A CLASSIFICATION OF CERTAIN SYMMETRIC IDEALS

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ABSTRACT. The infinite symmetric group S_∞ acts on the ring $\mathbb{C}[x_1, x_2, \dots]$ by permuting the formal variables. Even though this ring is not Noetherian, if one only considers the ideals stable under the S_∞ action, it is known that those ideals satisfy the ascending chain condition. In this paper, we classify the "S_∞-prime" radical ideals of a similar ring, $\mathbb{C}[x_1, x_2, \dots, y_1, y_2, \dots]$, where S_∞ acts on each x_i and y_i. This is accomplished by reducing the problem to classifying S_∞-stable Zariski closed irreducible sets. We also discuss applications of the theory and future directions.

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1. Introduction

Let S_∞ denote the infinite symmetric group. Let $k[X] := k[x_1, x_2, \dots,]$, for k some algebraically closed, characteristic 0 field. Similarly, denote $k[X,Y] := k[x_1, x_2, \dots, y_1, y_2, \dots,]$. S_∞ has a natural action on both $k[X]$ and $k[X,Y]$, via permuting the formal variables. In the $k[X,Y]$ case, S_∞ permutes the variables x_i and y_i simultaneously.

Let I be an S_∞-stable ideal of $k[X,Y]$. The goal of this paper is to classify specific such I. In particular, we study the S_∞-equivariant analogues of prime ideals, and show that they have a rich structure and geometric interpretation. For an S_∞-stable ideal I (contained in either $k[X]$ or $k[X,Y]$), we have the following definition:

Definition 1.1. We say that I is ***c*-*prime*** provided that for any S_∞-stable subsets $A, B \subset k[X]$ (resp. $k[X,Y]$), $AB \subset I$ implies that $A \subset I$ or $B \subset I$.

Observe that this is a similar, but weaker condition than being prime. In particular, being a prime ideal implies that the ideal is radical, but being c-prime does not (see Example 2.28 for an example of a c-prime, non-radical ideal). These c-prime ideals, radical or otherwise, have been completely classified by A. Snowden and R. Nagpal in [1]. We turn our attention to the c-prime radical ideals of $k[X, Y]$.

1.1. **Statement of Results.** Recall that from classical algebraic geometry, we have the following bijection for $R = k[x_1, \dots, x_n]$:

$$\mathrm{Spec}(R) \longleftrightarrow \{\text{irreducible sets in } \mathbb{A}^n\}$$

Similar results hold for the cases where $R = k[X]$ and $R = k[X, Y]$, but over infinite affine space, $\mathbb{A}^\infty$. This bijection allows us to study the properties of prime ideals by reducing the problem to classifying their zero sets. We would like a similar bijection to hold for c-prime radical ideals, but to do so would require an S_∞ equivariant notion of irreducibility:

Definition 1.2. An S_∞-stable algebraic set X is ***c-irreducible*** if it cannot be written as the union of two proper S_∞-stable algebraic subsets.

Just as in the case of S_∞-stable ideals and being c-prime, when restricted to S_∞-stable algebraic sets, being c-irreducible is a weaker condition than conventional irreducibility. One convenient result about c-irreducible sets is that any S_∞-stable algebraic set can be decomposed in the union of finitely many c-irreducible sets. This differs from the irreducible case over $k[X]$ or $k[X, Y]$; while any closed set can be decomposed into a union of irreducibles, this union need not be finite. We prove that c-irreducible sets and irreducible sets relate in the following way:

Suppose $X \subset \mathbb{A}^\infty$ is S_∞-stable and closed. Let $X = \bigcup X_i$ denote its decomposition into irreducible sets. If the S_∞ orbit of an irreducible component is dense in X (with respect to the Zariski Topology), then X is c-irreducible.

As one would expect, in both the cases where $R = k[X]$ and $R = k[X, Y]$, we have the following bijection (discussed further in section 2.4):

$$\{c\text{-prime radical ideals of } R\} \longleftrightarrow \{c\text{-irreducible sets in } \mathbb{A}^\infty\}$$

Thus, classifying c-prime radical ideals is equivalent to classifying c-irreducible sets.

We prove that for any algebraic set $X \subset \mathbb{A}^\infty$, for each $(a_1, \dots, b_1, \dots,) \in X$, $\exists\ f \in k[x, y]$ such that $f(a_i, b_i)$ takes on $\leq n$ values for all i. We show this by showing that the following set

$$Z_{n,d} = \{(a_1, a_2, \dots, b_1, b_2, \dots) \in \mathbb{A}^\infty \mid \exists f \in k[x, y] \text{ of degree} \leq d \text{ such that } f(a_i, b_i) \text{ takes} \leq n \text{ distinct values}\}$$

is both closed and cofinal, implying that for any algebraic S_∞-stable set X, $X \subset Z_{n,d}$ for some $n, d \in \mathbb{N}$.

In addition, we provide some tools for computing if an S_∞-stable algebraic set is closed (see section 4.2).

1.2. **Applications.** Many of the tools discussed here are part of the larger realm of representation stability, a burgeoning new subfield of commutative algebra in which one studies algebraic objects and sub-objects stable under some G-action (usually, $G = S_\infty, \mathrm{GL}_\infty, O_\infty$, etc.). Our direction of research in particular (specifically, the study of S_∞-stable ideals of infinite variable polynomial rings) has found applications in algebraic statistics (see [4], [5]) as well as computational algebra and Gröbner theory (see: [6], [7], [8]).

1.3. **Future Directions.** The primary direction we focus on next is strengthening our classification. For example, let $P^d \subset k[x,y]$ denote the subset of polynomials of degree $\leq d$. Then, we have that

$$Z_{n,d} = \bigcup_{f \in P^d} Z_{n,f}$$

With $Z_{n,f}$ in example 4.4. Perhaps there is a strengthening of the classification which categorizes the sets $Z_{n,Y^d} \subset Z_{n,d}$ where

$$Z_{n,Y^d} = \bigcup_{f \in Y^d} Z_{n,f}$$

Where $Y^d \subset P^d$. There are some Y^d which we already know to be closed. For example, we can impose algebraic constraints on the coefficient of the polynomial $f \in k[x,y]$, and only consider the $f \in P^d$ where the coefficients $\alpha_1, \dots, \alpha_d$ satisfy some algebraic equation.

In addition, much of the work done in 2 sets of variables can be extended to n sets of variables. Thus, a natural direction to move in is to classify all the c-prime radical ideals of $k[X_1, X_2, \dots, X_n]$. Even more generally, we can think of the corresponding c-irreducible sets of the above c-prime radical ideals as sets in $\mathbb{A}^\infty = (\mathbb{A}^n)^\infty$. What if we replace $\mathbb{A}^n$ with any finite dimensional variety $\mathcal{X}$? The case where $\mathcal{X} = \mathbb{A}^1$ was already solved in [1], and this paper focuses on the case where $\mathcal{X} = \mathbb{A}^2$.

Finally, we can also classify the c-prime ideals in full generality, without relying on the radical condition. Non-radical ideals do not have as nice of a geometric intuition, and are quite difficult to classify. They have already been classified in the $k[X]$ case in [1], but there is no such classification for $k[X,Y]$.

2. Background

We begin with some necessary background material. The first section is independent from the rest, and in it we define the determinant (in the context of multivariate polynomial rings) and provide some useful invariants. The next two sections cover more general background that one would see in a first semester commutative algebra or first semester algebraic geometry course, and can be skipped by readers familiar with this material. In the final section we define S_∞-equivariant analogues of the concepts defined in prior two sections, and show that the general theory specializes to the S_∞-equivariant case.

2.1. The Discriminant.

Definition 2.1. Define

$$\Delta_n := \prod_{i \leq j}^{n} (x_i - x_j) \in k[X]$$

Δ_n is the ***Discriminant*** on n variables over $k[X]$.

The reader may have encountered the discriminant before in the context of field theory or algebraic number theory. Given a polynomial $f \in F[x]$ of degree n for F a field, its discriminant is precisely $\prod_{i \leq j}^{n} (\alpha_i - \alpha_j)$, where $\alpha_1, \ldots, \alpha_n$ are the roots of f in its splitting field. Note that our definition of the discriminant is different. Rather than defining the discriminant over roots of a polynomial, the discriminant is defined in terms of finitely many formal variables of a polynomial ring in infinitely many variables, and is itself a polynomial.

Example 2.2. $\Delta_3 = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$.

The discriminant will be a very useful construction later in the paper. To get a glimpse of why, notice that if $\Delta_3(a_1, a_2, a_3) \neq 0$, then a_1, a_2, a_3 are all distinct. Equivalently, if $\Delta_3(a_1, a_2, a_3) = 0$, then two of a_1, a_2, a_3 are equal. This generalizes to Δ_n; $\Delta_n(a_1, \ldots, a_n) = 0$ only if $a_1, \ldots, a_n$ are not all distinct.

