CHILDREN'S DRAWINGS AND THE RIEMANN-HILBERT PROBLEM

DRIMIK ROY CHOWDHURY

ABSTRACT. Dessin d'enfants (French for *children's drawings*) serve as a unique standpoint of studying classical complex analysis under the lens of combinatorial constructs. A thorough development of the background of this theory is developed with an emphasis on the relationship of monodromy to Dessins, which serve as a pathway to the Riemann Hilbert problem. This paper investigates representations of Dessins by permutations, the connection of Dessins to a particular class of Riemann surfaces established by Belyi's theorem and how these combinatorial objects provide another perspective of solving the discrete Riemann-Hilbert problem.

CONTENTS

1. History and Motivation

The enigmatic world of Dessins actually started in another form under the explorations of Felix Klein (with some of his work relating to the material investigated here [2]). Felix Klein had used the ideas behind Dessins and Belyi maps for investigating an 11-fold cover of the Riemann sphere with monodromy group $PSL(2, 11)$. The actual term and the establishment of the theory of Dessins were introduced by Alexander Grothendieck in 1984 with his proposal *Esquisse d'un Programme*. We present this short introduction by mainly as a motivation for the study of Dessins.

Dessin d'enfants are graphs that provide a combinatorial viewpoint of studying objects of classical complex analytical and topological constructs. The paper aims to provide the core relationships of Dessins to these branches of mathematics. Towards the end, we introduce the reader to the Riemann-Hilbert problem, which is one of the famous 23 problems of Hilbert, and specifically try to understand the view of this problem in a specific case that can be analyzed from the perspective of a Dessin. Another motivating reason for the development of Dessins is the deep relationship between these combinatorial objects and the Absolute Galois group of the rationals; the connection may allow individuals to identify the latter's orbits and study the invariants from the action of the group on these Dessins. This will be briefly explored in Section 3.5, but is not the focus of the paper. The following quote is translated into English for Grothendieck's reaction to uncovering the theory of Dessins, its possible consequences, and why it was aptly named *children's drawings* [8].

> This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my centre of interest in mathematics, which suddenly found itself strongly focused. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact. This is surely because of the very familiar, non-technical nature of the objects considered, of which any child's drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a dessin we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke.

Often, there may be elements of the paper that refer to notes I had taken across my REU from a multitude of sources. Sometimes, they may present a more in-depth investigation into certain topics that are slightly off tangent for our purposes of the paper (such as the fundamental group and Galois theory) that the reader may be interested to inspect at his or her will. The notes are presented here [1].

Section 2 serves as a holistic overview of the main concepts from areas of group theory, covering theory, combinatorics and complex analysis that shall be utilized in our exploration of Dessins and theorems regarding their development. Section 3 introduces the reader to the construction of Dessins, the underlying information Dessins can encode of meromorphic functions and Riemann surfaces, and identification of what class of Riemann surfaces are in fact most interesting to be studied from the viewpoint of Dessins (by Belyi's theorem). Section 4 ends the paper by investigating how Dessins in fact can be seen as a method to solve the discrete Riemann-Hilbert problem and explicitly analyzing their connections to monodromy.

2. Background

The key properties of understanding the fundamentals behind Dessin d'enfants and the theorems that apply to them require some knowledge from group theory, covering theory, complex analysis, and combinatorics. This section serves to introduce the notions needed from each of the aforementioned disciplines that are relevant for our exploration. The background for differential equations and monodromy is left later in Section 4.1.

2.1. **Group Theory.**

Definition 2.1. A *group* $(G, \cdot)$ is a set G and binary operation $\cdot : G \times G \to G$ defined on the elements of G that satisfies the three following conditions:

- *Identity* There exists an $e_G \in G$ such that $\forall g \in G, g \cdot e_G = e_G \cdot g = g$
- *Inverse* For each $g \in G$, there exists an $h \in G$ such that $gh = hg = e_G$. In this case, h is said to be the inverse of g and identified with g^{-1}.
- *Associativity* For $x, y, z \in G$, we have that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

A *subgroup* $(H, \cdot|_H)$ of a group $(G, \cdot)$ is a set $H \subseteq G$ such that the restriction of the binary operation $\cdot$ to H forms a group.

One can prove that the inverse of an element of a group is unique (hence g^{-1} as notation makes sense) and so is the identity element. With the structure of groups established, we desire to understand what functions can be studied between groups that serve to preserve their structure.

Definition 2.2. Let $(G, \cdot)$ and $(H, *)$ be groups. The map $\mu : G \to H$ is a *group homomorphism* if for all $g_1, g_2 \in G$, $\mu(g_1 \cdot g_2) = \mu(g_1) * \mu(g_2)$

Some facts that can be proven with relative ease is that group homomorphisms send the identity of one group to the other and that for each $g \in G, \mu(g^{-1}) = \mu(g)^{-1}$.

When dealing with groups, they may be identified by solely their set in writing and in this case, the binary operation is implicit in the arithmetic. For example, group $(G, \cdot)$ with operation $g \cdot h$ may be equally represented without ambiguity of the group G with operation gh. This notation is adopted wherever it may be applicable.

Definition 2.3. Let $(G, \cdot)$ be a group and X a finite set. We say that the group G *acts* on X if every $g \in G$ determines a permutation of X i.e. the action of G on X is a group homomorphism from G to the group of permutations on X.

We state without proving that the group of permutations of a finite set X is in fact a group; the reader may wish to prove this for him or herself. The theory of group actions allow us to study the objects that the group acts on with close detail based on the particular type of action that the group admits. Group actions may be characterized as faithful, transitive etc. on objects such as vector spaces, topological spaces and of course, sets. For a more conceptual understanding, these group actions serve as a manner to study the symmetries of the object being acted on, which in fact is one of the motivating factors of studying group theory. We shall encounter the group action on a vector space in Section 4.1.2 resulting in group representations.

What follows are the definitions of few more terms that will be utilized for our development of Riemann's Existence Theorem and for the fundamental group (which we shall define in Section 2.3.1) of particular topological spaces investigated throughout the paper.

Definition 2.4. Let $(G, \cdot)$ be a group and X a finite set and let G act on X. The *stabilizer* of $x \in X$ with respect to G is precisely the set of elements in G that fix x. In other words,

$$G_x = \{g \in G | gx = x\}$$

Definition 2.5. Let $(G, \cdot)$ be a group and $X \subseteq G$ such that if $x \in X$, then $x^{-1} \in X$. A *group word* in X of G is an expression of the form $x_{i_1} x_{i_2} ... x_{i_n}$ where each $x_{i_j} \in X$. We say that a group word in X is *reduced* if the word cannot be further simplified by the elements of X in the product.

