mathbb{P}, \mathbb{F}^Y)-$martingale.*

Proof. In order to determine the process $\langle S \rangle$, we will perform some preliminary computations. More precisely, we will prove that the process S^2 is an $(\mathbb{P}, \mathbb{F}^Y)-$submartingale.

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(S_{n+1}^2|\mathcal{F}_n) &= \mathbb{E}_{\mathbb{P}}((S_n + S_{n+1} - S_n)^2|\mathcal{F}_n) \\
&= S_n^2 + 2\mathbb{E}_{\mathbb{P}}(S_n(S_{n+1} - S_n)|\mathcal{F}_n) + \mathbb{E}_{\mathbb{P}}((S_{n+1} - S_n)^2|\mathcal{F}_n) \\
&= S_n^2 + 2S_n\mathbb{E}_{\mathbb{P}}(S_{n+1} - S_n)|\mathcal{F}_n) + \mathbb{E}_{\mathbb{P}}((S_{n+1} - S_n)^2|\mathcal{F}_n) \\
&= S_n^2 + \mathbb{E}_{\mathbb{P}}((S_{n+1} - S_n)^2|\mathcal{F}_n) \\
&\geq S_n^2
\end{aligned}$$

From above, we know

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(S_{n+1}^2|\mathcal{F}_n) &= S_n^2 + \mathbb{E}_{\mathbb{P}}((S_{n+1} - S_n)^2|\mathcal{F}_n) \\
&= S_n^2 + \mathbb{E}_{\mathbb{P}}(Y_{n+1}^2|\mathcal{F}_n)
\end{aligned}$$

Notice that since $\mathbb{E}_{\mathbb{P}}(Y_n|\mathcal{F}_n) = 0$,

$$Var(Y_n) = \mathbb{E}_{\mathbb{P}}(Y_n^2|\mathcal{F}_n) = 1$$

Then

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(S_{n+1}^2 - \langle S\rangle_{n+1}|\mathcal{F}_n) &= \mathbb{E}_{\mathbb{P}}(S_{n+1}^2|\mathcal{F}_n) - \langle S\rangle_{n+1}\\ &= S_n^2 + Var(Y_{n+1}) - \langle S\rangle_{n+1}\\ &= S_n^2 - \langle S\rangle_n\end{aligned}$$

Then, $\forall n \in \mathbb{N}$

$$\langle S\rangle_{n+1} - \langle S\rangle_n = Var(Y_{n+1}) = 1$$

It is obvious that $\langle S\rangle_n = n$ satisfies the condition.

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(S_{n+1}^2 - \langle S\rangle_{n+1}|\mathcal{F}_n) &= \mathbb{E}_{\mathbb{P}}(S_{n+1}^2|\mathcal{F}_n) - (n+1)\\ &= S_n^2 + 1 - (n+1)\\ &= S_n^2 - n\end{aligned}$$

Therefore, M is a martingale if $\langle S\rangle_n = n$, □

From Proposition 1, we know S is a $(\mathbb{P}, \mathbb{F}^Y)$-martingale and we also know that $\langle S\rangle_n = n$ since $\mathbb{E}[Y_i] = 0$ and $Var[Y_i] = 1$. Now, we want to define a new measure $\mathbb{P}^a$ over $(\Omega, \mathcal{F})$ by $\frac{d\mathbb{P}^a}{d\mathbb{P}}\Big|_{\mathcal{F}_N} = \mathcal{E}(Z^a)$ for a discrete process $Z = (Z_n)_{n\in\mathbb{N}}$, where

$$\mathcal{E}(Z)_n := \exp(Z_n)\prod_{i=1}^{n}(1 + \Delta Z_i)e^{-\Delta Z_i}$$

Before we proceed, we want to prove some properties of $\mathcal{E}(Z)$.