In the infinite variable polynomial ring, it may be useful to discuss the discriminant of a collection of formal variables. Let $\Delta_{i_1,\ldots,i_n}$ denote the discriminant of $x_{i_1}, \ldots, x_{i_n}$. Similarly, for some $S \subset \mathbb{N}$, let Δ_S denote the discriminant of all x_i for $i \in S$.

We now prove some useful results about the discriminant over a polynomial ring in finitely many variables. Let S_n act on $k[x_1, \ldots, x_n]$ by permuting the formal variables, and choose $f \in k[x_1, \ldots, x_n]$.

Lemma 2.3. $\sigma \cdot \Delta_n = \text{sgn}(\sigma)\Delta_n$.

Proof.
$$\sigma \cdot \Delta_n = \sigma \cdot \prod_{i \leq j}^{n} (x_i - x_j) = \prod_{i \leq j}^{n} (x_{\sigma^{-1}(i)} - x_{\sigma^{-1}(j)})$$
Recall that any $\sigma \in S_n$ can be written as a product of transpositions. Observe that each transposition $(i\ j)$ only affects the term $(x_i - x_j)$, as while there are other x_i, x_j coefficients in the product, the transposition will merely swap terms $(x_k - x_j)$ and $(x_k - x_i)$ around, which due to commutativity of multiplication, is actually a trivial action. If $(i\ j)$ is a transposition in this product, then $(x_{\sigma^{-1}(i)} - x_{\sigma^{-1}(j)}) = (x_j - x_i) = -(x_i - x_j)$. If not, then $(x_{\sigma^{-1}(i)} - x_{\sigma^{-1}(j)}) = (x_i - x_j)$. Thus, for every transposition in the decomposition, a negative sign is added, thus we can conclude that
$$\sigma \cdot \prod_{i \leq j}^{n} (x_i - x_j) = (-1)^{\#\text{ of transpositions in }\sigma} \prod_{i \leq j}^{n} (x_i - x_j) = \text{sgn}(\sigma) \prod_{i \leq j}^{n} (x_i - x_j) = \text{sgn}(\sigma)\Delta_n$$
□

Lemma 2.3 easily generalizes below.

Lemma 2.4. *Suppose that $\sigma \cdot f = \text{sgn}(\sigma) f\ \forall \sigma \in S_n$. Then , $\Delta_n | f$.*

Proof. We prove the case where $n = 2$ explicitly. We need to show that for $f \in k[x_1, x_2]$ such that $(1\ 2) \cdot f = -f$, $(x_1 - x_2)|f$. Observe that, by hypothesis, if $x_1 = x_2$, then $f(x_1, x_2) = 0$. Thus, considering $f \in k(x_2)[x_1]$, it follows that x_2 is a root of f. Implying that $(x_1 - x_2)|f$ in $k(x_2)[x_1]$. Since $k[x_1, x_2]$ and $k(x_2)[x_1]$ are UFDs, it follows that this carries over to $k[x_1, x_2]$, so $(x_1 - x_2)|f$.

We now induct on n. Suppose that $\sigma \cdot f = \text{sgn}(\sigma) f\ \forall \sigma \in S_n$. Since $S_{n-1} \subset S_n$, it follows from the inductive hypothesis that $\Delta_{n-1}|f$. Thus, we now show that $(x_i - x_n)|f$ for each $i \leq n$. Well, we know that $(i\ n) \cdot f = -f$, so from the $n = 2$ case, it follows that $(x_i - x_n)|f$ for all $i \leq n$. Since
$$\Delta_n = \Delta_{n-1} \prod_{i \leq n} (x_i - x_n)$$
It follows that $\Delta_n | f$. □

Lemma 2.5. *If $\deg(f) < \deg(\Delta_n) = \binom{n}{2}$, then $\sum_{\sigma \in S_n} \text{sgn}(\sigma)(\sigma \cdot f) = 0$.*

Proof. Observe that for any $\tau \in S_n$,
$$\tau \cdot \left(\sum_{\sigma \in S_n} \text{sgn}(\sigma)(\sigma \cdot f) \right) = \text{sgn}(\tau) \left(\sum_{\sigma \in S_n} \text{sgn}(\sigma)(\sigma \cdot f) \right)$$

From Lemma 2.4, this implies that

$$\Delta_n \Big| \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(\sigma \cdot f)$$

However, note that

$$\deg(\Delta_n) = \binom{n}{2} > \deg(f) = \deg\left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(\sigma \cdot f)\right)$$

If $\deg\left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(\sigma \cdot f)\right) < \deg(\Delta_n)$, and the latter divides the former, this implies that

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(\sigma \cdot f) = 0$$

□

Lemma 2.6. $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(\sigma \cdot \Delta_{n-1} x_n^{n-1}) = (n-1)!\Delta_n$.

Proof. First, observe that

$$\Delta_n = \Delta_{n-1} x_n^{n-1} + \{\text{lower order terms of } x_n\}$$

To see why, note that the only terms with x_n that show up in Δ_n are precisely those of the form $(x_i - x_n)$ for $i = 1, \ldots, n-1$. This shows that the highest order term of x_n has degree $n-1$. The coefficient of x_n^{n-1} is just the product of the terms in Δ_n which don't contain an x_n term, which is just Δ_{n-1}.

Furthermore, observe that

$$\tau \cdot \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(\sigma \cdot \Delta_{n-1} x_n^{n-1})\right) = \operatorname{sgn}(\tau) \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(\sigma \cdot \Delta_{n-1} x_n^{n-1})$$

Implying via Lemma 2.4 that $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(\sigma \cdot \Delta_{n-1} x_n^{n-1})$ divides Δ_n. From our initial observation, we know that both $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(\sigma \cdot \Delta_{n-1} x_n^{n-1})$ and Δ_n are of same degree, implying that

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(\sigma \cdot \Delta_{n-1} x_n^{n-1}) = c\Delta_n$$

for some $c \in k$. To compute this c explicitly, we just compare the coefficients of x_n^{n-1} in both $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(\sigma \cdot \Delta_{n-1} x_n^{n-1})$ and Δ_n. When σ acts on $\Delta_{n-1} x_n^{n-1}$, if σ does not fix x_n, then there will be no x_n^{n-1} term in the summation. Thus, the only elements of the summation that contribute to the x_n^{n-1} term's coefficient are those which fix x_n, of which there are $|S_{n-1}| = (n-1)!$ many. It follows that in $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(\sigma \cdot \Delta_{n-1} x_n^{n-1})$, x_n^{n-1} has coefficient $\Delta_{n-1}(n-1)!$, implying that $c = (n-1)!$, completing the proof.

□

2.2. **Varieties and Algebraic Sets.** Let $I \subset k[x_1, \dots, x_n]$ be an ideal for k an algebraically closed field, and let $\mathbb{A}^n$ be ***affine** n**-space*** over k. For our purposes, we can treat $\mathbb{A}^n$ as an n-dimensional k-vector space.

Definition 2.7. Define

$$V(I) := \{a \in \mathbb{A}^n \mid f(a) = 0 \; \forall f \in I\}$$

This is an ***(Affine) Variety*** over I.

For the rest of this paper, as we are mostly dealing with affine varieties, we will simply refer to these as varieties. A variety can also be thought of as the intersection of all the ***zero loci*** (or zero sets) of polynomials in a given ideal.

Example 2.8. Let $I'_n = (\Delta_n)$. Recall that $\Delta_n(a_1, \dots, a_n) = 0$ precisely when $a_1, \dots, a_n$ are not all distinct. Thus,

$$V(I'_n) = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid a_1, \dots, a_n \text{ have at most } n-1 \text{ distinct values}\}$$

This example, as well as the concept of a variety, generalizes to the infinite variable case. Let $I \subset k[X]$ be an ideal. Here, $\mathbb{A}^\infty$ can be thought of as an infinite dimensional k-vector space, or alternatively as the set of sequences with coefficients in k. It is also the inverse limit of a sequence of finite dimensional k vector spaces, a characterization we return to later in this chapter. We define

$$V(I) := \{(a_1, a_2, \dots) \mid a_i \in k, f(a_1, a_2, \dots) = 0 \; \forall f \in I\}$$

Let us look at an ideal similar to example 2.8.

Example 2.9. Consider $I_n = (\Delta_n)_{S_\infty}$, or the ideal generated by the S_∞ orbit of Δ_n. This is still an ideal, and furthermore, is S_∞-stable. Suppose $(a_1, a_2, \dots) \in V(I_n)$. Then, for any choice of n a_i in the sequence, 2 must be equal. Thus, in the (infinitely long) sequence of a_i terms, there are at most $n-1$ distinct values. This allows us to conclude that

$$V(I_n) = \{(a_1, a_2, \dots,) \in \mathbb{A}^\infty \mid \text{all } a_i \text{ have at most } n-1 \text{ distinct values}\}$$

We will revisit example 2.9 again later in the paper.