A group $(G, \cdot)$ is said to be *free* if there exists a $X \subseteq G$ such that if $x \in X$, then $x^{-1} \in X$ where every non-empty reduced group word from X creates a non-trivial element of G.

Definition 2.6. Let $(G, \cdot)$ be a group. A set of *generators* of G is a set of elements within G such that any combination of them creates any element of G.

2.2. **Introduction to Topology.** A topological space provides a space with points and defines the intuition of what it means for points in the space to be "close". This serves as a pathway for a more generalized interpretation of continuity and convergence from the definition of limits in metric spaces. The definition of a topological space formally stated is:

Definition 2.7. A *topological space* (X, τ_X) is a set X with a collection of subsets of X called a *topology* τ_X such that the following conditions are held:

(1) $\varnothing, X \in \tau_X$
(2) The union of an arbitrary collection of elements in τ_X is in τ_X
(3) The intersection of a finite collection of elements in τ_X is in τ_X

By definition, the elements of τ_X are said to be the *open* sets of X with respect to this topology. A set $U \subseteq X$ is said to be *closed* with respect to this topology if its complement $X \setminus U$ is in τ_X; that is, the complement is open.

Definition 2.8. Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let $f : X \to Y$ be a well-defined function. We say that f is *continuous* if for all $V \in \tau_Y$, $f^{-1}(V) \in \tau_X$. In other words, the pre-image of an open set under f is open.

For most of the arguments presented in this paper, when dealing with topological spaces that have a well established and implied metric structure such as in the complex plane $\mathbb{C}$ or the Riemann sphere $\hat{\mathbb{C}}$, the topology will be taken as the Euclidean topology τ_E in each of the respective spaces. Since most if not all points of our following discussions are dealt in metric spaces, the following $\epsilon - \delta$ definition is given for continuity, which often serves as a more intuitive manner to understand this notion:

Definition 2.9. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ be a function. We say that f is *continuous at* $x_0 \in X$ if $\forall \epsilon > 0$, there exists an $\delta > 0$ such that whenever $d_X(x, x_0) < \delta$, we have $d_Y(f(x), f(x_0)) < \epsilon$.

The function f is said to be *continuous* if it is continuous at each point in its domain space.

Definition 2.10. Let (X, τ_X) and (Y, τ_Y) be topological spaces with $f : X \to Y$ as a continuous function. We say that f is a *homeomorphism* between X and Y if there exists a continuous function $g : Y \to X$ such that

$$f \circ g = id_Y$$
$$g \circ f = id_X$$

If there exists such a function f (and therefore g) satisfying the above properties, we say that X and Y are *homeomorphic* to each other.

The paper deals with functions in the contexts of different spaces with different properties endowed. For the sake of the reader, we shall be explicitly clear on the properties

of the function in relation to the domain and target spaces whenever possible unless it is well-established or clear.

2.3. **Covering Theory.** The notion of the fundamental group will be critical for the monodromy representation and its analysis in the role of Dessin d'enfants in the Riemann Hilbert problem. Furthermore, covering spaces are utilized in the definition of a Dessin d'enfant and are needed for several concepts of the paper such as the interpretation of Riemann's existence theorem and Belyi's theorem.

2.3.1. *Fundamental Group.*

Definition 2.11. We say that (X, x_0) is a *pointed topological space* if X is a topological space with x_0 as a *basepoint* of interest.

We note that in Definition 2.11, the underlying topologies are not stated but are present. This format is implicit in all following arguments. Let (X, τ_X) and (Y, τ_Y) be topological spaces for the following definitions.

Definition 2.12. We say that α is a *path* in X from $x_0 \in X$ to $x_1 \in X$ if $\alpha : [0, 1] \to X$ is a continuous map such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$.

Definition 2.13. Let α, β be two paths in X that begin at x_0 and end at x_1. We say that α is *path-homotopic* to β if there exists a continuous function $H : [0, 1] \times [0, 1] \to X$ such that

- $H(x, 0) = \alpha(x)$ and $H(x, 1) = \beta(x)$, $\forall x \in [0, 1]$
- $H(0, t) = x_0$ and $H(1, t) = x_1$ for each $t \in [0, 1]$

The second point simply states that the starting point and endpoints of the paths must be fixed in the continuous deformation from one path to the other by H.

We now look at paths with the same start and endpoint i.e. loops. Note the convention that for two loops at a basepoint to be homotopic, the basepoint is fixed in the continuous deformation that maps a loop to the other just as it is for paths in general.

Definition 2.14. The *fundamental group* $\pi_1(X, x_0)$ of a pointed topological space (X, x_0) is a group represented by the set of equivalence classes with respect to the equivalence relation of path-homotopy of loops at the basepoint x_0 and the binary operation of concatenation of loops.

Intuitively, the fundamental group characterizes the basic structure of the topological surface X being inspected with respect to the point at x_0 in regards to the number of holes there are. The reader is advised to visit [1] for a more in depth investigation of the fundamental group, proof that it is in fact a group with respect to concatenation of loops, and several of its properties.

A very important note is that often the basepoint may not be specified for the fundamental group on a surface X i.e. $\pi_1(X)$. This is done when the fundamental group at any two points on the surface are isomorphic to one another and so the group can simply be referred to as a single entity of the entire space irrespective of the basepoint. It turns out that when X is path-connected, this aforementioned property holds. For example, $\hat{\mathbb{C}}$ is path connected and it can be seen that the fundamental group at any point on the Riemann Sphere is only the homotopy class of loops equivalent to the identity loop. For more information on this and the proof, we advise the reader to visit [1].

Suppose for example that $m + 1$ points are removed on $\hat{\mathbb{C}}$, the fundamental group of $\hat{\mathbb{C}}$ for any point is the free group on m generators. We will use this fact in our exploration of the discrete Riemann-Hilbert problem with regards to the monodromy representation, which will be explained in Sections 4.1 and 4.2.

2.3.2. *Unramified covering spaces.*

Definition 2.15. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A *covering map* $p : X \to Y$ is a continuous, surjective map such that for each $y \in Y$, there exists a $V \subseteq Y$ such that $p^{-1}(V)$ is a disjoint collection of open neighborhoods in X, each of which is projected homeomorphically onto V by p. In other words, $p^{-1}(V) = \bigsqcup_{x \in p^{-1}(y)} U_x$ where each $U_x \subseteq X$ and $p|_{U_x} : U_x \to V$ is a homeomorphism for each $x \in p^{-1}(y)$.

