SYMBOLIC POWERS OF DEFINING IDEALS OF VERONESE RINGS

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Abstract. The d-th Veronese ring is the ring of polynomials such that the degree of every term is a multiple of d. We can represent these Veronese rings as the quotient ring of another polynomial ring. We are interested in the defining ideal of this quotient ring. The n-th symbolic power of a radical ideal I can be thought of as the set of polynomials which vanish up to order n along the solution set of I. Our research is concerned with the symbolic powers and the primary decomposition of the defining ideals of Veronese rings.

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1. Definitions

Definition 1.1. Let k be a field, $S_n = k[x_1, \dots, x_n]$, and $S_{n,d} = (k[x_1, \dots, x_n])_d$ be the d-th Veronese in n variables. Let

$$\Lambda = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n | \sum_i \alpha_i = d, \alpha_i \geqslant 0\},$$

and write $R = k[x_\alpha : \alpha \in \Lambda]$. There is a surjective ring homomorphism

$$R \longrightarrow S_{n,d}\ .$$

$$t_\alpha \longrightarrow x^\alpha$$

We write $I_{n,d}$ for the kernel of this map.

Note that since $S_{n,d}$ is an integral domain, $I_{n,d}$ is a prime ideal.

Definition 1.2. An ideal $I \neq (1)$ in a ring R is primary if $fg \in I$ implies $f \in I$ or $g^m \in I$ for some $m \in \mathbb{N}$.

Definition 1.3. The radical of an ideal I in a ring R is the ideal $\sqrt{I} = \{f \in R | f^n \in I \text{ for some } n \in \mathbb{N}\}$; we say that an ideal is radical if $\sqrt{I} = I$.

If Q is a primary ideal in a ring R, then $\sqrt{Q} = P$ is a prime ideal and we say that Q is P-primary.

An ideal in a Noetherian ring can always be written as a finite intersection of primary ideals.

Definition 1.4. A primary decomposition $I = \bigcap_{i=1}^{n} Q_i$ is irredundant if $\bigcap_{i \neq j} Q_i \neq I$ for each $j \in \{1, \ldots, n\}$ and $\sqrt{Q_i} \neq \sqrt{Q_j}$ for all $i \neq j$.

If Q_1 and Q_2 are P-primary, then $Q_1 \cap Q_2$ is also a P-primary ideal, so every primary decomposition can be simplified into an irredundant one.

Definition 1.5. Let R be a notherian ring and I an ideal in R with no embedded primes. The n-th symbolic power of I is the ideal defined by

$$I^{(n)} = \bigcap_{P \in Ass(R/I)} (I^n R_p \cap R) = \{f \in R | \exists s \notin \cup_{P \in Ass(R/I)} P \text{ such that } sf \in I^n\}$$

Theorem 1.6 (Zariski-Nagata)**.** Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and I a radical ideal in R. Then the n-th symbolic power of I is $I^{(n)} = \{f \in R | f$ vanishes to order $\geq n$ at every $x \in V(I)\} = \{f \in R | \frac{\partial^{k_1}}{\partial x_1^{k_1}} \ldots \frac{\partial^{k_n}}{\partial x_n^{k_n}} f \in I$ for all $\sum_{i=1}^{n} k_i < n\}$.

Note that we can use this theorem to describe the ideals $I_{n,d}$, because they are prime and thus, radical.

Definition 1.7. Let k be a field and $R = k[x_1, \ldots, x_n]$. The initial monomial $in_<(f)$ of $f \in R$ is the largest (with respect to chosen order) x^α which appears in the expression $f = \sum_{c_\alpha \neq 0} c_\alpha x^\alpha$. A subset $\{g_1, \ldots, g_m\}$ of an ideal I is a Groebner basis for I if the ideal generated by initial monomials of elements of I (denoted $in_<(I)$) is generated by $\{in_<(g_1), \ldots, in_<(g_m)\}$.

Notation 1.8. Let k be a field and R a polynomial ring over k. Let M be a matrix whose entries are polynomials in R. The ideal generated by all the n-minors of m is denoted with $I_n(M)$.

2. Basic Facts

The ideal $I_{n,d}$ is generated by elements of the form

$$t_\alpha t_\beta - t_\gamma t_\delta$$

where $t_i \mapsto x^i$ for $i = \alpha, \beta, \gamma, \delta$.

Its initial ideal $in_<(I_{n,d})$ is generated by the initial monomials of these quadratic binomials.