We now define algebraic sets, which are subsets of affine space that are very similar to varieties.

Definition 2.10. An ***Algebraic Set*** is a subset of affine n-space over k which is the zero locus of a set of polynomials in $k[X] = k[x_1, x_2, \dots,]$. In other words, algebraic sets are subsets of affine space of the form

$$V(S) := \{x \in \mathbb{A}^\infty \mid f(x) = 0 \; \forall f \in S\}$$

When this collection of polynomials is an ideal of R, the algebraic set is just a variety. All the varieties above are examples of algebraic sets.

Lemma 2.11. *Suppose I, J are ideals of $k[X]$, and $I \subset J$. Then $V(I) \supset V(J)$.*

Proof. Choose $(a_1, a_2, \ldots,) \in V(J)$. Then, all $f \in J$ vanish on $(a_1, a_2, \ldots,)$, implying that they vanish on all $f \in I$, since all $f \in I \subset J$. It follows then that $V(J) \subset V(I)$. □

Lemma 2.12. *Let S be a set of polynomials, and let $I_S = (S)$. Then, $V(S) = V(I_S)$.*

Proof. First, note that if $f \in S$, then $f \in I_S$. Choose $(a_1, a_2, \ldots,) \in V(S)$. For any $f \in S$, $f(a_1, a_2, \ldots,) = 0$. The products of any elements of S still vanish under $(a_1, a_2, \ldots,)$, so it follows that $V(S) \subset V(I_S)$. From Lemma 2.11, since $S \subset I_S$, $V(I_S) \subset V(S)$, implying that $V(S) = V(I_S)$. □

This implies that any algebraic set can be considered the zero loci of an ideal (namely the ideal generated by the collection of polynomials), and thus any algebraic set is also a variety. For the remainder of the paper, we will use these terms interchangeably. Similarly, for any subset $X \subset \mathbb{A}^\infty$, we can consider the set of polynomials in R which vanish on X. This is an ideal, and we denote it $I(X)$.

Remark 2.13. $I(S)$ and $V(I)$ relate nicely. $I(V(S)) \supset S$ and $V(I(V(S))) = V(S)$ for S a set of polynomials, and $V(I(X)) \supset X$ for X a set of points in affine space.

Definition 2.14. An algebraic set X is ***irreducible*** if it cannot be written as the union of two proper algebraic subsets.

We would like to know that any algebraic set can be written as a union of irreducible sets. This is true over polynomial rings in finitely many variables, and a proof of the following lemma can be found in [3], page 8.

Lemma 2.15. *Let V be an algebraic set. Then, $V = V_1 \cup \cdots \cup V_m$ where each V_i is irreducible, and $V_i \not\subset V_j$ for $i \neq j$.*

Proof. [3] Since k is a field (and thus a Noetherian ring), it follows from Hilbert's Basis Theorem that $k[x_1, \ldots, x_n]$ is Noetherian. Thus, by Zorn's Lemma, every nonempty collection of ideals has a maximal element with respect to inclusion. From Lemma 2.11, this implies that and collection of algebraic sets has a minimal element with respect to inclusion. Define

$$\mathcal{P} := \{V \subset \mathbb{A}^n \mid V \text{ is algebraic and not the union of a finite number of irreducible sets}\}$$

We wish to show that $\mathcal{P}$ is empty, and proceed by contradiction. Choose $V \in \mathcal{P}$ minimal with respect to inclusion. V is not irreducible, so $V = V' \cup V''$ for some V', V'' closed and proper subsets of V. Since V is minimal, $V', V'' \notin \mathcal{P}$, implying that they have a decomposition into finitely many irreducibles. Let V_i', V_i'' be these irreducible sets.

$$V' = \bigcup_{i=1}^{n} V_i' \text{ and } V'' = \bigcup_{j=1}^{m} V_j'' \quad \Rightarrow \quad V = \left(\bigcup_{i=1}^{n} V_i'\right) \cup \left(\bigcup_{j=1}^{m} V_j''\right)$$

This is a union of finitely many proper closed subsets of V, a contradiction (the second condition follows from us just ignoring duplicates in the decomposition). □

In the case of infinitely many variables, there exists such a decomposition, but the number of irreducible sets need not be finite. This is because $k[X]$ is not Noetherian, so we cannot choose a minimal element in $\mathbb{A}^\infty$.

2.3. **Prime Ideals and The Zariski Topology.** Let R be a (commutative) ring and let I be an ideal.

Definition 2.16. for subsets $A, B \subset R$, we say I is a ***Prime Ideal*** if $AB \subset I$ implies that either $A \subset I$ or $B \subset I$.

Equivalently, I is prime if and only if for any $a, b \in R$, $ab \in I$ implies that $a \in I$ or $b \in I$.

Definition 2.17. The ***Spectrum*** of R, denoted $\text{Spec}(R)$, is the set of prime ideals of R.

Definition 2.18. For I an ideal, the ***Radical*** of I is defined as

$$\sqrt{I} := \{r \in R \mid r^n \in I \text{ for some } n\}$$

If $I = \sqrt{I}$, then I is a ***Radical Ideal***.

Remark 2.19. All prime ideals are radical.

The remark follows from the fact that in any prime ideal I, $x^n \in I$ implies that $x \in I$.

The remainder of this section will be devoted to developing the Zariski Topology on $\text{Spec}(R)$. We define the topology first on affine space, then pull back to Spec(R). We define the Zariski Topology in terms of its closed sets as below.

Definition 2.20. Define $X \subset \mathbb{A}^\infty$ as closed if it is an algebraic set.

Theorem 2.21. *The above definition defines a topology on Affine space.*

Proof. First, observe that

$$V(k[X]) = \emptyset$$

$$V(\emptyset) = \mathbb{A}^\infty$$

Thus, it is sufficient to check that the finite union and arbitrary intersection of closed sets is closed. Due to Lemma 2.12, we can assume that closed sets are zero sets of ideals. It is sufficient to check that the following identities hold for ideals I, J of $k[X]$:

(1) $V(I) \cup V(J) = V(IJ)$.
(2) $V(I) \cap V(J) = V(I + J)$

As the product of finitely many ideals is still an ideal, and the sum of arbitrarily many ideals is still an ideal, we can conclude from (1) and (2) that our definition defines a topology. (1) follows almost immediately; $I, J \subset IJ$, so $V(IJ) \subset V(I), V(J)$ by Lemma 2.11. Thus, $V(IJ) \subset V(I) \cup V(J)$. Also, if $(a_1, a_2, \dots)$ vanishes on I and J, it also vanishes on products of elements of I and J, so $V(I) \cup V(J) \subset V(IJ)$, and thus $V(I) \cup V(J) = V(IJ)$. Similar logic can be used to prove claim (2), which completes the proof. □

The above theorem, along with Lemma 2.15, tell us that any closed set can be decomposed into a union of irreducible closed sets. We now want to characterize the ideal who's zero loci is irreducible.

Lemma 2.22. *Fix I a radical ideal. I is a prime ideal if and only if $V(I)$ is irreducible.*

Proof. We prove the reverse direction first. We want to check that if $ab \in I$, then $a \in I$ or $b \in I$. Observe that

$$V((I, a)) \cup V((I, b)) = V((I, a)(I, b)) = V(I)$$

Thus, one of $a, b \in I$. We now prove the forward direction. Suppose that $V(I) = X_1 \cup X_2$, for X_1, X_2 closed sets. Then, since $X_1, X_2 \subsetneq V(I)$, it follows that $I(X_1) \supsetneq I$, and $I(X_2) \supsetneq I$. Choose $f_1 \in I(X_1), f_2 \in I(X_2)$, but $f_1, f_2 \notin I$. However, $f_1 f_2 \in I$, violating the condition that I is prime. □

Remark 2.23. Because we are working over an algebraically closed field, we know that for any ideal J, $I(V(J)) = \sqrt{J}$.

This is a nontrivial statement that relies on Hilbert's *Nullstellensatz*, but we assume this for the sake of brevity. The remark along with the above lemma yields the following theorem:

Theorem 2.24. *There is an order reversing (with respect to inclusion) bijection between* $\operatorname{Spec}(k[X])$ *and the set of irreducible sets in affine space.*

Proof. Fix J a prime ideal. From Lemma 2.22, we can conclude that $V(J)$ is irreducible if and only if $I(V(J))$ is prime. Since our field is algebraically closed, we know that $I(V(J)) = \sqrt{J}$, from the remark. Since J is prime, $\sqrt{J} = J$, so $I(V(J)) = J$. Thus, $V : \operatorname{Spec}(k[X]) \to \{\text{irreducible sets}\}$ and $I : \{\text{irreducible sets}\} \to \operatorname{Spec}(k[X])$ are well defined and inverses of each other, establishing the bijection. Lemma 2.11 guarantees that this bijection is inclusion reversing. □

This bijection allows us to define the Zariski Topology on $\operatorname{Spec}(k[X])$.