X is said to be the *covering space* of the *base space* Y and the pre-image $p^{-1}(y)$ of $y \in Y$ is known as the *fiber over y*. The open neighborhoods U_x referred to in the definition are known as the *sheets* of V for y. When the covering map has the property that the fiber over any point in the base space is a finite set and the fibers for any two points have the same cardinality, n, we say that n is the *degree* of the map p and that p is an *unramified covering map*.

Covering theory and the existence of covering maps are crucial for the understanding and application of lifting properties, deck transformations, fundamental groups etc.. The reader is deferred to [1] for a quick overview and to [3] for a thorough exploration of the subject. For a more intuitive understanding of the covering map, note once again that p is a local homeomorphism. So the sheets of Y over X given by p can be interpreted as homeomorphic copies of subset V in question. Furthermore, the covering space always receives the local properties from the base space.

2.3.3. *Ramified covering spaces.* For the purposes of the paper, we shall inspect particular covering maps that do not have the degree property satisfied; that is, there may exist some *finite* many points in the base space Y that have a different number of pre-images from the remaining points. If $p : X \to Y$ satisfies Definition 2.15 and has this property, we say that p is a *ramified covering map*.

Let us try to build some familiarity and introduce the terminology regarding these covering maps with an example. Let $f : \mathbb{C} \to \mathbb{C}$ be defined as $f(z) = z^n + 1$. We note then for any $z_0 \in \mathbb{C} \setminus \{1\}$, there are exactly n pre-images of z_0 under f or $|f^{-1}(z_0)| = n$ and the pre-image of 1 is only 0. To introduce the property of the degree for ramified covering maps, we assign 0 with a *multiplicity* of n so that now the pre-images for any $z \in \mathbb{C}$ under f contains n points, counting multiplicities. In this case, we note that 1 is a *critical value* of f and the pre-images of a critical value with multiplicity greater than 1 are the *critical points* of f (which is just 0 in this case). Often, critical values are called *ramification points*.

2.4. **Complex Analysis.**

2.4.1. *Meromorphic functions.* Simply put, complex analysis studies functions of complex variables. The notions of continuity and differentiability extend into the complex case from familiar real space with continuity of a complex function being interpreted as continuity of vector-valued functions of two real-variables. However, differentiability is a whole other story.

Definition 2.16. Let $f : \Omega \to \mathbb{C}$ be a function where $\Omega \subseteq \mathbb{C}$. f is said to be *complex-differentiable at $z_0 \in \Omega$* or *holomorphic at $z_0 \in \Omega$* if the following limit exists in Ω:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

We say f is *complex-differentiable* or *holomorphic* on Ω if the above limit exists for each $z_0 \in \Omega$.

Although complex differentiability appears similarly defined as that of its real-counterpart, the former is a stronger property than that of the latter because a function being holomorphic at z_0 implies that the previous limit exists for any direction that the function can approach z_0 in the complex plane. A major theorem in complex analysis is that a function f is complex differentiable in Ω if and only if f is analytic in Ω and analytic functions on its own have several valuable properties associated with them. These facts are left for the reader to verify. Steven J. Miller's lectures on Complex Analysis are a fantastic and rigorous introduction to this subject and are recommended as a resource on the material [6].

What is pertinent for this paper is the notion of meromorphic functions, which can be seen as "relaxing" the notion of holomorphicity.

Definition 2.17. A function $f : \Omega \to \mathbb{C}$ where $\Omega \subseteq \mathbb{C}$ is said to be *meromorphic* on Ω if f is holomorphic on Ω except at discrete many points $z_0, z_1, z_2, ..., z_n \in \Omega$. We say $z \in \Omega$ is a *pole* of f if it is a *zero* of $1/f$. In this example, each z_i is a pole of f.

To concretize this, take the function $f : \mathbb{C} \to \mathbb{C}$ as $f(z) = 1/z$. Then f is holomorphic on $\mathbb{C} \setminus \{0\}$ and has a single pole at 0. Note that by the above definition, a function f being holomorphic implies its meromorphicity but not vice versa.

Informally speaking, at these poles the function f can be seen as "blowing up"; that is, the function approaches ∞. A useful method of studying these meromorphic functions is to extend its domain and codomain to that of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where we can now assign a mapping to each of the poles z_i under f to be ∞. It is then useful to see that meromorphic functions on $\mathbb{C}$ can be extended to be holomorphic functions on $\hat{\mathbb{C}}$.

An important result follows for holomorphic functions, meromorphic functions and of poles because of the analytic property mentioned above: we say that a meromorphic function f has a pole of *order* $n \in \mathbb{N}$ at z_0 if n is the smallest natural number such that $(z - z_0)^n f(z)$ is holomorphic. The concept of poles and zeroes of meromorphic functions are utilized as we shall see in the definition of a Dessin d'enfant in Section 3.3.

2.4.2. *Riemann surfaces.* Riemann surfaces are often the heart of study in complex analysis because they are often viewed as deformations of the complex plane.

Definition 2.18. A *Riemann surface* is a 1-dimensional complex manifold; that is, a space that appears locally as the complex plane.

Definition 2.18 is what is needed for most of the concepts to follow in this paper regarding these objects. Riemann surfaces are here referenced under the section of complex

analysis, but note that they are an extension of a topological construct by adding some complex properties. We will not delve more into details behind Riemann surfaces and manifolds; for more information, visit the REU notes for a detailed understanding on Riemann surfaces and examples [1]. Just for a quick, intuitive illustration, we can see that the Riemann sphere and complex plane are Riemann surfaces.

What is most important to understand is that Riemann surfaces serve as a general mathematical structure with complex analytic properties. In addition, with the local property of being "similar" to the complex plane, it is natural to extend the notion of meromorphic and holomorphic functions from above onto these Riemann surfaces. In fact, these surfaces are often the domains of study for holomorphic functions [7]. A key theorem of this paper, Belyi's theorem, states an important property of Riemann surfaces and meromorphic functions on them, which we shall encounter in Section 3.4.

2.5. **Combinatorics: Hypermaps and Maps.**

Definition 2.19. A *graph* $\Gamma = (V, E)$ is a tuple consisting of *vertices* V and *edges* E such that elements of E are incident to two (not necessarily different) vertices v_1, v_2 in V.