3. Useful exercises

Lemma 3.1. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring and $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_t})$ be generated by a subset of the variables. Then $\mathfrak{p}^{(a)} = \mathfrak{p}^a$ for all $a \geq 1$.

Proof. Without loss of generality, let $\mathfrak{p} = (x_1, \ldots, x_t)$. It is clear that $\mathfrak{p}^a \subseteq \mathfrak{p}^{(a)}$. Suppose $f = \sum c_{a_1,\ldots,a_n} x_1^{a_1} \ldots x_n^{a_n} \in \mathfrak{p}^{(a)}$. By Zariski-Nagata theorem, since $\mathfrak{p}$ is a prime ideal, we have $\frac{\partial^{\underline{b}} f}{\partial x^{\underline{b}}} \in \mathfrak{p}$ for all $|\underline{b}| < a$. If $\sum_{i=1}^{t} a_i < a$ for some monomial term of f, then there is a term $c_{a_1,\ldots,a_n} a_1! \ldots a_t! x_{t+1}^{a_{t+1}} \ldots x_n^{a_n}$ in $\frac{\partial^{a_1}}{\partial x_1^{a_1}} \ldots \frac{\partial^{a_t}}{\partial x_t^{a_t}} f$, contradicting $\frac{\partial^{a_1}}{\partial x_1^{a_1}} \ldots \frac{\partial^{a_t}}{\partial x_t^{a_t}} f \in \mathfrak{p}$. Therefore, $\sum_{i=1}^{t} a_i \geq a$ for all monomial terms of f, which implies $f \in \mathfrak{p}^a$. $\square$

Lemma 3.2. Let $X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}$ be an $m \times n$ matrix of indeterminates over $\mathbb{C}$, and $R = \mathbb{C}[X]$ be the polynomial ring in these indeterminates. Let $\mathfrak{p} = I_2(X)$ be the ideal generated by the 2×2 minors of X. Then, for all $t \leq \min\{m,n\}-1$, we have $I_{t+1}(X) \subseteq \mathfrak{p}^{(t)}$.

Proof. For $t = 1$, $I_2(X) = \mathfrak{p} \subseteq \mathfrak{p}^{(1)}$. Assume $I_t(X) \subseteq \mathfrak{p}^{(t-1)}$ for some $2 \leq t < \min\{m,n\}-1$. Let f be a $(t+1)\times(t+1)$ minor of X. By Laplace expansion, we have $f \in I_t(X) \subseteq \mathfrak{p}^{(t-1)} \subseteq \mathfrak{p}$ and $\frac{\partial f}{\partial x_{ij}} \in I_t(X) \subseteq \mathfrak{p}^{(t-1)} \subseteq \mathfrak{p}$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$. Then by Zariski-Nagata theorem, $\frac{\partial^{\underline{k}} f}{\partial x^{\underline{k}}} \in \mathfrak{p}$ for all $1 < |\underline{k}| < t$. Therefore, $f \in \mathfrak{p}^t$. □

Corollary 3.3. Let $R = \mathbb{C}[y_1, \ldots, y_l]$ be a polynomial ring and $X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}$ be an $m \times n$ matrix where each coordinate $x_{ij} = x_{ij}(y_1, \ldots y_l)$ is a polynomial in R. Let $\mathfrak{p} = I_2(X)$ be the ideal generated by the 2×2 minors of X. If $\mathfrak{p}$ is prime, then for all $t \leq \min\{m,n\}-1$, we have $I_{t+1}(X) \subseteq \mathfrak{p}^{(t)}$.

Proof. Let $f = \det \begin{bmatrix} x_{i_1 j_1} & \cdots & x_{i_1 j_{t+1}} \\ \vdots & \ddots & \vdots \\ x_{i_{t+1} j_1} & \cdots & x_{i_{t+1} j_{t+1}} \end{bmatrix}$ be a $(t+1)\times(t+1)$ minor of X. Then apply chain rule, we have $\frac{\partial f}{\partial y_s} = \sum_{u=1}^{t+1} \sum_{v=1}^{t+1} \frac{\partial f}{\partial x_{i_u j_v}} \frac{\partial x_{i_u j_v}}{\partial y_s}$ where $\frac{\partial f}{\partial x_{i_u j_v}} \in I_t(X)$ by Laplace expansion and $\frac{\partial x_{i_u j_v}}{\partial y_s} \in R$ for each integers $1 \leq u, v \leq t+1$ and $1 \leq s \leq l$. The rest of the proof is the same as in lemma 3.2. □

Lemma 3.4. Let I be a radical ideal in a polynomial ring R. Then $I^n \subseteq I^{(n)}$ for all $n \in \mathbb{N}$.