Proposition 3.

$$\mathcal{E}(Z)_n = \mathcal{E}(Z)_0 + \sum_{i=1}^{n}\mathcal{E}(Z)_{i-1}(Z_i - Z_{i-1}).$$

Proof. Prove by induction. For $n = 1$,

$$
\begin{aligned}
\mathcal{E}(Z)_1 &= e^{Z_1}(1+\Delta Z_1)e^{-\Delta Z_1} \\
&= e^{Z_0}(1+Z_1-Z_0) \\
&= e^{Z_0} + e^{Z_0(Z_1-Z_0)} \\
&= \mathcal{E}(Z)_0 + \mathcal{E}(Z)_0(Z_1-Z_0)
\end{aligned}
$$

Assume that for some $m \in \mathbb{N}$ the assumption is true. Then, we have for $m+1$

$$
\begin{aligned}
\mathcal{E}(Z)_{m+1} &= e^{Z_m} e^{Z_{m+1}-Z_m} \left(\prod_{i \leq m} (1+\Delta Z_i) e^{-\Delta Z_i} \right) ((1+\Delta Z_{m+1}) e^{-\Delta Z_{m+1}}) \\
&= \mathcal{E}(Z)_m (1+Z_{m+1}-Z_m) \\
&= \mathcal{E}(Z)_m + \mathcal{E}(Z)_m \Delta Z_{m+1} \\
&= \mathcal{E}(Z)_0 + \sum_{i=1}^{m} [\mathcal{E}(Z)_{i-1}(Z_i - Z_{i-1})] + \mathcal{E}(Z)_m (Z_{m+1}-Z_m) \\
&= \mathcal{E}(Z)_0 + \sum_{i=1}^{m+1} [\mathcal{E}(Z)_{i-1}(Z_i - Z_{i-1})]
\end{aligned}
$$

□

Proposition 4. *If Z is a martingale, then $\mathcal{E}(Z)$ is a martingale*

Proof. Since Z is a martingale,

$$E(Z_n|F_{n-1}) = Z_{n-1}$$

Thus,

$$E(Z_n - Z_{n-1}|F_{n-1}) = 0$$

To prove $\mathcal{E}(Z)$ is a martingale, it is sufficient to show

$$
\begin{aligned}
E(\mathcal{E}(Z)_n|F_{n-1}) &= E(\mathcal{E}(Z)_0 + \sum_{i=1}^{n} (\mathcal{E}(Z)_{i-1}(Z_i - Z_{i-1}))|F_{n-1}) \\
&= \mathcal{E}(Z)_0 + \sum_{i=1}^{n-1} [\mathcal{E}(Z)_{i-1}(Z_i - Z_{i-1})] + E(\mathcal{E}(Z)_{n-1}(Z_n - Z_{n-1})|F_{n-1}) \\
&= E(Z)_{n-1}
\end{aligned}
$$

So $\mathcal{E}(Z)$ is a martingale.

□

Now, let's get back to the process we discussed before. Define $h : \mathbb{R} \to \mathbb{R}$ be a bounded function and $a = (a_n)_{n\in\mathbb{N}}$ be a predictable process. Define a process S^a where $S^a = S - \frac{1}{\sigma}h(a) \bullet \langle S \rangle$. With the properties we have proved for $\mathcal{E}(Z)$, we are able to prove a very important property of the process S^a under probability measure $\mathbb{P}^a$. Notice that a represents the efforts exerted by the Agent, which change the original distribution of outcome process.

Proposition 5. *The process S^a is a $\mathbb{P}^a$-martingale*

Proof. First, we know that

$$h(a_n) \bullet \langle S_n \rangle = \sum_{k=1}^{n} h(a_k)(\langle S_k \rangle - \langle S_{k+1} \rangle)$$

Since $\sigma^2 = 1$

$$h(a_n) \bullet \langle S_n \rangle = \sum_{k=1}^{n} h(a_k)$$

which is a predictable process.