Note that it is possible for loops and multiple edges between vertices in graphs and we include those possibilities in the graphs we study in this paper. However, we always require the graphs be connected (that is, there exists a path of traversal from any vertex to any other in the graph) for our arguments. We say that that the *degree* of a vertex v $deg(v)$ is equal to the number of edges incident to it. Note that for an edge as a loop at a vertex, the edge contributes twice to the degree of the vertex.

Definition 2.20. A *map* is a graph embedded onto a two-dimensional, compact, oriented manifold such that the edges don't intersect and the complement of the graph on the surface is a disjoint union of regions homeomorphic to open disks.

These regions mentioned in the 2nd property of maps are the *faces* of the map and the *genus* of the map is the genus of the underlying surface. Figure 1 provides some examples for comprehension.
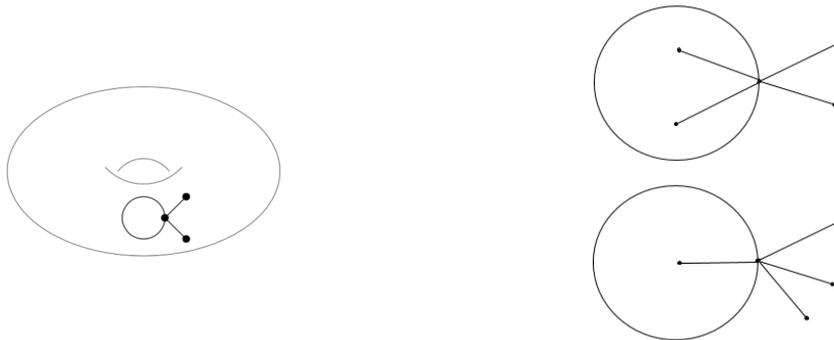

FIGURE 1. The left image is an example of a graph on the torus that is not a map [9] because of the second property of maps not being satisfied. The right image represents one graph but two different maps: the top map has 2 faces of degree 5 and the bottom map has 2 faces of degree 3 and 7.

Definition 2.21. A *hypermap* is a bipartite map. That is, it is possible to separate the elements of V into two disjoint collections such that every edge in E connects vertices between the two collections only.

Now, finally we are in the presence of a combinatorial objects, which are Dessins in their bare essence.

This somewhat extensive setup of different areas of mathematics and the fundamental tools that will be necessary in the development and definition of the Dessins indicates a unifying view of these concepts under this one roof, which is a key motivating factor for this research.

3. Dessin d'enfants: Construction

3.1. **Encoding hypermaps by triple of permutations.** Now with the plethora of definitions established, let's see what we can construct and what properties of Dessins are revealed in the process. First, let us directly work with the combinatorial objects of Section 2.5. We can obtain a hypermap H from a map M by labeling all the vertices of M as belonging to one set (let us call them the black vertices) and then inserting a white vertex in between every edge. Hence, every white vertex will have degree 2 and we say by convention for any hypermap that each edge between a black and white vertex is a *half-edge*.

Hence, we can rewrite the Euler characteristic of H with n half-edges as

$$\chi(H) = 2 - 2g = B + W + F - n$$

with B, W, F as the number of black vertices, white vertices and faces, respectively, and g as the genus of the map.

This section will present the fact that there is a bijective relationship between hypermaps and with a triple of permutations with some specific properties. Here we shall be utilizing the fact that the surface on which the hypermap is embedded is oriented as stated in the map Definition 2.20 and by extension to Definition 2.21. Every half-edge associated with each vertex has a left and right bank and a positive and negative orientation with positive represented in the counterclockwise motion around the vertex.

Let H be a hypermap with n half-edges and label each half-edge from 1 to n. Furthermore, "place" the label on the left-bank of the half-edge in the direction from the black to white vertex associated to the half-edge. With this formality defined, a triple of permutations $\langle \sigma, \alpha, \phi \rangle$ associated with the n labels of H can be defined as follows:

- A cycle of σ contains the labels of the half-edges incident to a particular black vertex taken in the positive i.e. counterclockwise direction. Note then there are as many cycles as there are black vertices and the length of each cycle corresponds to the number of half-edges incident to the black vertex inspected.
- As σ is defined for the black vertices, we can similarly define α for white vertices.
- A cycle of ϕ contains labels placed inside a face taken in the positive direction (counterclockwise) from the center of face. There are as many cycles in ϕ as there are faces of the hypermap.

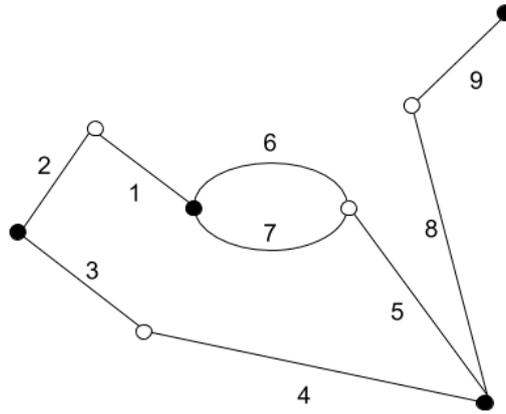To illustrate this construction, let's work with the following example.

FIGURE 2. Example of hypermap labelling

Based on Figure 2, we arrive at the following permutations from definition:

$$\sigma = (176)(23)(485)(9)$$
$$\alpha = (12)(34)(567)(89)$$
$$\phi = (135)(7)(26894)$$

For emphasis, the left and right bank placement of the labels are utilized in the construction of ϕ as the faces consider edge labels that are incident to it. Furthermore, the positive and negative orientation around the vertex are used in the cycles of each σ, α, and ϕ. Note that for the outer face, the cycle (26894) appears to be in the negative direction, but recall that the cycle is created in the counterclockwise orientation from the center of the face. This immediately addresses the surface on which the hypermap is defined on. Recall that a map is embedded onto a surface or manifold – in this particular case, we haven't said what it is and we can just imagine it being the Riemann sphere as it is compact and can be oriented. If we were to take the perspective of being at the center of the sphere with the hypermap on the surface, the cycle (26894) is in fact in the counterclockwise direction from this viewpoint.

Now that we have constructed this triple of permutations, the interesting property that arises is that $\sigma\alpha\phi = (1)$. Incidentally, because of this property it is in fact the case that we only need 2 of the 3 permutations to encode the information of a hypermap.

For some foreshadowing, this can precisely be viewed as a specific case of the *monodromy* for a ramified covering onto the Riemann sphere over 3 ramification points. In fact, the monodromy can be viewed for *any* 3 points of $\hat{\mathbb{C}}$ because of the property that fractional linear transformations allow the placement of any 3 points to any other 3 points on $\hat{\mathbb{C}}$. This is a case of Riemann's Existence Theorem, which will be stated more thoroughly and serve as a pathway to Belyi's theorem in Section 3.4.