Proof. For all $f \in I^n$, $1 \cdot f \in I^n$ where 1 is not in any prime ideal. □

Lemma 3.5. Let I be a radical ideal in a polynomial ring R. Then $I^{(a)}I^{(b)} \subseteq I^{(a+b)}$ for all $a, b \geq 1$.

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be associated primes of I. Suppose $f \in I^{(a)}$ and $g \in I^{(b)}$. Then there exist $s \notin \cup_{i=1}^k \mathfrak{p}_i$ and $t \notin \cup_{i=1}^k \mathfrak{p}_i$ such that $sf \in I^a$ and $tg \in I^b$. Then $st \notin \cup_{i=1}^k \mathfrak{p}_i$ and $stfg \in I^{a+b}$, which gives $fg \in I^{(a+b)}$. Therefore, $I^{(a)}I^{(b)} \subseteq I^{(a+b)}$. □

Lemma 3.6. Let k be a field and $R = k[x_1, \ldots, x_n]$ be a polynomial ring. Let I and J be ideals in R. Then $in_<(I)in_<(J) \subseteq in_<(IJ)$.

Proof. Let $\{f_1, \ldots, f_m\}$ and $\{g_1, \ldots, g_l\}$ be Groebner basis for I and J, respectively. Then $in_<(I) = (in_<(f_1), \ldots, in_<(f_m))$ and $in_<(J) = (in_<(g_1), \ldots, in_<(g_l))$, so $in_<(I)in_<(J)$ is generated by $\{in_<(f_i)in_<(g_j) | 1 \leq i \leq m, 1 \leq j \leq l\}$ where each generator $in_<(f_i)in_<(g_j) = in_<(f_i g_j) \in in_<(IJ)$. □

Corollary 3.7. Let k be a field and $R = k[x_1, \ldots, x_n]$ be a polynomial ring. Let I be an ideal in R. Then $in_<(I)^b \subseteq in_<(I^b)$ for all $b \geq 1$.

Lemma 3.8. Let I be a radical ideal in a polynomial ring $R = \mathbb{C}[x_1, \dots, x_n]$. If $in_<(I)$ is radical, then $in_<(I^{(a)}) \subseteq in_<(I)^{(a)}$ for all $a \geq 1$.

Proof. Suppose $g \in I^{(a)}$ and let $f = in_<(g) = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. For any $k = (k_1, \dots, k_n)$ with $\sum_{i=1}^n k_i < a$, by Zariski-Nagata theorem, we have $\frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}} g \in I$.

(1) $\frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}} f = 0 \in in_<(I)$.

(2) $\frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}} f \neq 0$, then for each nonzero monomial term in $\frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}} g$ with orignial vector exponent $(\beta_1, \dots, \beta_n) \leq (\alpha_1, \dots, \alpha_n)$, its vector exponent after differentiation is $(\beta_1 - k_1, \dots, \beta_n - k_n) \leq (\alpha_1 - k_1, \dots, \alpha_n - k_n)$, so $\frac{\partial^{k_n}}{\partial x_n^{k_n}} f$ is the initial term of $\frac{\partial^{k_n}}{\partial x_n^{k_n}} g$ and then $\frac{\partial^{k_n}}{\partial x_n^{k_n}} f \in in_<(I)$.

Therefore, by Zariski-Nagata theorem, since $in_<(I)$ is radical, we have $f \in in_<(I)^{(a)}$. □

4. Macaulay2 computations

Macaulay2 computation verifies the following primary decompositions: $I_{3,3}^2 = I_{3,3}^{(2)} \cap \mathfrak{m}^4$; $I_{3,3}^3 = I_{3,3}^{(3)} \cap \mathfrak{m}^6$; $I_{3,4}^2 = I_{3,4}^{(2)} \cap \mathfrak{m}^4$; $I_{4,3}^2 = I_{4,3}^{(2)} \cap \mathfrak{m}^4$.