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^a}(S_{n+1} - \frac{1}{\sigma}h(a_{n+1}) \bullet \langle S_{n+1} \rangle | \mathcal{F}_n) &= \mathbb{E}_{\mathbb{P}^a}(S_{n+1}|\mathcal{F}_n) - \frac{1}{\sigma}h(a_{n+1}) \bullet \langle S_{n+1} \rangle \\
&= \mathbb{E}_{\mathbb{P}^a}(S_n + Y_{n+1}|\mathcal{F}_n) - \frac{1}{\sigma}h(a_{n+1}) \bullet \langle S_{n+1} \rangle \\
&= \mathbb{E}_{\mathbb{P}^a}(Y_{n+1}|\mathcal{F}_n) + S_n - \frac{1}{\sigma}h(a_{n+1}) \bullet \langle S_{n+1} \rangle \\
&= \frac{\mathbb{E}_{\mathbb{P}}(Y_{n+1}\mathcal{E}(Z^a)_{n+1}|\mathcal{F}_n)}{\mathcal{E}(Z^a)_n} + S_n - \frac{1}{\sigma}h(a_{n+1}) \bullet \langle S_{n+1} \rangle
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(Y_{n+1}\mathcal{E}(Z^a)_{n+1}|\mathcal{F}_n) &= \mathbb{E}_{\mathbb{P}}(Y_{n+1}\mathcal{E}(Z^a)_0 + \sum_{i=1}^{n+1} \mathcal{E}(Z)_{i-1}(Z_i^a - Z_{i-1}^a)|\mathcal{F}_n) \\
&= \mathbb{E}_{\mathbb{P}}(Y_{n+1}\mathcal{E}(Z^a)_0 + Y_{n+1}\sum_{i=1}^{n} \mathcal{E}(Z)_{i-1}(Z_i^a - Z_{i-1}^a) + Y_{n+1}\mathcal{E}(Z^a)_n(Z_{n+1}^a - Z_n^a)|\mathcal{F}_n) \\
&= \mu\mathcal{E}(Z^a)_0 + \mu\sum_{i=1}^{n} \mathcal{E}(Z)_{i-1}(Z_i^a - Z_{i-1}^a) + \mathcal{E}(Z)_n h(a_{n+1})\mathbb{E}_{\mathbb{P}}(Y_{n+1}^2 - Y_{n+1}|\mathcal{F}_n) \\
&= \mu\mathcal{E}(Z^a)_n + \mathcal{E}(Z^a)_n h(a_{n+1})\sigma^2
\end{aligned}$$

So we have

$$\mathbb{E}_{\mathbb{P}^a}(S_{n+1} - \frac{1}{\sigma}h(a_{n+1}) \bullet \langle S_{n+1}\rangle|\mathcal{F}_n) = \frac{1}{\sigma}h(a_{n+1}) + S_n - h(a_{n+1}) \bullet \langle S_{n+1}\rangle$$
$$= S_n - \frac{1}{\sigma}h(a_n) \bullet \langle S_n\rangle$$

Therefore the process $S - \frac{1}{\sigma}h(a) \bullet \langle S\rangle$ is a $(\mathbb{P}^a, \mathbb{F}^Y)-$martingale □

We present now the main result of this section, which corresponds to the weak formulation of the agent's controlling actions.

Theorem 1. *The process X satisfies*

$$X_{n+1} = X_n + h(a_{n+1}) + \sigma(S^a_{n+1} - S^a_n), \quad \mathbb{P}^a - a.s.,$$

where $S^a = S - \frac{1}{\sigma}h(a) \bullet \langle S\rangle$ is a $(\mathbb{P}^a, \mathbb{F})-$martingale.

Proof. From Proposition 2, we know S^a is a $\mathbb{P}^a$-martingale. Finally, note that

$$\begin{aligned}
\sigma(S_{n+1} - S_n) &= h(a_{n+1}) + \sigma(S_{n+1} - S_n) - h(a_{n+1}) \\
&= h(a_{n+1}) + \sigma(S_{n+1} - S_n - \frac{1}{\sigma}h(a_{n+1})) \\
&= h(a_{n+1}) + \sigma(S_{n+1} - \frac{1}{\sigma}\sum_{n=1}^{n+1} h(a_i)(\langle S_i\rangle - \langle S_{i-1}\rangle) - S_n + \frac{1}{\sigma}\sum_{n=1}^{n} h(a_i)(\langle S_i\rangle - \langle S_{i-1}\rangle) \\
&= h(a_{n+1}) + \sigma(S^a_{n+1} - S^a_n)
\end{aligned}$$

We conclude by noting that the measures $\mathbb{P}$ and $\mathbb{P}^a$ are equivalent. □