A logical question follows the construction from above. We derived a triple of permutations for a hypermap based on some criteria, but what about the converse? Is it possible to create a hypermap from a triple of permutations? It is not difficult to see that the answer is yes given the following conditions are satisfied:

(1) the permutation group $G = \langle \sigma, \alpha, \phi \rangle$ is transitive
(2) $\sigma\alpha\phi = (1)$

The need for the transitive action guarantees that the graph is connected. The formal name of $G = \langle \sigma, \alpha, \phi \rangle$ is the *cartographic group* associated with the hypermap.

3.2. Critical points and critical values for complex polynomials.

To extend on the discussion of concepts in Section 2.3.3, we will state critical points and critical values now in the context of polynomials.

Let $P \in P_n(\mathbb{C})$ i.e. a polynomial of degree n having complex-valued coefficients. We note that because $\mathbb{C}$ is algebraically closed and thus, with polynomials defined over the space, the roots of the equation $P(z) = y_0$ can be divided into at most n linear factors. However, when this is not the case, some roots have a multiplicity greater than 1. We note then for these roots of $P(z) = y_0$, they are also roots of the equation $P'(z) = 0$.

Definition 3.1. Let $P \in P_n(\mathbb{C})$. If $z_0 \in \mathbb{C}$ is a solution to $P'(z_0) = 0$, we say that z_0 is a *critical point* of P. The *multiplicity* of z_0 is said to be the first $n \in \mathbb{N}_{>1}$ such that $P^{(n)}(z_0) \neq 0$. The *critical value* associated with z_0 is $P(z_0) \in \mathbb{C}$.

Let us note the parallelism between the notion of critical points and values established for ramified covering spaces in Section 2.3.3. For the points in the target space where the multiplicity property was not satisfied with the number of pre-images, we denoted them as the critical values and the critical points as the respective pre-images that have order at least 2. That notion is similarly reflected in the polynomial definition of this concept under the multiplicity interpretation with the decomposition of a polynomial into its linear factors.

For example, take the polynomial $P(z) = z^n + 1$ just as in Section 2.3.3. The conclusion to be made is just as it was before. The polynomial has a single critical value at 1 with a single respective critical point 0 of multiplicity n.

3.2.1. *Shabat polynomials.*

Definition 3.2. Let P be a polynomial with complex coefficients. We say that P is a *Shabat polynomial* if it has at most 2 critical values. That is, if $P'(z) = 0$ then it must be the case that $P(z) \in \{y_0, y_1\}$ where $\{y_0, y_1\} \subseteq \mathbb{C}$.

These are a particular class of polynomials that are of interest to us for our exploration into the relationship of Dessins and differential equations. The major consequence of Theorem 3.3 allows us to inspect a set of Shabat polynomials under a unique bipartite plane tree, where *plane* indicates the tree lives on a surface of genus zero e.g. $\hat{\mathbb{C}}$. The set of Shabat polynomials in question is precisely an equivalence class of Shabat polynomials based on the following equivalence relation.

Let $P, Q \in P_n(\mathbb{C})$ be Shabat polynomials with critical values z_1, z_2 and y_1, y_2, respectively. For the pairs $(P, [z_1, z_2])$ and $(Q, [y_1, y_2])$ to be *equivalent*, there must exist constants $A, B, a, b \in \mathbb{C}$ with $A, a \neq 0$ where [4]

$$Q(z) = AP(az + b) + B \text{ with } y_1 = Az_1 + B, y_2 = Az_2 + B$$

Instead of stating that $(P, [z_1, z_2])$ and $(Q, [y_1, y_2])$ are equivalent, we may say that the polynomials P and Q (by abuse of language) are equivalent to convey the same notion.

Theorem 3.3. *The set of bicolored plane trees is in bijection with the set of equivalence classes of Shabat polynomials with the notion of equivalence defined above.*

The theorem is established and proved as Theorem 2.2.9 of [4]. An important consequence of Theorem 3.3 is that every plane tree has in fact a unique geometric form that it can be associated with. In other words, more specifically, for every tree placed on the Riemann sphere, there exists a corresponding set of Shabat polynomials uniquely associated

with it up to an affine transformation. These Shabat polynomials will serve as the main algebraic objects of study in Section 4.3.

3.3. **Dessins and Belyi Pairs.** The theory of Dessin d'enfants and Esquisse d'un Programme is fundamentally developed due to the major results that were established by Belyi. This section introduces these facts and will lead us to finally understand why Dessins are created in the way that they are.

Definition 3.4. A pair (X, f) is known as a *Belyi pair* if X is a Riemann surface and $f : X \to \hat{\mathbb{C}}$ is a meromorphic function that is unramified outside of $\{0, 1, \infty\}$. The function f is known as a *Belyi map*.

Definition 3.5. A *dessin d'enfant* is a hypermap representation $H_{(X,f)}$ of a Belyi pair (X, f). The edges of the graph are essentially the pre-images of the unit interval $[0, 1]$ where

- $f^{-1}(0)$ and $f^{-1}(1)$ are the black and white vertices, respectively
- Degree of vertex is the multiplicity of the associated pre-image of 0 or 1 under f
- Each face corresponds to a pole (i.e. element of $f^{-1}(\infty)$) called a *center*
- Degree of face is the multiplicity of the associated pole under f

So by definition, for every Belyi pair (X, f) there exists a Dessin (or hypermap) representation $H_{(X,f)}$. In the converse direction, it turns out that for every hypermap, there exists a Belyi pair [9]. Note that for emphasis, the hypermap defines both the meromorphic function and the Riemann surface.