We use Macaulay2 to find the smallest a such that $I_{n,d}^{(a)} \subseteq I_{n,d}^b$ for some n, d, b.

n	d	b	a	n	d	b	a
2	4	2	3	3	2	2	3
		3	4			3	4
		4	5			4	5
		5	7			5	7
		6	8			6	8
		7	9			7	9
2	5	2	3			8	11
		3	4	3	3	2	3
		4	5			3	4
2	6	2	3	4	2	2	3
		3	4			3	4
2	7	2	3	5	2	2	3
		3	4				
2	8	2	3				

5. Conjectures and Results

Proposition 5.1. The ideal $I_{n,d}$ where $n, d \geq 2$ can be written as the ideal generated by 2-minors of a matrix of n rows and $\binom{n+d-2}{d-1}$ columns, where the columns of this matrix are of the form

$$\begin{bmatrix} t_{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n}} \\ t_{x_1^{\alpha_1 - 1} x_2^{\alpha_2 + 1} x_3^{\alpha_3} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n}} \\ t_{x_1^{\alpha_1 - 1} x_2^{\alpha_2} x_3^{\alpha_3 + 1} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n}} \\ \cdots \\ t_{x_1^{\alpha_1 - 1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots x_{n-1}^{\alpha_{n-1} + 1} x_n^{\alpha_n}} \\ t_{x_1^{\alpha_1 - 1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n + 1}} \end{bmatrix}$$

where $\sum_{i=1}^{n} \alpha_i = d$.

Proof. Consider the $n\times\binom{n+d-2}{d-1}$ matrix $M_{n,d} = \begin{bmatrix} t_{x_1^d} & t_{x_1^{d-1}x_2} & \cdots & t_{x_1x_n^{d-1}} \\ t_{x_1^{d-1}x_2} & t_{x_1^{d-2}x_2^2} & \cdots & t_{x_2x_n^{d-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x_{x_1^{d-1}x_n} & t_{x_1^{d-2}x_2x_n} & \cdots & t_{x_n^d} \end{bmatrix}$.

The i-th row of $M_{n,d}$ consists of all t_α where α is a degree d monomial in (x_i).
$I_2(M_{n,d}) \subseteq I_{n,d}$ because $x_1^{a_1}\dots x_i^{a_i}\dots x_j^{a_j}\dots x_n^{a_n}x_1^{b_1}\dots x_i^{b_i-1}\dots x_j^{b_j+1}\dots x_n^{b_n}$ $- x_1^{a_1}\dots x_i^{a_i-1}\dots x_j^{a_j+1}\dots x_n^{a_n}x_1^{b_1}\dots x_i^{b_i}\dots x_j^{b_j}\dots x_n^{b_n} = 0$ for any $1 \le i < j \le n$ and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = d$.
We know $I_{n,d}$ is generated by quadratic binomials of the form $f = t_{x_1^{p_1}\dots x_n^{p_n}}t_{x_1^{q_1}\dots x_n^{q_n}} - t_{x_1^{r_1}\dots x_n^{r_n}}t_{x_1^{s_1}\dots x_n^{s_n}}$, where $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} s_i = d$ and $p_i + q_i = r_i + s_i$ for all $1 \le i \le n$.