Example 2.25. $\operatorname{Spec}(k)$ for k a (algebraically closed) field is just the topological space with one element.

Example 2.26. Since $k[x]$ (the polynomial ring in just one variable) is a PID, its prime ideals are just the ideals generated by prime elements of $k[x]$. Since k is algebraically closed, it follows that any irreducible, and thus prime, element of the ring is simply a linear function of the form of $x - a$ for $a \in k$. Thus, $\operatorname{Spec}(k[x])$ consists of one closed point for each $a \in k$, and a distinguished point corresponding to the zero ideal. Because of this, it is easy to see that closed sets in $\operatorname{Spec}(k[x])$ are just finite unions of points, the empty set, and the whole space. $\operatorname{Spec}(k[x])$ is sometimes referred to as the ***affine line*** in k. As one would expect, $\operatorname{Spec}(k[x, y])$ is the affine plane $\mathbb{A}^2$, and so on.

Furthermore, this allows us to reduce studying prime ideals to just studying irreducible sets in affine space. In the next section, we will prove that a similar bijection holds between S_∞-stable analogues of prime ideals and irreducible sets.

2.4. c-Prime Ideals and c-Irreducible Sets.

Recall that S_∞ acts on the polynomial ring $k[X] = k[x_1, x_2, \dots]$ (where k is algebraically closed) by permuting the formal variables. We say that a subset S is S_∞-stable (or S_∞-equivariant) provided that $\sigma \cdot S = S$ $\forall \sigma \in S_\infty$.

Definition 2.27. Let $I \subset k[X]$ be an S_∞-stable ideal. we say that I is ***c-prime*** provided that for any S_∞-stable subsets $A, B \subset k[X]$, $AB \subset I$ implies that $A \subset I$ or $B \subset I$.

Equivalently, I is c-prime if for any $f, g \in R$, if $(f)(\sigma \cdot g) \in I$ for all $\sigma \in S_\infty$, then $f \in I$ or $g \in I$.

Note that for any S_∞-stable ideal, being c-prime is weaker than being prime. In particular, an ideal being c-prime does not necessarily imply that it is radical, as in the example below.

Example 2.28. The ideal $I_2 = (x_1^2, x_2^2, \dots)$ is c-prime. To see why, choose $f, g \in R$ such that $(f)(\sigma \cdot g) \in I$ for all $\sigma \in S_\infty$. Since polynomials have finitely many terms, we know that f has a maximal formal variable with respect to their index. Let x_m denote this maximal formal variable appearing in f. Choose σ such that g is written in coefficients $x_{m+1}, \dots, x_{m+n}$. Since the product $(f)(\sigma \cdot g) \in I$, we know that the product is written entirely in terms $x_1^2, \dots x_{m+n}^2$. Since f is written only in $x_1, \dots, x_m$, and g is not written in any of these terms, it follows that f is written in $x_1^2, \dots, x_m^2$, and $f \in I$. Using similar logic, $I_\ell = (x_1^\ell, x_2^\ell, \dots)$ is c-prime for any ℓ.

This ideal is not radical because $x_1 \in \sqrt{I_2}$, since $x_1^2 \in I_2$, but clearly $x_1 \notin I_2$.

Example 2.29. $(\Delta_n)_{S_\infty}$, as in example 2.9, is c-prime.

We also have a S_∞-equivariant notion of irreducible.

Definition 2.30. An S_∞-stable algebraic set X is ***c-irreducible*** if it cannot be written as the union of two proper S_∞-stable algebraic subsets.

Lemma 2.31. *Any S_∞-stable algebraic set V can be decomposed into the union of finitely many c-irreducible sets $V = V_1 \cup \cdots \cup V_m$, where $V_i \not\subset V_j$ for $i \neq j$.*

Proof. Recall that from [2], we know that S_∞-stable ideals satisfy the ascending chain condition, so a nonempty collection of these ideals has a maximal element by Zorn's Lemma. It follows by Lemma 2.11 that there must be a minimal element of any collection of S_∞-stable closed subsets of $\mathbb{A}^\infty$. We then proceed identically as in the proof of Lemma 2.15 to conclude. □

Observe that in the general case, algebraic sets over $k[X]$ don't necessarily decompose into finitely many irreducible sets. The finiteness condition here comes from the S_∞-noetherianity of $k[X]$.

The rest of the section will be focused on proving the following theorem relating these two concepts:

Theorem 2.32. *There exists an order reversing (with respect to inclusion) bijection between the set of c-prime radical ideals and the set of c-irreducible sets.*

The proof of this relies heavily on the following Lemma:

Lemma 2.33. *Suppose I is an S_∞-stable radical ideal. I is a c-prime ideal if and only if $V(I)$ is c-irreducible.*

Proof. We prove the reverse direction first. We want to check that if $a(\sigma \cdot b) \in I$ for all $\sigma \in S_\infty$, then $a \in I$ or $b \in I$. Observe that

$$V((I, a)) \cup V((I, \sigma \cdot b)) = V((I, a)(I, \sigma \cdot b)) = V(I)$$

Thus, one of $a, \sigma \cdot b \in I$. If $a \in I$, then we're done. If $\sigma \cdot b \in I$, recall that I is S_∞-stable. Thus, it follows that $\sigma^{-1} \cdot (\sigma \cdot b) = b \in I$.

We now prove the forward direction. Suppose that $V(I) = X_1 \cup X_2$, for X_1, X_2 S_∞-stable proper closed subsets. Then, since $X_1, X_2 \subsetneq V(I)$, it follows that $I(X_1) \supsetneq I$, and $I(X_2) \supsetneq I$. Choose $f_1 \in I(X_1), f_2 \in I(X_2)$ (which implies that $\sigma \cdot f_2 \in I(X_2)$ for all $\sigma \in S_\infty$), but $f_1, f_2 \notin I$. However, $f_1(\sigma \cdot f_2) \in I$ $\forall \sigma$, violating the condition that I is c-prime. □

Now we are ready to prove the theorem.

Proof. Fix J a c-prime radical ideal. From the previous Lemma, we know that $V(J)$ is c-irreducible if and only if $I(V(J))$ is c-prime. Since our field is algebraically closed, $I(V(J)) = \sqrt{J}$. Since J is radical, $\sqrt{J} = J$, so $I(V(J)) = J$. Thus, as in Theorem 2.24, V and I are well defined and are natural inverses of each other. As before, Lemma 2.11 guarantees that this bijection is inclusion reversing. □

The goal of this paper is to classify c-prime radical ideals of $k[X, Y]$, where S_∞ acts on each x_i and y_i. Defining c-prime and c-irreducible in the context of this ring still yields the above result, by similar logic. Through the above result, we have reduced the problem of classifying c-prime radical ideals to classifying c-irreducible sets. We give a criterion for showing a set is c-irreducible below:

Lemma 2.34. *Suppose $X \subset \mathbb{A}^\infty$ is S_∞-stable and closed. Let $X = \bigcup X_i$ denote its decomposition into irreducible sets. If the S_∞ orbit of an irreducible component is dense in X, then X is c-irreducible.*

Proof. First, we verify that our criterion is well defined. Recall that X has a decomposition into irreducible sets, though this decomposition need not be finite. S_∞ acts on the set of irreducible components, since if X_i is irreducible, it is clear that $\sigma \cdot X_i$ is still irreducible (if it wasn't, and was the union of X_1', X_2', then the union of the inverse image of these would equal X_i, a contradiction to X_i's irreducibility).

Now, suppose that X is not c-irreducible. Then, $X = Y \cup Y'$ for Y, Y' S_∞-stable closed proper subsets. let $\bigcup Y_i, \bigcup Y_i'$ denote their respective decomposition into irreducibles. Now, note that each irreducible component of Y and Y_i are still irreducible components of X. Furthermore, $X_i \subset Y$ or $X_i \subset Y'$. If not, then $(X_i \cap Y) \cup (X_i \cap Y') = X_i$, where $X_i \cap Y$ and $X_i \cap Y'$ are both closed and proper subsets of X_i, contradicting that X_i is irreducible. Choose X_i to be an irreducible component who's orbit is dense, and without loss of generality $X_i \subset Y$. Take the S_∞ orbit of X_i. Since Y is S_∞-stable, $(X_i)_{S_\infty} \subset Y$. By hypothesis, $(X_i)_{S_\infty}$ is dense in X, and by X_i being closed, it follows that $Y = X$. This is a contradiction, as Y is a proper subset of X. It follows that X is c-irreducible. □

From this lemma, it is sufficient to verify that the S_∞ orbit of some irreducible component of a set is dense to verify that the set itself is c-irreducible. We now develop a tool to determine density.