Let's look at a few examples to build some familiarity with these Dessins and their construction.
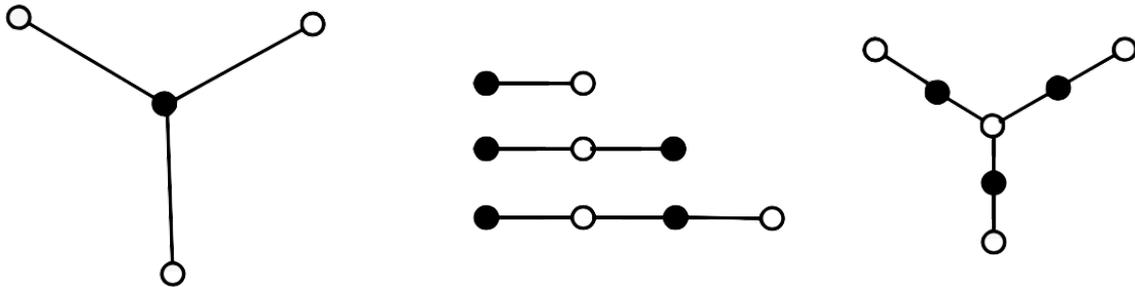
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$

FIGURE 3. Examples of Dessins from stars and Chebyshev polynomials

Figure 3 showcases a few Dessins constructed from the following functions (as ordered left to right):

(1) $y = x^n$ or a *star with n arms*. The figure displays the Dessin for $n = 3$.
(2) The *Chebyshev polynomials* are of the form $\cos(nx) = T_n(\cos(x))$ given by $z = \cos(x)$. Alternatively, we could represent these polynomials as $T_n(x) = \cos(n \arccos(x))$. By recursively unpacking the relationship between $\cos(nx)$ and $\cos x$ for $n \in \mathbb{N}$, we arrive at those first few Chebyshev polynomials until $n = 3$ in the figure above.
(3) $y = (x^n - 1)^2$ or a *2-star*. The figure displays the Dessin for $n = 3$.

3.4. **Riemann's Existence Theorem into Belyi's Theorem.** Let $M = \{y_1, y_2, ..., y_k\}$ denote a set of k ramification points and S^2 the familiar 2-sphere. For any $y_0 \in S^2 \setminus M$, the fundamental group of $S^2 \setminus M$ at this basepoint y_0 is the free group on $k - 1$ generators. However, k generators will be taken which then allows us to associate each generator uniquely to a ramification point [4]. We can precisely define the generator γ_i to correspond to the behavior around the respective ramification point y_i for $i = 1, ..., k$. We can define $g_i \in S_n$ where n is the degree of a finite-sheeted covering of the punctured sphere $f : X \to S^2 \setminus M$ where the g_i's act on the pre-image of y_0 under f (that is, permute elements of $f^{-1}(y_0)$). With that, we arrive at the *monodromy group* $\langle g_1, ..., g_k \rangle$ of the ramified covering map f. This will be further explained in Section 4.1.2. The fact that $\gamma_1 \gamma_2, ... \gamma_k = id$ indicates that $g_1 g_2 ... g_k = (1)$ as well as transitivity of the group. See Construction 1.2.13 of [4] for more information.

Claim 3.6. *For any $G = \langle g_1, ..., g_k \rangle$ with each $g_i \in S_n$ for a fixed $n \in \mathbb{N}$ and the properties that $g_1 g_2 ... g_k = (1)$ and that G acts transitively on n symbols, there exists an unramified covering map of the punctured sphere with the ramification points specified by M.*

Proof. Let $y_0 \in S^2 \setminus M$ and K be the underlying set of points permuted by G. Suppose we have the mapping $h : \pi_1(S^2 \setminus M, y_0) \to G$ taking the generator γ_i to the corresponding g_i from above. With $\gamma_1 \gamma_2, ... \gamma_k = id$ and $g_1 g_2 ... g_k = (1)$ as the only required properties to be satisfied for h, a unique group homomorphism can be formed with h.

Let $x \in K$ and let M_x be the pre-image under h of the stabilizer of x in G. We note that M_x is a subgroup of $\pi_1(S^2 \setminus M, y_0)$. Now, we will investigate the set of oriented paths in $S^2 \setminus M$ that begin at y_0 and end at some $y \in S^2 \setminus M$. We define an equivalence relation between paths η, ϕ beginning at y_0 based on the following properties

- η, ϕ have the same endpoint y
- $\eta \phi^{-1} \in M_x$

The reader should prove for him or herself that this in fact is is an equivalence relation if he or she desires. Define X to be the set of equivalence classes of such paths and we say that $f : X \to S^2 \setminus M$ maps a set of equivalence paths to their common endpoint. It is in fact the case that f is an unramified covering with the properties we desired. Refer to [4] Construction 1.2.8 for more on this. Lastly, we note that G having a transitive action causes the space X to be connected. $\square$

It turns out that a method to represent a Riemann surface is by a ramified covering over $\hat{\mathbb{C}}$, which can be done with the two following pieces of information [9]:

(1) k ramification points $z_1, ..., z_k$ on $\hat{\mathbb{C}}$
(2) k permutations $g_1, ..., g_k \in S_n$ that act transitively on n points with $g_1 g_2 ... g_k = (1)$.

The g_i's once again represent the monodromy or the behavior around the point z_i on Riemann surface.

Theorem 3.7 (Riemann's existence theorem). *For any two sequences defined with the properties above, there exists a Riemann surface X and meromorphic function $f : X \to \hat{\mathbb{C}}$ with z_i's as the ramification points and g_i's as the respective monodromy permutations. The surface X is unique up to an automorphism.*

Proof. With Claim 3.6 established, we have a ramified covering $f : X \to S^2$ with the ramification points of $Z = \{z_1, ..., z_k\}$ and $G = \langle g_1, ..., g_k \rangle$ from above; the only thing that remains is to add the complex structure to f to create a meromorphic function. By puncturing $\hat{\mathbb{C}}$ at the ramification points Z, we can create an unramified covering from f as $p : X \setminus f^{-1}(Z) \to \hat{\mathbb{C}} \setminus Z$ where the complex structure is "reconstructed" from $\hat{\mathbb{C}} \setminus Z$ or

in more appropriate jargon, lifted from $\hat{\mathbb{C}} \setminus Z$. For more information on liftings in covering theory, visit [1].

Take some $z_i \in Z$ and a preimage x_i of z_i under f. For these pre-images, we shall "fill" in the disks corresponding to the points using f. Formally, let z be a complex coordinate formed in the neighborhood of x_i by $z = (f(x) - f(x_i))^{1/d}$ where d is the degree of the map f at x_i. This reconstruction in fact satisfies the complex structure desired (and is unique). $\qquad\square$

We note again that the automorphisms of the Riemann sphere (i.e. fractional linear transformations) have the ability to map any 3 points to any other 3 points, so there are precisely $k - 3$ continuous parameters of choice with k representing the number of ramification points. Now, when $k = 3$ we say the Riemann surface is *rigid* in the sense that there are no continuous parameters to be chosen. Crucially, when we have these 3 ramification points we arrive at an example of the cartographic group mentioned in Section 3.1 corresponding to hypermaps [9]. In other words, we can create Dessins for the case that $k = 3$. The main question that remains is which Riemann surfaces are those that can be represented as ramified coverings of the Riemann sphere over 3 points. Do they have any special properties? Belyi's theorem serves to reveal some of this information.