Let m be the smallest integer such that $p_m \neq r_m$. Without loss of generality, assume $p_m > r_m \ge 0$. Then $s_m > q_m \ge 0$, and $\sum_{i=m}^{n} p_i = \sum_{i=m}^{n} r_i$, so there is $l > m$ such that $r_l > p_l \ge 0$ and $q_l > s_l \ge 0$.
Since $p_m > 0$ and $q_l > 0$, we can find $t_{x_1^{p_1}\dots x_n^{p_n}}$ on the m-th row of $M_{n,d}$ and $t_{x_1^{q_1}\dots x_n^{q_n}}$ on the l-th row of $M_{n,d}$.
If these two variables $t_{x_1^{p_1}\dots x_n^{p_n}}$ and $t_{x_1^{q_1}\dots x_n^{q_n}}$ are not on the same column of $M_{n,d}$, then we have $g = t_{x_1^{p_1}\dots x_n^{p_n}}t_{x_1^{q_1}\dots x_n^{q_n}} - t_{x_1^{p_1}\dots x_m^{p_m-1}\dots x_l^{p_l+1}\dots x_n^{p_n}}t_{x_1^{q_1}\dots x_m^{q_m+1}\dots x_l^{q_l-1}\dots x_n^{p_n}} \in I_2(M_{n,d})$ and $f = g + f'$ where $f' = t_{x_1^{p_1}\dots x_m^{p_m-1}\dots x_l^{p_l+1}\dots x_n^{p_n}}t_{x_1^{q_1}\dots x_m^{q_m+1}\dots x_l^{q_l-1}\dots x_n^{p_n}} - t_{x_1^{r_1}\dots x_n^{r_n}}t_{x_1^{s_1}\dots x_n^{s_n}}$.
If these two variables $t_{x_1^{p_1}\dots x_n^{p_n}}$ and $t_{x_1^{q_1}\dots x_n^{q_n}}$ are on the same column of $M_{n,d}$, then $x_1^{q_1}\dots x_n^{q_n} = x_1^{p_1}\dots x_n^{p_n}\cdot\frac{x_l}{x_m}$. Consider the $t_{x_1^{s_1}\dots x_n^{s_n}}$ on the m-th row of $M_{n,d}$ and $t_{x_1^{r_1}\dots x_n^{r_n}}$ on the l-th row of $M_{n,d}$. If these two $t_{x_1^{s_1}\dots x_n^{s_n}}$ and $t_{x_1^{r_1}\dots x_n^{r_n}}$ are on the same column, then $x_1^{r_1}\dots x_n^{r_n} = x_1^{s_1}\dots x_n^{s_n}\cdot\frac{x_l}{x_m}$, and from $x_1^{r_1}\dots x_n^{r_n}x_1^{s_1}\dots x_n^{s_n} = x_1^{p_1}\dots x_n^{p_n}x_1^{q_1}\dots x_n^{q_n}$, it follows that $x_1^{s_1}\dots x_n^{s_n} = x_1^{p_1}\dots x_n^{p_n}$ and $x_1^{r_1}\dots x_n^{r_n} = x_1^{q_1}\dots x_n^{q_n}$, so $f = 0$. Otherwise, $g = t_{x_1^{r_1}\dots x_m^{r_m+1}\dots x_l^{r_l-1}\dots x_n^{r_n}}t_{x_1^{s_1}\dots x_m^{s_m-1}\dots x_l^{s_l+1}\dots x_n^{s_n}} - t_{x_1^{r_1}\dots x_n^{r_n}}t_{x_1^{s_1}\dots x_n^{s_n}} \in I_2(M_{n,d})$ and $f = g + f'$ where $f' = t_{x_1^{p_1}\dots x_n^{p_n}}t_{x_1^{q_1}\dots x_n^{q_n}} - t_{x_1^{r_1}\dots x_m^{r_m+1}\dots x_l^{r_l-1}\dots x_n^{r_n}}t_{x_1^{s_1}\dots x_m^{s_m-1}\dots x_l^{s_l+1}\dots x_n^{s_n}}$.
Let $f = f'$ and repeat the process until $f = 0$. This will terminate because each time $\sum_{i=1}^{n} |p_i - r_i|$ becomes smaller. Thus, $I_{n,d} \subseteq I_2(M_{n,d})$. $\square$