3 Solving the agent's problem

In this section, we fix a contract $(c_n)_{n\in\mathbb{N}}$, offered by the Principal. The goal is to find the optimal response of the agent to such a generic contract. At any intermediate time i, define the continuation utility of the agent, if he chooses to exert effort a, as

$$U_i^A(a) = \mathbb{E}^{\mathbb{P}^a}\left[\sum_{n=i}^{N} e^{-r(n-i)}u_A(c_n) - \sum_{n=i+1}^{N} e^{-r(n-1-i)}g(a_n)\middle|\mathcal{F}_i\right]$$

At the last period, we have $U_N(a) = u_A(c_N)$. We define now the following auxiliary process, which is going to be useful because of the next Proposition

$$M_i^a := e^{-ri}U_i^A(c,a) + \sum_{n=0}^{i-1} e^{-rn}u_A(c_n) - \sum_{n=0}^{i} e^{-r(n-1)}g(a_n).$$

Proposition 6. *M^a is a $\mathbb{P}^a$-martingale.*

Proof.

$$\begin{aligned} M_i^a &= e^{-ri} \cdot \mathbb{E}^{\mathbb{P}^a}\left[\sum_{n=i}^{N} e^{-r(n-i)}u_A(c_n) - \sum_{n=i+1}^{N} e^{-r(n-1-i)}g(a_n)\middle|\mathcal{F}_i\right] + \sum_{n=0}^{i-1} e^{-rn}u_A(c_n) - \sum_{n=0}^{i} e^{-r(n-1)}g(a_n) \\ &= \mathbb{E}^{\mathbb{P}^a}\left[\sum_{n=i}^{N} e^{-rn}u_A(c_n) - \sum_{n=i+1}^{N} e^{-r(n-1)}g(a_n)\middle|\mathcal{F}_i\right] + \sum_{n=0}^{i-1} e^{-rn}u_A(c_n) - \sum_{n=0}^{i} e^{-r(n-1)}g(a_n) \\ &= \mathbb{E}^{\mathbb{P}^a}\left[\sum_{n=0}^{N} e^{-rn}u_A(c_n) - \sum_{n=0}^{N} e^{-r(n-1)}g(a_n)\middle|\mathcal{F}_i\right] \end{aligned}$$

Thus, it is a $\mathbb{P}^a$-martingale, as the process of conditional expectation of a fixed random variable (see next Lemma). □

Lemma 1. *The process of conditional expectation of a fixed random variable is a martingale.*

Proof. Let $M_i = \mathbb{E}[A|\mathcal{F}_i]$. By the tower property

$$\mathbb{E}\left[M_{i+1}|\mathcal{F}_i\right] = \mathbb{E}\left[\mathbb{E}\left[A|\mathcal{F}_{i+1}\right]|\mathcal{F}_i\right] = \mathbb{E}\left[A|\mathcal{F}_i\right] = M_i$$

So the process M_i is a martingale. □

Since M^a and S^a are $\mathbb{P}^a$-martingales. We assume $(Y_i)_{i\in\mathbb{N}}$ only takes two values, then by the Martingale Representation Property (MRP), we can write it as the stochastic integral of some process. Indeed, there exists a process $\hat{H}^a$ such that

$$M^a = \hat{H}^a \bullet S^a = \hat{H}^a \bullet S - \frac{1}{\sigma}\hat{H}^a h(a) \bullet \langle S\rangle.$$

where $S^a = S - \frac{1}{\sigma}h(a) \bullet \langle S\rangle$.

At time n,

$$\begin{aligned} M_n^a &= \sum_{i=1}^{n} \hat{H}_i^a (S_i - S_{i-1}) - \frac{1}{\sigma} \sum_{i=1}^{n} \hat{H}_i^a h(a) \\ &= e^{-rn} U_n^A(c, a) + \sum_{i=0}^{n-1} e^{-ri} u_A(c_i) - \sum_{i=0}^{n} e^{-r(i-1)} g(a_i) \end{aligned}$$

Then we let,

$$\begin{aligned} M_{n+1}^a - M_n^a &= e^{-rn}(e^{-r} U_{n+1}^A(a) - U_n^A(a) + u_A(c_n) - g(a_n)) \\ &= \hat{H}_{n+1}^a (S_{n+1} - S_n) - \frac{1}{\sigma} \hat{H}_{n+1}^a h(a_{n+1}) \end{aligned}$$