Recall that $\mathbb{A}^\infty$ can be realized as the inverse limit of finite dimensional vector spaces. Thus, we can induce the inverse limit topology over $\mathbb{A}^\infty$ and relate it to the Zariski Topology. In the inverse limit topology, the sequence $\{a_i\}_{i\in I}$ converges to some a if $\forall$ finite $J \subset I$ $\exists N \in \mathbb{N}$, such that $a_i|_J = a|_J$ for all $i \geq N$ (recall that a_i, a are all sequences with coefficients in k). This can be thought of as a point-wise limit of sequences.

Lemma 2.35. *Any Zariski-closed set over $\mathbb{A}^\infty$ is closed under the inverse limit topology.*

Proof. Suppose $X \subset \mathbb{A}^\infty$ is Zariski closed. Choose $a_1, a_2, \dots \in X$ such that $a_i \to a$ as $i \to \infty$ in the inverse limit topology. Choose $f \in I(X)$, and suppose that f uses variables $\{x_i\}_{i\in J}$, where J must be finite, as f is a polynomial. $\exists N$ such that $a|_J = a_i|_J$ for all $i \geq N$, implying that $f(a_i) = f(a)$ for these a_i. Thus, $f(a) = 0$, since $f(a_i) = 0$. Since this is true $\forall f \in I(X)$, it follows that $a \in X$. Thus, X contains all of its limit points, and is closed. □

This yields a useful corollary.

Lemma 2.36. *If a set $X \subset \mathbb{A}^\infty$ is dense in Y with respect to the inverse limit topology, then it is dense with respect to the Zariski Topology.*

Proof. If X is closed under the inverse limit topology, then $X = Y$, so X is clearly dense under any topology, namely the Zariski Topology. Thus, we consider the case where X is open. The closure of X under the inverse limit of topology contains Y; we want to show this holds in the Zariski case. Let $\overline{X}$ denote the Zariski Closure of X. Well, $\overline{X}$ is closed under the Zariski Topology, so it is closed under the inverse limit topology by the previous Lemma. Thus, $\overline{X}$ is a closed set which contains X. Since Y is contained in the closure of X under the inverse limit topology, it follows that Y is contained the smallest closed set that contains X. Thus, Y is contained in $\overline{X}$, and X is dense in Y with respect to the Zariski Topology. □

Thus, to check denseness under the Zariski Topology, it is sufficient to check denseness with respect to the inverse limit topology.

3. Univariate case

We first review the classification of c-prime radical ideals in the case where $k[X] = k[x_1, x_2, \ldots]$. This will be referred to as the "univariate" case, and the "bivariate" denotes the case where $R = k[X, Y]$. The univariate case has been solved by A. Snowden and R. Nagpal in [1]. We discuss some of their findings and provide some illustrative examples below.

3.1. **A Cofinal Closed Set.** Recall the ideal $I_n = (\Delta_n)_{S_\infty}$ from example 2.9. We can prove that $V(I_n)$ is a cofinal closed, c-irreducible set in affine space. That is, for any X S_∞-stable and closed, $X \subset V(I_n)$ for some n. Because I_n is c-prime (and radical), we can cite the bijection in 2.32 and conclude that $V(I_n)$ is cofinal from the following Lemma:

Lemma 3.1. *For any nonzero S_∞-stable ideal $I \subset R$, $\Delta_n \in I$ for some n.*

Proof. Choose $f \in I$ nonzero, and let x_m denote the maximal formal variable (with respect to index) contained in f. We can rewrite

$$f = c_n x_1^n + \{\text{lower order terms}\}$$

for $c_n \in k[x_2, x_3, \ldots, x_m]$. It follows then that

$$\sum_{\sigma \in S_{1,m+1,\ldots,m+n}} \operatorname{sgn}(\sigma)\sigma(\Delta_{m+1,\ldots,m+n} f) \in I$$

Where $S_{1,m+1,\ldots,m+n}$ is the symmetric group on $1, m+1, \ldots, m+n$ and $\Delta_{m+1,\ldots,m+n}$ is the discriminant on $x_{m+1} \ldots, x_{m+n}$. This being contained in I follows from the fact that I is S_∞-stable. Well, by Lemma 2.6,

$$\sum_{\sigma \in S_{1,m+1,\ldots,m+n}} \operatorname{sgn}(\sigma)\sigma(\Delta_{m+1,\ldots,m+n}f) = c_n n! \Delta_{1,m+1,\ldots,m+n}$$

This implies that I contains a product of discriminants of disjoint sets of variables. It follows then that the discriminant of ALL these variables is contained in I, since this product divides it. $\square$

It follows from this lemma that for any c-irreducible set $X \subset \mathbb{A}^\infty$, if $(a_1, a_2, \ldots,) \in X$, then each a_i can take only finitely many distinct values (this follows from example 2.9). Let $a_1, \ldots, a_r$ denote these distinguished values. It makes sense then that we can apply relations on these a_i.

(1) First, we impose combinatorial relations on each a_i. That is, we say there can only be at most n_i appearances of a_i for each i.

Example 3.2. If $X \subset V(I_4)$ and has distinguished values a_1, a_2, a_3, we can impose that a_3 is in the sequence at most 2 times, and a_2 is in the sequence at most 5 times.

These can be parameterized by partitions of ∞.

Definition 3.3. We say $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition of ∞ provided that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$, and $\lambda_1 + \cdots + \lambda_r = \infty$. Note that this implies that $\lambda_1 = \infty$, but every other λ_i could be finite, or infinite.

In Example 3.2, X is parameterized by the partition $\lambda = (\infty, 5, 2)$. In this instance we would say that X is ***of type*** λ. By maximality of $V(I_n)$, we know that any closed X is of type λ for some arbitrary λ.

(2) Second, we consider applying algebraic relations to each a_i.

Example 3.4. If $X \subset V(I_5)$ and has distinguished values a_1, a_2, a_3, a_4, we can impose that $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$.

This example in particular requires the distinguished variables to satisfy a symmetric polynomial, and thus is easily verified to be closed. In fact, X is the zero set of the S_∞ orbit of the following polynomial:

$$\Delta_4(x_1^2 + x_2^2 + x_3^2 + x_4^2 - 1)$$

To show this, choose $(c_1, c_2, \ldots,)$ in this polynomial's S_∞ orbit's zero set. If this polynomial is zero, either Δ_4 is zero, or $c_1^2+c_2^2+c_3^2+c_4^2-1 = 0$. Note that since we are taking the S_∞ orbit, this relation is applied on any 4 c_i we choose in the sequence. If Δ_4 is nonzero, then each of

c_i, c_j, c_k, x_ℓ are distinct for any c_i, c_j, c_k, c_ℓ in the sequence. Since there are only 4 distinguished variables, we can identify each c_i with each a_i. Then, we force these c_i to obey the polynomial relation we want. Since the polynomial is symmetric, it does not matter which c_i is identified with which a_i. In general, though, this is a concern, and is treated in a more general context in section 4.2.

It remains to be shown that any set of type λ is c-irreducible, and that imposing algebraic relations (as in, forcing the distinguished variables to satisfy a polynomial) yield an irreducible set. The former was proven in [1], which we cite, and the latter was also proven in [1], but we will prove it in a more general context in section 4.2.

3.2. **A Classification Theorem for c-Prime Radical Ideals.** In [1], A. Snowden and R. Nagpal proved that the two relations above are the only relations one can place on c-irreducible sets. This gives us the following classification. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition, and let X be of type λ Let G_λ be a subgroup of S_r that preserves λ, and let $U_r \subset \mathbb{A}^r$ be the (Zariski Open) subset consisting of points where all coordinates are distinct.

Theorem 3.5. *There exists a bijection between S_∞-stable closed sets of $\mathbb{A}^\infty$ and the set of pairs (λ, Z) for λ a partition of infinity and Z is a G_λ-stable closed subset of U_r.*

One way to think of this is that any S_∞-stable variety (of infinite dimension) can be identified with a finite dimensional variety that is parameterized by a partition.

4. Bivariate case

We now move on to classifying the c-radical prime ideals of $k[X, Y] = k[x_1, x_2, \dots, y_1, y_2, \dots]$. Recall that S_∞ acts on $k[X, Y]$ by permuting both sets of formal variables simultaneously. While $k[X, Y] \cong k[X]$ as rings (just map $x_i \to x_{2i-1}, y_i \to x_{2i}$), they are **not** isomorphic as G-sets.

Example 4.1. The ideal $I = (x_1, x_2, \dots) \in k[X, Y]$ is S_∞-stable, but under the isomorphism above, its image $(x_1, x_3, x_5, \dots)$ is clearly not S_∞-stable in $k[X]$.