We can rewrite the variables $t_{x_1^{b_1}\dots x_n^{b_n}}$ where $\sum_{i=1}^{n} b_i = d$ as t_m and $m = 1 + \sum_{k=1}^{n-1}[\binom{n-k+d-b_1-\cdots-b_k}{n-k} - \binom{n-k+d-b_1-\cdots-b_k-1}{n-k-1}]$. If $1 \le m \le \binom{n+d-2}{d-1}$, then the m-th column of the matrix $M_{n,d}$ whose 2-minors generate $I_{n,d}$ is

$$\begin{bmatrix} t_m \\ t_{m+\binom{n+d-b_1-2}{n-2}} \\ t_{m+\binom{n+d-b_1-2}{n-2}+\binom{n+d-b_1-b_2-3}{n-3}} \\ \vdots \\ t_{m+\binom{n+d-b_1-2}{n-2}+\binom{n+d-b_1-b_2-3}{n-3}+\cdots+\binom{n+d-b_1-b_2-\cdots-b_{n-1}-n}{0}} \end{bmatrix}$$

Proposition 5.2. Let $I = I_{2,d}$ where $d \ge 4$. Then $[I^{(a)}]_{\ge 2a} \subseteq I^a$.

Proof. It is enough to show that $[in_<(I^{(a)})]_{\geq 2a} \subseteq in_<(I^a)$, because it will follow that, for any $g \in I^{(a)}$ homogeneous of degree $\geq 2a$, $in_<(g) \in in_<(I^a)$ and we can choose $g' \in I^a \subseteq I^{(a)}$ such that g' is homogeneous and $in_<(g') = in_<(g)$, then $g - g' \in I^{(a)}$ is homogeneous and $in_<(g - g') < in_<(g)$. If $g - g' \neq 0$, then repeat the process and such algorithm terminates by well ordering property.
Choose graded lexicographic order for the monomials so that the initial ideal $in_<(I) = (t_1t_3, t_1t_4, \ldots, t_1t_{d+1}, t_2t_4, t_2t_5, \ldots, t_2t_{d+1}, \ldots\ldots, t_{d-1}t_{d+1})$ is radical. By lemma 3.8 and 3.7, we have $in_<(I^{(a)}) \subseteq in_<(I)^{(a)}$ and $in_<(I)^a \subseteq in_<(I^a)$, so it is enough to show $[in_<(I)^{(a)}]_{\geq 2a} \subseteq in_<(I)^a$.
We have $in_<(I) = \cap_{i=1}^d (t_1, t_2, \ldots, t_{i-1}, t_{i+2}, t_{i+3}, \ldots, t_{d+1})$ so its symbolic power

$$\begin{aligned} in_<(I)^{(a)} &= \cap_{i=1}^d (t_1, t_2, \ldots, t_{i-1}, t_{i+2}, t_{i+3}, \ldots, t_{d+1})^{(a)} \\ &= \cap_{i=1}^d (t_1, t_2, \ldots, t_{i-1}, t_{i+2}, t_{i+3}, \ldots, t_{d+1})^a \end{aligned} \tag{1}$$

For $a = 1$, $[in_<(I)^{(1)}]_{\geq 2} \subseteq in_<(I)^1$ since $in_<(I)^{(1)} = in_<(I)^1$.
Suppose $[in_<(I)^{(a)}]_{\geq 2a} \subseteq in_<(I)^a$ for $a = k \in \mathbb{N}$.
Let $a = k + 1$ and $f = t_1^{a_1} \ldots t_{d+1}^{a_{d+1}} \in [in_<(I)^{(a)}]_{\geq 2a}$. Then the degree of f is $D = a_1 + \cdots + a_{d+1} \geq 2a = 2k + 2$. For each integer $1 \leq i \leq d$, by 1 we have $a_1 + a_2 + \cdots + a_{i-1} + a_{i+2} + \cdots + a_{d+1} \geq a$, which is equivalent to $a_i + a_{i+1} \leq D - a$. The goal is to find $1 \leq l \leq m \leq d + 1$ such that $f = t_lt_mf'$ where $t_lt_m \in in_<(I)$ (i.e. $m - l > 1$) and $f' = t_1^{a'_1} \ldots t_{d+1}^{a'_{d+1}} \in in_<(I)^{(k)}$ (i.e. $a'_i + a'_{i+1} \leq (D - 2) - k$ for all integer $1 \leq i \leq d$).
For at most two distinct pair of consecutive integers, the equality $a_i + a_{i+1} = D - a$ can hold because if $a_{i_1} + a_{i_1+1} = a_{i_2} + a_{i_2+1} = a_{i_3} + a_{i_3+1} = D - a$ for some integers $1 \leq i_1 < i_1 + 1 < i_2 < i_2 + 1 < i_3 < i_3 + 1 \leq d + 1$, then $3(D - a) = a_{i_1} + a_{i_1+1} + a_{i_2} + a_{i_2+1} + a_{i_3} + a_{i_3+1} \leq a_1 + \cdots + a_{d+1} = D$, which implies $D \leq 3a/2$ and contradicts $D \geq 2a$. Then among all the ineqaulities $a_i + a_{i+1} \leq D - a$, at most four equailities $a_i + a_{i+1} = D - a$ can hold because if $1 \leq i_1 < i_2 < i_3 < i_4 < i_5 \leq d$, then $i_1, i_1 + 1, i_3, i_3 + 1, i_5, i_5 + 1$ are distinct, so it is impossible to have $a_{i_1} + a_{i_1+1} = a_{i_3} + a_{i_3+1} = a_{i_5} + a_{i_5+1} = D - a$.