Our goal here is to establish differential equation form $dU = dS + f$, where f is a function. To eliminate the coefficient in front of $U^A(a)$, we multiply e^{rn} both forms of the equation, and define $H^a = \hat{H}^a \cdot e^{rn}$. So we have

$$\begin{aligned} e^{-r} U_{n+1}^A(a) - U_n^A(a) + u_A(c_n) - g(a_n) &= U_{n+1}^A(a) - U_n^A(a) + (e^{-r} - 1) U_{n+1}^A(a) + u_A(c_n) - g(a_n) \\ &= H_{n+1}^a (S_{n+1} - S_n) - \frac{1}{\sigma} H_{n+1}^a h(a_{n+1}) \end{aligned}$$

Then, we will have

$$U_{n+1}^A(a) - U_n^A(a) = H_{n+1}^a (S_{n+1} - S_n) - \frac{1}{\sigma} H_{n+1}^a h(a_{n+1}) - ((e^{-r} - 1) U_{n+1}^A(a) + u_A(c_n) - g(a_n))$$

which is

$$dU^A(a) = H_{n+1}^a dS - \frac{1}{\sigma} H_{n+1}^a h(a_{n+1}) - ((e^{-r} - 1) U_{n+1}^A(c, a) + u_A(c_n) - g(a_n))$$

Our plan is to solve the differential equation by comparison result for Backward Stochastic Differential Equations (BSDE). For every effort a, let $Y_n^a := U_n^A(a)$ and define the generator of the BSDE as

$$f(a_n, a_{n+1}, y, H) = -\frac{1}{\sigma} H h(a_{n+1}) - ((e^{-r} - 1) y + u_A(c_n) - g(a_n))$$

Then we have,

$$Y^a_{n+1} - Y^a_n = f(Y^a_{n+1}, H^a_{n+1}, a) + H^a_{n+1}(S_{n+1} - S_n) \tag{1}$$

$$Y^a_N = u_A(c_n) \tag{2}$$

Theorem 2 (Comparison Result)**.** *Let* (Y^1, H^1) *be the solution to the BSDE in discrete time*

$$Y^1_{n+1} - Y^1_n = f_1(n, Y^1_{n+1}, H^1_{n+1}) - H^1_{n+1}(S_{n+1} - S_n)$$

$$Y^1_N = \xi_1$$

and let (Y^2, H^2) *be the solution to the BSDE*

$$Y^2_{n+1} - Y^2_n = f_2(n, Y^2_{n+1}, H^2_{n+1}) - H^2_{n+1}(S_{n+1} - S_n)$$

$$Y^2_N = \xi_2$$

If $\xi_1 \geq \xi_2$, $\mathbb{P} - a.s.$, *and* $\forall n$, $f_1(n, Y, H) \geq f_2(n, Y, H)$, $\mathbb{P} - a.s.$ *Then it holds that* $Y^1_n \geq Y^2_n$ *for every* n, $\mathbb{P} - a.s.$

Therefore, to solve the agent's problem, we only need to maximize the generator of the BSDE at each time n.

4 Future Research

In the previous sections, we have built up a discrete-time model for a discrete–time Principal–Agent Problem with N periods. Based on our work, we outline the path to follow for solving the full problem in future research.

On the one hand, by the comparison results, we reformulated the agent's problem into maximizing the generator of a BSDE. Given the explicit functions $h(a_{n+1}), u_A(c_n), and\ g(a_n)$, it is easy to find the maximal generator which will be associated to the optimal effort $a^\star$ and the maximal utility process of the agent $Y^{a^\star}_n$. This means we can find the optimal $a^\star$ without any past dependence, by using the process H obtained from the (MRP). In this way, the agent will decide the optimal effort he would make for maximize his well-being.

On the other hand, given the optimal effort only as a function $a^\star(H)$, we can reformulate the problem of the principal as a control problem in which she controls H. By using the BSDE (1)-(2) and this auxiliary process, her problem can be equivalently written by omitting the c and any past dependence.

Eventually, the Principal's and Agent's problem can be solved explicitly in our discrete time model with N periods, by using standard dynamic programming techniques which allow to associate a Markovian control problem to a Hamilton-Jacobi-Bellman equation. By solving the HJB equation, we will be able to find the optimal control H and therefore the optimal contract c.

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