Theorem 3.8 (Belyi's Theorem). *A compact Riemann surface X admits a model over the algebraic numbers $\bar{\mathbb{Q}}$ if and only if there exists a covering map $f : X \to \hat{\mathbb{C}}$ that is unramified outside of the set $\{0, 1, \infty\}$.*

The proof here is presented in [4] rigorously in Section 2.6. We note then that by definition of Dessins and Belyi's theorem, we have the following correspondence: exactly those Riemann surfaces that can be defined over $\bar{\mathbb{Q}}$ have a Dessin representation!

3.5. Other applications of Dessins: Galois group of rationals.
The following short section serves for the reader if he or she desires to learn of other applications of Dessins. A reason to study the theory of Dessins arises because of its relationship with the Absolute Galois group of the rationals $\mathrm{Aut}(\bar{\mathbb{Q}}|\mathbb{Q})$. The terminology and more detailed facts are provided in the REU notes [1]. This group has been widely inspected and studied by number theorists because it has never been fully described and remains largely a mystery. If one were able to characterize the elements of the group, by the Fundamental Theorem of Galois theory, all finite field extensions of $\mathbb{Q}$ (i.e. algebraic number fields) would be known, which would undoubtedly lead to major developments in algebraic number theory.

By Belyi's theorem, we have that Riemann surfaces represented as ramified coverings of $\hat{\mathbb{C}}$ over 3 points are those that can be written as algebraic curves with coefficients over $\bar{\mathbb{Q}}$. Thus, the Absolute Galois group acts on the coefficients of these Riemann surfaces and by extension, the $\mathrm{Aut}(\bar{\mathbb{Q}}|\mathbb{Q})$ acts on Dessins as well. Moreover, the action is *faithful* on these Dessins. As a result, these simple combinatorial structures allow mathematicians to *see* elements of this puzzling group, which is a very appealing reason for the development of the theory of Dessins.

As a sort of checkpoint, we remind the reader that the exploration thus far has unraveled intriguing connections between more or less classical topology and complex analysis to much more modern developments in algebraic and arithmetic geometry, which by the above discussion provide new ways to look at the Galois group of rationals.

4. Dessins and Fuchsian Differential Equations

The remainder of the paper aims to analyze the viewpoint of Dessins from the perspective of Fuchsian differential equations and to identify the connections that can be established

in these disciplines. Examples of solutions to the Riemann-Hilbert problem from a Dessin will be presented.

4.1. **Differential Equations and Monodromy.** First, some terminology and background will be introduced for Fuchsian differential equations and the monodromy representation for understanding the Riemann-Hilbert problem.

4.1.1. *Fuchsian ODEs and Hypergeometric functions.* The Euler-Gauss hypergeometric equation is given by

$$
{}_2F_1(a,b,c;z) = \sum_{k=0}^{\infty} \frac{a(a+1)(a+2)...(a+k-1)b(b+1)(b+2)...(b+k-1)}{c(c+1)(c+2)...(c+k-1)} \frac{z^k}{k!}
$$

where a, b, c are scalars.

The hypergeometric function serves as a solution to Euler's *hypergeometric differential equation*, which is given by

$$
z(1-z)\frac{d^2y}{dz^2} + [c - (a+b+1)z]\frac{dy}{dz} - aby = 0
$$

Let $\sum_{i=0}^{n} p_i(z)f^{(i)}(z) = 0$ be an n-th order ordinary linear homogeneous differential equation where $p_i(z)$ are each meromorphic functions. By convention, we set $p_n(z) = 1$ and if this is not the case, we may simply divide throughout by $p_n(z)$ (thereby writing the ODE in its *standard form*). For the remainder of this section, we are only concerned with ordinary homogeneous linear differential equations; we may substitute differential equations as a shorter placeholder.

Definition 4.1. Suppose we carry forward with the n-th order differential equation defined as $\sum_{i=0}^{n} p_i(z)f^{(i)}(z) = 0$. We say that $z_0 \in \mathbb{C}$ is an *ordinary point* of the differential equation if all the meromorphic functions p_k are analytic at z_0. If this is not the case i.e. for some $k \in \mathbb{N}_{\leq n} \cup \{0\}$, $p_k(z)$ has a pole at z_0, then we say z_0 is a *singular point* of the differential equation.

Definition 4.2. Singular points of a differential equation can be either characterized as regular or irregular. A *regular singular point* z_0 is a singular point of a differential equation if each p_i has a pole of order at most $n - i$ at z_0 for each $i \in \mathbb{N}_{\leq n} \cup \{0\}$s. Otherwise, z_0 is an *irregular singular point*.

Thus, a regular singular point of a differential equation states that the meromorphic coefficients for the differential equation are in some sense bounded in a small region. That is, every solution in a region of a regular singular point grows at most polynomially at the regular singular point.

Definition 4.3. An ordinary homogeneous linear differential equation with only regular singular points is known as a *Fuchsian ordinary differential equation*.

4.1.2. *Representation Theory: Monodromy.* Now, we shall understand more formally the monodromy representation that has been loosely mentioned and used in some context in Section 3.4. Let G be a group and V a finite-dimensional vector space over $\mathbb{C}$. Let $\mathrm{GL}(V)$ denote the group of invertible linear transformations on V.

Definition 4.4. A *(group) representation* (π, V) of G on V is a group homomorphism $\pi : G \to \mathrm{GL}(V)$.

Let $\Omega \subseteq \mathbb{C}$ and suppose we have an n-th order differential equation of the form $(\partial^n + a_{n-1}\partial^{n-1} + ... + a_0)f = 0$ where the coefficients a_i are each holomorphic on Ω. With a basepoint $z_0 \in \Omega$, let V_0 denote the vector space of local holomorphic solutions around z_0. Similarly, define V_1 for $z_1 \in \Omega$. Let γ be a path in Ω from z_0 to z_1. It turns out that we can analytically continue these solutions along γ and this only depends on the path-homotopy equivalence class associated with γ in Ω, $[\gamma]$. Recall in Section 2.3.1 under Definition 2.13 that these are precisely paths in Ω from z_0 to z_1 that can be continuously deformed from one into the other while preserving the endpoints in the deformation.

Using this, we can define a *monodromy operator*

$$M([\gamma]) : V_0 \to V_1$$

When we restrict our scope to only loops in Ω with regards to the basepoint z_0, we can define a *monodromy representation*

$$M : \pi_1(\Omega, z_0) \to \mathrm{GL}(V_0)$$

The map into $\mathrm{GL}(V_0)$ rather than into the broader class of endomorphisms of V_0 is precisely because of analytic continuation.