(1) No equalities hold in the d inequalities $a_i + a_{i+1} \leq D - a$.
    Let p be the smallest i such that $a_i > 0$. Then since $a_p + a_{p+1} < D - a < D$, there is $p + 1 < q \leq d + 1$ such that $a_q > 0$, then $f = t_pt_qf'$ where $t_pt_q \in in_<(I)$.
(2) The only equalities hold in the d inequalities $a_i + a_{i+1} \leq D - a$ are consecutive $a_p + a_{p+1} = a_{p+1} + a_{p+2} = \cdots = a_{p+j} + a_{(p+j)+1} = D - a$ where $1 \leq p \leq d - j$ and $j \in \{0, 1, 2, 3\}$.
    (a) $j = 0$, $a_p + a_{p+1} = D - a$.
        If $a_{p+1} = 0$, then $a_p = D - a$, which implies $p = 1$ because otherwise $a_{p-1} + a_p \geq D - a$. Since $a_p + a_{p+1} = D - a < D$, there is $q > p + 1$ such that $a_q > 0$. Then $f = t_pt_qf'$ where $t_pt_q \in in_<(I)$.
        If $a_p = 0$, then $a_{p+1} = D - a$, which implies $p = d$. Since $a_p + a_{p+1} = D - a < D$, there is $q < p$ such that $a_q > 0$. Then $f = t_qt_{p+1}f'$ where $t_qt_{p+1} \in in_<(I)$.
        If $a_p > 0$ and $a_{p+1} > 0$, then since $a_p + a_{p+1} = D - a < D$, there is $q < p$ or $q > p + 1$ such that $a_q > 0$. If $q < p$, then $f = t_qt_{p+1}f'$; if $q > p + 1$, then $f = t_pt_qf'$.

(b) $j = 1$, $a_p + a_{p+1} = a_{p+1} + a_{p+2} = D - a$.
If $a_p = a_{p+2} = 0$, then since $a_{p+1} = D - a < D$, there is $q < p$ or $q > p + 1$ such that $a_q > 0$, so $f = t_{p+1}t_q f'$ where $t_{p+1}t_q \in in_<(I)$.
If $a_p = a_{p+2} > 0$, then $f = t_p t_{p+2} f'$ where $t_p t_{p+2} \in in_<(I)$.

(c) $j = 2$, $a_p + a_{p+1} = a_{p+1} + a_{p+2} = a_{p+2} + a_{p+3} = D - a$.
If $a_p = a_{p+2} = 0$, then $a_{p+1} = a_{p+3} > 0$ so $f = t_{p+1}t_{p+3}f'$ where $t_{p+1}t_{p+3} \in in_<(I)$.
If $a_p = a_{p+2} > 0$, then $f = t_p t_{p+2} f'$ where $t_p t_{p+2} \in in_<(I)$.

(d) $j = 3$, $a_p + a_{p+1} = a_{p+1} + a_{p+2} = a_{p+2} + a_{p+3} = a_{p+3} + a_{p+4} = D - a$.
If $a_{p+1} = a_{p+3} = 0$, then $a_p = a_{p+2} = a_{p+4} = D - a$, which implies $3(D - a) = a_p + a_{p+2} + a_{p+4} \leq D$, leading to contradiction.
Thus, $a_{p+1} = a_{p+3} > 0$ and $f = t_{p+1}t_{p+3}f'$ where $t_{p+1}t_{p+3} \in in_<(I)$.

(3) The equalities hold in the d inequalities $a_i + a_{i+1} \leq D - a$ are $a_{(p-\varepsilon_1)-1} + a_{p-\varepsilon_1} = a_{p-1} + a_p = a_q + a_{q+1} = a_{q+\varepsilon_2} + a_{(q+\varepsilon_2)+1} = D - a$ where $\varepsilon_1 + 2 \leq p < q \leq d - \varepsilon_2$ and $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$.
We have $a_{p-1} + a_p = D - a > a_p + a_{p+1}$ and $a_q + a_{q+1} = D - a > a_{q-1} + a_q$, so $a_{p-1} > a_{p+1} \geq 0$ and $a_{q+1} > a_{q-1} \geq 0$. Then $f = t_{p-1}t_{q+1}f'$ where $t_{p-1}t_{q+1} \in in_<(I)$.

In all cases, $f' = t_1^{a_1'} \dots t_{d+1}^{a_{d+1}'}$ has degree $D - 2 \geq 2k$ and $a_i' + a_{i+1}' \leq D - a - 1 = (D - 2) - k$ for all integers $1 \leq i \leq d$, so $f \in [in_<(I)^{(k)}]_{\geq 2k}$ and by induction hypothesis, $f' \in in_<(I)^k$. Therefore, $f \in in_<(I)^{k+1} = in_<(I)^a$. ☐

Here is an easier proof of 5.2 by showing $[in_<(I^{(a)})]_{\geq 2a} \subseteq in_<(I)^a$ in graded lexicographic order.