It is clear that any S_∞-stable ideal in $k[X]$ is stable in $k[X, Y]$, but this example illustrates that the converse does not hold. However, if we relate $k[X]$ and $k[X, Y]$ by inclusion and projection maps, then any S_∞-stable ideal in $k[X]$ injects into an S_∞-stable ideal in $k[X,Y]$, and any S_∞-stable ideal in $k[X,Y]$ projects down to an S_∞-stable ideal in $k[X]$.

Lemma 4.2. *If a set $X \subset \mathbb{A}^\infty$ is c-irreducible under the Zariski Topology of $k[X]$, it is c-irreducible under the Zariski Topology of $k[X,Y]$.*

Proof. If X is c-irreducible, then $X = V(I)$ for some $I \subset k[X]$ c-prime and radical. $V(I) = V(i(I))$, where i is the inclusion map $i : k[X] \hookrightarrow k[X,Y]$. $i(I)$ is clearly still c-prime radical. Thus, X is c-irreducible under the Zariski Topology of $k[X,Y]$. □

Thus, we can conclude that if X is the vanishing set of some polynomial in just x_i terms (or just y_i terms), they fall under the classification in the univariate case. Thus, we only need to apply restraints on those which are bivariate.

4.1. **c-Irreducible Sets.** We provide some examples of bivariate c-irreducible sets.

Example 4.3. Define

$$X := \{(a_1, a_2, \dots, b_1, b_2, \dots) \in \mathbb{A}^\infty \mid [a_1, a_2, \dots], [b_1, b_2, \dots] \text{ are linearly dependent}\}$$

Recall that the two vectors are linearly dependent if only if $\det \begin{bmatrix} a_i & b_i \\ a_j & b_j \end{bmatrix} = 0$ $\forall i, j \in \mathbb{N}$. This is equivalent to saying that each sequence satisfies the polynomial $x_i y_j - x_j y_i$ for all i, j. Thus, the S_∞ orbit of the following polynomial

$$x_1 y_2 - x_2 y_1$$

has zero set X, so X is closed. Furthermore, $x_1 y_2 - x_2 y_1$ is prime in $k[X,Y]$, implying that its corresponding zero set is irreducible. Irreducibility implies c-irreducibility for S_∞-stable ideals, so X is c-irreducible.

Example 4.4. Define

$$Z_{n,f} = \{(a_1, a_2, \dots, b_1, b_2, \dots) \in \mathbb{A}^\infty \mid f(a_i, b_i) \text{ takes} \leq n \text{ values}\}$$

For some fixed $f \in k[x,y]$ non-constant. This is closed; here's a proof for $n = 1$. If $f(a_i, b_i) = c$ for all a_i, b_i, then we have

$$f(a_1, b_1) = f(a_2, b_2)$$

And this holds for any a_i, b_i, a_j, b_j. Thus the S_∞ orbit of the polynomial

$$f(x_1, y_1) - f(x_2, y_2)$$

has the zero set $Z_{1,f}$. In the case for $n=2$, $f(a_1,b_1)=c_1,c_2$, as does $f(a_2,b_2)$. Thus, $f(a_3,b_3)$ must equal one of $f(a_2,b_2)$ or $f(a_1,b_1)$, or $f(a_1,b_1)=f(a_2,b_2)$. Thus, we have the S_∞ orbit of the polynomial

$$(f(x_1,y_1)-f(x_2,y_2))(f(x_2,y_2)-f(x_3,y_3))(f(x_1,y_1)-f(x_3,y_3))$$

This is just the 3 discriminant on $f(x_i,y_i)$. This procedure works in the general case.

Suppose $f(a_i,b_i)$ can take on one of n values, $c_1,\dots,c_n$. Then, if $f(a_1,b_1),\dots,f(a_n,b_n)$ are all distinct, $f(a_{n+1},b_{n+1})$ must equal $f(a_i,b_i)$ for some $i\le n$. Define the set

$$[N+1]:=\{f(x_i,y_i)\mid i\le n+1\}$$

Let Δ_{N+1} denote the discriminant of the elements of the set $[N+1]$. The S_∞ orbit of this has a zero locus equal to $Z_{n,f}$. This shows that $Z_{n,f}$ is closed, so we now only need to check irreducibility. Choose $A_1,A_2,\dots,A_n\subset\mathbb{N}$ such that

$$\bigsqcup_{i=1}^{n} A_i=[\infty]$$

Using these, define

$$X_A:=\{(a_1,a_2,\dots,b_1,b_2,\dots,)\mid f(a_i,b_i)\text{ is constant on each }i\in A_1,\dots,i\in A_n\}$$

X_A is specifically the set of sequences where the n distinguished values are taken in specific indices in the sequence. $X_A\subset Z_{n,f}$, and X_A is closed and irreducible. Taking the union over all such possible A_i such that $A:=A_1\sqcup\cdots\sqcup A_n=[\infty]$, we find that

$$Z_{n,f}=\bigcup_A X_A$$

Choose A such that each $|A_i|=\infty$. Taking the S_∞ orbit of X_A yields the union of all X_A where each $|A_i|=\infty$. The set of all these is dense with respect to the inverse limit topology in $Z_{n,f}$, and by Lemma 2.36, it is dense with respect to the Zariski topology. By Lemma 2.34, we can conclude that $Z_{n,f}$ is irreducible.

4.2. **Defining the Identification Polynomial.** As seen in example 3.4, it may be useful to apply algebraic constraints to sets of type λ and verify that they are still closed. Note that in the bivariate case, being type λ means that the x_i terms and the y_i terms are both of type λ, as in the univariate case.

Example 4.5. Choose $X \subset \mathbb{A}^\infty$ such that each x_i term and y_i term can take on only r many distinct values. Let $a_1, \dots, a_r$ and $b_1, \dots, b_r$ be these distinguished values. Suppose that a_1, b_1 show up at most λ_1 times in the sequence, and constrain a_i, b_i by λ_i similarly. In this case, for $\lambda = (\lambda_1, \dots, \lambda_r)$, X is of type λ.

Suppose we want to apply constraints on these $a_1, \dots, a_r, b_1 \dots, b_r$. Our goal is to find a polynomial in $k[X, Y]$ who's zero set satisfies those constraints. By defining the polynomial, we can only constrain x_i terms and y_i terms. Thus, it would be useful to have a polynomial such that, when it is nonzero over its S_∞ orbit, x_i is identified with a_i and y_i with b_i for each $i \leq r$. Thus, we can apply those constraints to x_i, y_i within the polynomial. We develop the theory behind this identification polynomial now, and give some examples after.

The construction is quite complicated, so we illustrate it first with an example. Let $\lambda = (\infty, \infty, 5, 2, 2, 1)$. Label these distinguished values $a_1, \dots, a_6$ respectively. We prove this case then move onto the general case, where X is of type λ for any λ. First, we roughly describe the process.

(a) distinguish sets of x_i such that at least 1 member of the set is of finite type.
(b) distinguish specific x_i as being of finite type, and others of being of infinite type.
(c) distinguish finite type x_i as being $a_3, \dots, a_6$ specifically, and distinguish infinite type x_i of being a_1, a_2 specifically.
(d) proceed similarly in the y case.

Here, a_i being of finite type means that λ_i is finite, and if $\lambda_i = \infty$, then a_i is said to be of infinite type. First, we proceed with step (a), introducing some notation along the way. Let $\Delta^x_{i,j,k}$ be the discriminant of x_i, x_j, x_k. If this is nonzero, then x_i, x_j, x_k are all distinct. Similarly, for $S \subset \mathbb{N}$, define Δ^x_S as the discriminant of all x_i such that $i \in S$. Define $\Delta^y_{i,j,k}, \Delta^y_S$ similarly as well.

Note that there are 2 values of infinite type. Thus, if $\Delta^x_{1,2,3}$ is nonzero, then at least one of x_1, x_2, x_3 must be of finite type. Similarly, if $\Delta^x_{1,2,3}\Delta^x_{4,5,6}$ is nonzero, then at least one of x_1, x_2, x_3 and at least one of x_4, x_5, x_6 must be of finite type.

Now onto step (b). There are only $5 + 2 + 2 + 1 = 10$ terms of finite type, so we can consider the product

$$\prod_{i=0}^{9} \Delta^x_{3i+1,3i+2,3i+3} \tag{4.1}$$

If this is nonzero, then at least one of $x_{3i+1}, x_{3i+2}, x_{3i+3}$ is of finite type for each $0 \leq i \leq 9$. However, since there are only 10 possible terms of finite type, and there are 10 such sets of 3 terms, it follows that if equation (4.1) is nonzero, then precisely one of $x_{3i+1}, x_{3i+2}, x_{3i+3}$ is of finite type. Furthermore, $x_{31}, x_{32}, \dots$ are all of infinite type, since every finite type term must belong to indices ≤ 30.