4.2. **How Dessins play into the Riemann-Hilbert problem.** The Discrete Riemann Hilbert problem is stated as follows: Let E be a finite set and $S \subseteq \hat{\mathbb{C}}$ a finite set with $m+1$ elements. Then $F_m \cong \pi_1(\hat{\mathbb{C}} \setminus S) \to \mathrm{Aut}(E)$ be a group homomorphism into the group of permutations of E where F_m is the free group on m generators. Assume that group homomorphism has a transitive image.

The big question lies in whether or not we can view E as holomorphic solutions to a Fuchsian differential equation whose regular singularities lie in S and the group homomorphism is exactly the monodromy representation arising by analytically continuing solutions at a base point $x_0 \in \hat{\mathbb{C}} \setminus S$ around S. With some facts from covering theory and application of Theorem 3.7, the data of the discrete Riemann-Hilbert problem does in fact create an unramified covering over $\hat{\mathbb{C}} \setminus S$ corresponding to the monodromy group given by the image of the group homomorphism $\pi_1(\hat{\mathbb{C}} \setminus S) \to \mathrm{Aut}(E)$.

Continuing what we have developed, it is obvious that we would like to inspect the discrete Riemann Hilbert problem for the case of $|S| = 3$ i.e. 3 regular singular points. It turns out that for any basepoint $x_0 \in \hat{\mathbb{C}} \setminus S$, the group homomorphism $F_2 \cong \pi_1(\hat{\mathbb{C}} \setminus S, x_0) \to \mathrm{Aut}(E)$ and the information from E provide us with a ramified covering $p : X \to \hat{\mathbb{C}}$ where X is a Riemann surface up to isomorphism because the covering has exactly $|S|-3$ continuous parameters modulo Möbius transformations (and so, X is rigid). In addition, by Belyi's theorem, we know that these Riemann surfaces are exactly the covering spaces of $\hat{\mathbb{C}}$ defined over $\bar{\mathbb{Q}}$ and furthermore, have all their information encoded into a Dessin d'enfant.

Another crucial point is that the information of the discrete Riemann Hilbert problem is in fact encoded into two permutations acting transitively on a finite set E with the cartographic property satisfied (recall that it is only 2 of the 3 permutations really needed to encode the information of the Dessin). To do this, label the first and second permutation as black and white, respectively. Construct a graph with each cycle in each permutation as a vertex (corresponding to its assigned color) with elements of E as edges; this graph is precisely a Dessin d'enfant [5] by our exploration in Section 3.3. These permutations can be viewed as the permutations σ and α of the cartographic group of a Dessin mentioned in Sections 3.1 and 3.4.

To wrap the main points of this section together, the image of the group homomorphism $F_2 \to \mathrm{Aut}(E)$ is precisely the familiar permutations $\langle \sigma, \alpha \rangle$ we have described. When $|S| = 3$, by the preceding paragraph, what is done is that the elements of E or the solution space

of a supposed homogeneous Fuchsian differential equation with the given properties of the discrete Riemann-Hilbert problem are identified by the edges of the Dessin or simply the pre-images of $[0, 1]$ under the ramified covering map p.

4.3. **Riemann Hilbert problem for Dessins as Trees.**
Note than in the case of a tree as the hypermap, we have that $v = e + 1$ and $f = 1$ and given by Euler's characteristic, $v - e + f = 2 - 2g$, we have $g = 0$. Thus, the covering surface can be taken to be the Riemann sphere $\hat{\mathbb{C}}$. Furthermore, because of the single face of a tree (i.e. outer face) and the fact that by definition that the faces correspond to pre-images of ∞ for a Dessin, we may take ∞ itself to be the pre-image (via Möbius transformations if necessary). Thus, there are at most 2 critical values for the Belyi map; in other words, we arrive at the Belyi map being a Shabat polynomial. We could have also simply attained this fact by the correspondence presented in Theorem 3.3.

The following are examples of solutions to the Riemann Hilbert problem from Dessins:

(1) $y = x^n$ or a *star with n arms* has the solution

$$y' - \frac{1}{nx}y = 0$$

satisfied by the inverses $x^{1/n}$.

(2) $y = (x^n - 1)^2$ or a *2-star* has the solution

$$x(1 - x)y'' + ((\frac{1}{n} - \frac{3}{2})x + \frac{1}{2})y' + \frac{1}{4n}(1 - \frac{1}{n})y = 0$$

satisfied by the inverses $(x^{1/2} + 1)^{1/n}$.

(3) The Chebyshev polynomials $T_n(x)$ have the solution

$$x(1 - x)y'' + (\frac{1}{2} - x)y' + \frac{1}{n^2}y = 0$$

satisfied by the inverses $\cos(\frac{1}{n}\arccos x)$. We note that this particular form of the differential equation is achieved by an affine change of variables permitted by the equivalence relation of Shabat polynomials mentioned in Section 3.2.1.

The reader is left to verify whether these differential equations are in fact solved by the respective inverses. In each of the cases, the differential equation is a hypergeometric equation. Recall that the corresponding Dessins for each of the 3 class of functions above is described in Figure 3 of Section 3.3.

It turns out that the Riemann Hilbert problem for a Dessin as a tree can in fact be explicitly solved and algorithmically calculated. The Fuchsian ODE associated with the Dessin in this case is a n-th order homogeneous linear differential equation $q_n\sigma^{(n)} + q_{n-1}\sigma^{(n-1)} + ... + q_0 = 0$ where σ are local inverses of the n-th degree Shabat polynomial P and q_i are each polynomials [5]. How they can be explicitly computed is presented in Section 5 of [5].

We end the paper on the following theorem.

Theorem 4.5. *Let T be a plane tree. If T is a star, 2-star or chain, then there exists a hypergeometric differential equation of order at most 2 serving as the solution to the Riemann Hilbert problem for T.*

Proof. Recall that a plane tree lives on a genus zero surface such as the Riemann Sphere $\hat{\mathbb{C}}$. Examples of star, 2-star and chains are presented in Section 3.3.

For either of the cases, there exists a Shabat polynomial of the form x^n, $(x^m - 1)^2$, or a Chebyshev polynomial in the same equivalence class. We note that because of Theorem

3.3 and the exhaustive set of examples showcased in Section 3.3 that T has a solution to the Riemann-Hilbert problem as a hypergeometric equation of order at most 2. $\square$

Acknowledgements

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