There are two terms of infinite type, so we want to distinguish those as well. Note that $(x_{31} - x_{32})$ is nonzero precisely when $x_{31} \neq x_{32}$. Thus, we multiply (4.1) by $(x_{31} - x_{32}) = \Delta^x_{31,32}$, implying that when the product is nonzero, the inequality holds, and x_{31}, x_{32} are distinct. Without loss of generality, we identify a_1 with x_{31}, and a_2 with x_{32}.

We now use these to distinguish the terms of finite type. For the sake of argument, let's say we want to distinguish each x_{3i+1} of being of finite type. In this case, we multiply by

$$(x_{31} - x_{3i+1})(x_{32} - x_{3i+1})$$

If this polynomial is nonzero, this ensures that x_{3i+1} is not of infinite type, since it is not a_1 or a_2. We thus multiply in this product for each $0 \leq i \leq 9$. To recap, we currently have the product

$$\left(\prod_{i=0}^{9} \Delta^x_{3i+1,3i+2,3i+3}\right) \left(\Delta^x_{31,32}\right) \left(\prod_{i=0}^{9} (x_{31} - x_{3i+1})(x_{32} - x_{3i+1})\right)$$

Allowing some redundancy, note that $\Delta^x_{31,32}(x_{31} - x_{3i+1})(x_{32} - x_{3i+1}) = \Delta^x_{31,32,3i+1}$ (there may be a negative sign here, but it doesn't matter, since $f = 0 \iff -f = 0$), implying that we can rewrite the equation as

$$\prod_{i=0}^{9} \Delta^x_{3i+1,3i+2,3i+3} \Delta^x_{31,32,3i+1} \tag{4.2}$$

This equation enforces that each x_{3i+1} for $0 \leq i \leq 9$ is of finite type, and x_{31} is identified with a_1 and x_{32} with a_2. We now wish to identify each x_{3i+1} with a_3, a_4, a_5, a_6.

We now move onto step (c), where we want to do exactly that. a_1, a_2 have already been identified, so we only have the finite cases left. Note that a_4, a_5 are both "the same", so once we show these are distinct, we can without loss of generality identify one variable with one, and one with the other. We identify a_6 first, as it is the easiest to do so and illustrative of the method used. Consider the polynomial

$$\prod_{i=1}^{9}(x_{3(0)+1}-x_{3i+1})$$

(Note the index change). If this is nonzero, then $x_{3(0)+1}$ must be distinct from the other 9 variables of finite type. Since only a_6 shows up exactly once, this would evaluate to zero if $x_{3(0)+1}=a_3,a_4,a_5$. Thus, this being nonzero identifies $x_{3(0)+1}$ with a_6. Similarly, if

$$\prod_{i=0,i\neq 1,2}^{9}(x_{3(1)+1}-x_{3i+1})$$

is nonzero, then $x_{3(1)+1}$ shows up at most twice. Since a_6 is already identified, $x_{3(1)+1}$ must show up twice, where $x_{3(1)+1}=x_{3(2)+1}$. Thus we identify these with a_5, and similarly with the polynomial

$$\prod_{i=0,i\neq 3,4}^{9}(x_{3(3)+1}-x_{3i+1})$$

Identify $x_{3(3)+1}=x_{3(4)+1}$ with a_4. Note that we don't even need to verify that $x_{3(3)+1}=x_{3(4)+1}\neq x_{3(1)+1}=x_{3(2)+1}$, since this is already taken care of by the above two polynomials. Thus, the remaining terms $x_{3(5)+1},x_{3(6)+1},x_{3(7)+1}$ $,x_{3(8)+1},x_{3(9)+1}$ must all be identified with a_3.

Taking the product of all the above equations with equation (4.2), we get

$$\left(\prod_{i=0}^{9}\Delta^x_{3i+1,3i+2,3i+3}\Delta^x_{31,32,3i+1}\right)\left(\prod_{i=1}^{9}(x_{3(0)+1}-x_{3i+1})\right)$$
$$\left(\prod_{i=0,i\neq 1,2}^{9}(x_{3(1)+1}-x_{3i+1})\right)\left(\prod_{i=0,i\neq 3,4}^{9}(x_{3(3)+1}-x_{3i+1})\right) \tag{4.3}$$

Denote $I^x_\lambda=(31),(32),(16,19,22,25),(10,13),(4,7),1$. The notation here signifies which indices are identified with each a_i. Let the polynomial above be denoted $P_{I^x_\lambda}$, to signify this as the polynomial which induces the identification I^x_λ. This is an ***Identification Polynomial*** of λ on x, who's specific identification is clarified by I. Since we are working up to symmetry (as in, we are taking the S_∞ orbit), these values all could be anything we want them to be. That is, an identical process can be used to establish any identification we want. Proceeding similarly with the y case, we have the polynomial

$$P_{I^x_\lambda}P_{I^y_\lambda} \tag{4.4}$$

Which fixes our identifications of each x_i, y_i to each $a_1, \dots, a_6$, $b_1, \dots, b_6$. This completes step (d). Observe that everything above can be generalized to an arbitrary partition. Let $\lambda = (\lambda_1, \dots, \lambda_{m+n})$ for $\lambda_1 = \dots, \lambda_m = \infty$, and the rest finite. The generalized version of equation (2) looks like

$$\prod_{i=0}^{\lambda_{m+1}+\cdots+\lambda_n} \Delta^x_{[m+1]_i^\lambda} \Delta^x_{S_i^\lambda}$$

For $[m+1]_i^\lambda = \{(m+1)i+1, \dots, (m+1)i+(m+1)\}$, and $S_i^\lambda = \{(\lambda_{m+1} + \cdots + \lambda_n)(m+1)+1, \dots, (\lambda_{m+1} + \cdots + \lambda_n)(m+1)+m, (m+1)i+1\}$

The first discriminant is just setting aside blocks of distinct elements of size $m+1$, implying that at least 1 element in each block is of finite type. Taking $\lambda_{m+1} + \cdots + \lambda_n$ blocks implies that exactly 1 element in each block is of finite type. The second discriminant enforces that the smallest index in each block is the element of finite type, since it is different than the m elements of infinite type.

and we can repeat the process in step (c) as many times as we want to complete the identification, then just copy the polynomial in terms of y and take the corresponding product to complete step (d).

Lemma 4.6. *Let x, y be of type $\lambda = (\lambda_1, \dots, \lambda_r)$, and let $a_1, \dots, a_r, b_1, \dots, b_r$ be the distinguished values, then the set defined by the constraint that*

$$\begin{bmatrix} a_i \\ \vdots \\ a_r \end{bmatrix}, \begin{bmatrix} b_i \\ \vdots \\ b_r \end{bmatrix}$$

are linearly dependent (where i for $i \leq r$ denotes the first index in the partition which is of finite type) is a closed set.

Proof. Let $P_{I^x_\lambda} P_{I^y_\lambda}$ denote an identification polynomial where each a_j, b_j have an identification with matching indices (i.e. x_j is identified with a_j if and only if y_j is identified with b_j). We now want to verify the linear dependence condition. This is equivalent to saying that $\det \left[\begin{smallmatrix} a_k & b_k \\ a_\ell & b_\ell \end{smallmatrix}\right] = 0$ for each $i \leq k, \ell \leq r$. Consider the set

$$V^0_{I_\lambda} := \{P_{I^x_\lambda} P_{I^y_\lambda}(x_k y_\ell - y_k x_\ell) \mid i \leq k, \ell \leq r\}$$

For a given $p \in V^0_{I_\lambda}$, $p = 0$ if and only if $a_k b_\ell - b_k a_\ell = 0$ for fixed k, ℓ. The intersection of all such zero loci thus satisfy the condition above. To finish the proof, we need to alleviate concerns of matching distinguished elements. If x_j, y_j are both identified to some a_k, b_ℓ, then the S_∞ action will affect a_k, b_ℓ similarly. This added condition on the identification needs to be alleviated; to show the above set is closed, we need a set of polynomials who's zero locus is precisely that set. If there are added restrictions, the zero set of our polynomial will not match. In the case of $V^0_{I_\lambda}$, the indices of the x identifications and y identifications match exactly. Thus, we introduce the following notation:

$$V^z_{I_\lambda} := \{P_{I^x_\lambda} P_{(I')^y_\lambda}(x_k y_\ell - y_k x_\ell) \mid i \leq k, \ell \leq r\}$$

Where the identification $(I')^y_\lambda$ differs from the identification I^y_λ by z values. Note that there are only $\lambda_{m+1}+\cdots+\lambda_n$ identifiable elements, implying that this is the maximal such z. Furthermore, setting $z = 0$ agrees with our definition of $V^0_{I_\lambda}$ as before. Taking the union of all such $V^z_{I_\lambda}$ gives us

$$V := \bigcup_{z=0}^{\lambda_{m+1}+\cdots+\lambda_n} V^z_{I_\lambda}$$

Taking the S_∞ orbit of V yields our desired zero set. □

Lemma 4.7. *Let x be of type $\lambda = (\lambda_1, \dots, \lambda_r)$, and let y be of type $\mu = (\mu_1, \dots, \mu_s)$. Let $a_1, \dots, a_r, b_1, \dots, b_s$ be their distinguished values. Fix $S \subset k[x_1, \dots, x_r, y_1, \dots, y_s]$ finite. The set $X \subset \mathbb{A}^\infty$ defined by $a_1, \dots, a_r, b_1, \dots, b_s$ satisfying every polynomial in S is closed.*

Proof. Let $P_{I^x_\lambda}, P_{J^y_\mu}$ denote the identification polynomials for x over λ and y over μ respectively, where x_i is identified with a_i and y_j with b_j for each $i \leq r, j \leq s$. It follows then that the product $P_{I^x_\lambda} P_{J^y_\mu}$, when nonzero, fixes both identifications. Fix $f \in S$. Thus, it follows that the polynomial

$$P_{I^x_\lambda} P_{J^y_\mu} f$$

is zero precisely when $a_1, \dots, a_r, b_1, \dots, b_s$ satisfy f. Thus, if our point satisfies the following set of polynomials:

$$T = \{P_{I^x_\lambda} P_{J^y_\mu} f \mid f \in S\}$$

It follows that $a_1, \dots, a_r, b_1, \dots, b_s$ satisfies all polynomials in S. Taking the S_∞ orbit of T yields our desired zero set. □

4.3. **A Cofinal Closed Set.** We now want to find a cofinal closed set, in the same vain as Section 3.1, but for the bivariate case. Consider a more general version of example 4.4:

$$Z_{n,d} = \{(a_1, a_2, \dots, b_1, b_2, \dots) \in \mathbb{A}^\infty \mid \exists f \in k[x,y] \text{ of degree} \leq d \text{ such that } f(a_i, b_i) \text{ takes} \leq n \text{ distinct values}\}$$

To prove this is closed, we need to rely on a general result about projection maps being proper.

Theorem 4.8. *Let $\pi : \mathbb{A}^\infty \times \mathbb{P}^d \to \mathbb{A}^\infty$ be the standard projection map. If $X \subset \mathbb{A}^\infty \times \mathbb{P}^d$ is closed, then $\pi(X)$ is closed in $\mathbb{A}^\infty$.*

Where $\mathbb{P}^d$ is d-projective space.

Lemma 4.9. *$Z_{n,d}$ is closed.*

Proof. Define

$$X = \left\{ (a_1, a_2, \dots, b_1, b_2, \dots, \alpha_0, \dots, \alpha_d) \;\middle|\; f_\alpha(a_i, b_i) \text{ takes} \leq n \text{ distinct values} \right\} \subset \mathbb{A}^\infty \times \mathbb{P}^d$$

Where f_α is the degree $\leq d$ polynomial with coefficients $\alpha_0, \dots, \alpha_d$. First we verify that this condition is well defined. Recall that $\mathbb{P}^d$ is just $\mathbb{A}^d$ quotiented by the following relation:

$$(\alpha_0, \dots, \alpha_d) \sim (\beta_0, \dots, \beta_d) \iff \alpha_i = \lambda \beta_i \; \forall i \leq d$$

for some $\lambda \neq 0$. It follows that for any nonzero λ, $f_{\lambda\alpha} = \lambda f_\alpha$. This implies that if f_α takes on at most n distinct values, namely $c_1, \dots, c_n$, then $f_{\lambda\alpha}$ also takes on at most n distinct values, namely $c_1/\lambda, \dots, c_n/\lambda$ (the converse of this statement also follows). Thus this restriction is well defined over $\mathbb{P}^d$.

We now verify that X is closed. Observe that X is a (projective) variety over $k[X, Y, z_0, \dots, z_d]$. Since f_α is fixed, this proof is nearly identical to Example 4.4. Define the set

$$[N+1]_z = \{ f_z(x_i, y_i) \mid i \leq n+1 \}$$

Where f_z is a degree d polynomial with "coefficients" $z_1, \dots, z_d$. Let $\Delta_{[N+1]_z}$ denote the discriminant of the elements in this set. If this is zero on $(a_1, a_2, \dots, b_1, b_2, \dots, \alpha_0, \dots, \alpha_d)$, then of any $n+1$ choices of $f_\alpha(a_i, b_i)$, 2 are equal, implying that $f_\alpha(a_i, b_i)$ takes on at most distinct n values for $i \leq n+1$. Taking the S_∞ orbit of $\Delta_{[N+1]_z}$ thus yields our desired zero set, implying that X is closed.

From the previous theorem, X being closed implies that $\pi(X)$ is closed. It is clear that $\pi(X) = Z_{n,d}$, so we conclude. □

We want to show that $Z_{n,d}$ is cofinal. More precisely, we want to prove the following theorem.

Theorem 4.10. *If $X \subset \mathbb{A}^\infty$ is closed and S_∞-stable, then $\exists n, d \in \mathbb{N}$ such that $X \subset Z_{n,d}$.*

Proof. It suffices to treat the case where X is the zero set of the S_∞ orbit of a fixed polynomial f. Since polynomials have finitely many terms, we can assume that this polynomial has maximal terms x_m, y_n (with respect to index), and thus $f \in k[x_1, \ldots, x_m, y_1, \ldots, y_n]$. Choose $(a_1, \ldots, a_m, \ldots, , b_1, \ldots, b_n, \ldots) \in V(f)$. Separating the terms a_1, b_1 and $a_2, \ldots, a_m, b_2, \ldots, b_n$, we know that there exists some polynomial F_1 such that $F_1(a_1, b_1) = C$, for $C \in k$ written in terms of $a_2, \ldots, a_m, b_2, \ldots, b_n$. Note here that in this context, $a_2, \ldots, a_m, b_2, \ldots, b_n$ are constants, and the polynomial is in the variables a_1, b_1, so $F_1 \in k[x, y]$.

Before proceeding, we tackle some edge cases. It may be the case that F_1 is the zero polynomial. If this the case, we can redefine F_1 as F_i, where $F_i(a_i, b_i) = C'$ for $C' \in k$ written in $a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m$ and $b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n$, such that F_i is now nonzero. If no such F_i exists, then it follows that $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$. If this is the case, by symmetry $(a_1, \ldots, b_1, \ldots,) = (0, \ldots, 0, \ldots)$, and we have a degenerate case where any nonzero polynomial with zero scalar term will annihilate (a_i, b_i) for each i. Thus, with these out of the way, we can say without loss of generality that F_1 is nonzero.

Now note that for $i > \max(m, n)$, $(a_i, a_2, \ldots, a_m, b_i, b_2, \ldots, b_m)$ also satisfies f. This implies that $F_1(a_i, b_i) = C$, for the same C as before. Thus, for any $(a_1, \ldots, b_1, \ldots,) \in V(f)$, we have that

$$F_1(a_i, b_i) = \begin{cases} C & i = 1 \\ C & i > \max(m, n) \\ \text{other values} & \text{otherwise} \end{cases}$$

Thus, $F_1(a_i, b_i)$ takes on at most $\max(m, n)$ distinct values, so there exists N, D such that

$$X \subset Z_{N,D}$$

□

References

1. A. Snowden, R. Nagpal, *On Symmetric Ideals*, In preparation.
2. J. Draisma *Noetherianity up to Symmetry*, In: Combinatorial Algebraic Geometry. Lecture Notes in Mathematics, **2108** (2014). Springer, Cham.
3. W. Fulton *Algebraic Curves* (2008).
4. Andries E. Brouwer, Jan Draisma *Equivariant Groebner bases and the Gaussian two-factor model*, In: Math. Comp. 80: 1123-1133 (2011) arXiv:0908.1530v3 [math.AC]
5. Christopher J. Hillar, Seth Sullivant *Finite Groebner bases in infinite dimensional polynomial rings and applications* (2011) arXiv:0908.1777v2 [math.AC]
6. Seth Sullivant, *Strongly robust toric ideals in codimension 2* (2017) arXiv:1610.07476v2 [math.AC]
7. Matthias Aschenbrenner, Christopher J. Hillar *An Algorithm for Finding Symmetric Grbner Bases in Infinite Dimensional Rings* (2008) arXiv:0801.4439v1 [math.AC]
8. Christopher J. Hillar, Robert Krone, Anton Leykin *Equivariant Grbner bases* (2017) arXiv:1610.02075v2 [math.AC]

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