Proof. In graded lexicographic order we have $t_i t_j \in in_<(I)$ if and only if $|i - j| > 1$. Suppose $f \in [in_<(I^{(a)})]_{\geq 2a}$ and $in_<(f) = t_{i_1}t_{i_2}\dots t_{i_D}$ where $1 \leq i_1 \leq i_2 \leq \dots \leq i_D \leq d + 1$ and $D \geq 2a$.
By Zariski-Nagata theorem, we have $\frac{\partial}{\partial t_{i_D}}\frac{\partial}{\partial t_{i_{D-1}}}\cdots\frac{\partial}{\partial t_{i_{D-(a-2)}}} f \in I$.
Then $t_{i_1}t_{i_2}\dots t_{i_{D-a+1}} = in_<(\frac{\partial}{\partial t_{i_D}}\frac{\partial}{\partial t_{i_{D-1}}}\cdots\frac{\partial}{\partial t_{i_{D-a+2}}} f) \in in_<(I)$, which implies $t_{i_1}t_{i_{D-a+1}} \in in_<(I)$.
For the same reason, we have $t_{i_2}t_{i_{D-a+2}}, t_{i_3}t_{i_{D-a+3}}, \dots, t_{i_a}t_{i_D} \in in_<(I)$.
Then $t_{i_1}t_{i_2}\dots t_{i_D} = (t_{i_1}t_{i_{D-a+1}})(t_{i_2}t_{i_{D-a+2}})\dots(t_{i_a}t_{i_D})(t_{i_{a+1}}t_{i_{a+2}}\dots t_{i_{D-a}}) \in in_<(I)^a$ since $D - a \geq a$. ☐

Proposistion 5.2 implies that for all $d \geq 4$, the $\mathfrak{m}$-primary component of $I_{2,d}$ is $\mathfrak{m}^{2a}$, so a primary decomposition of $I_{2,d}^a$ is

$$I_{2,d}^a = I_{2,d}^{(a)} \cap \mathfrak{m}^{2a}$$

Lemma 5.3. Let k be a field and $R = k[t_1, \dots, t_{d+1}]$. For $2 \leq a \leq \lfloor \frac{d}{2} \rfloor + 1$, we have

$$I_2\begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_d \\ t_2 & t_3 & t_4 & \cdots & t_{d+1} \end{bmatrix} = I_2(M_a) \text{ where } M_a = \begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_{d-a+2} \\ t_2 & t_3 & t_4 & \cdots & t_{d-a+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_a & t_{a+1} & t_{a+2} & \cdots & t_{d+1} \end{bmatrix}.$$

Proof. It is enough to show $I_2(M_a) = I_2(M_{a+1})$ for any $2 \leq a < \lfloor \frac{d}{2} \rfloor$.
Fix $2 \leq a < \lfloor \frac{d}{2} \rfloor$, it suffices to consider the 2-minors at column $1, d - a + 2$ of M_a and the 2-minors at row $1, a + 1$ of M_{a+1}.

For any $1 \le i < j \le a$, since $t_i t_{j+d-a+1} - t_j t_{i+d-a+1} = (t_i t_{j+1+d-a} - t_{j+1} t_{i+d-a}) - (t_j t_{i+d-a+1} - t_{j+1} t_{i+d-a})$ where in the matrix M_{a+1}, $t_i t_{j+1+d-a} - t_{j+1} t_{i+d-a}$ is the 2-minor at row $i, j+1$ and column $1, d-a+1$, and $t_j t_{i+d-a+1} - t_{j+1} t_{i+d-a}$ is the 2-minor at row $i, i+1$ and column $j-i+1, d-a+1$, we have $I_2(M_a) \subseteq I_2(M_{a+1})$. For any $1 \le i < j \le d-a+1$, since $t_i t_{j+a} - t_{i+a} t_j = (t_i t_{j+a} - t_{i+a-1} t_{j+1}) - (t_{i+a} t_j - t_{i+a-1} t_{j+1})$ where in the matrix M_a, $t_i t_{j+a} - t_{i+a-1} t_{j+1}$ is the 2-minor at row $1, a$ and column $i, j+1$, and $t_{i+a} t_j - t_{i+a-1} t_{j+1}$ is the 2-minor at row $j-i+1, a$ and column $i, i+1$, we have $I_2(M_{a+1}) \subseteq I_2(M_a)$. □

Proposition 5.4. Let $I = I_{2,d}$ where $d \ge 2$, the least degree of an element in $I^{(a)}$ is $\lceil \frac{(d+2)a}{d} \rceil$ for d even and $\lceil \frac{(d+1)a}{d-1} \rceil$ for d odd.

Proof. Choose graded lexicographic order for monomials and suppose $f \in I^{(a)}$ homogeneous of degree D. As is shown in the proof of 5.2, we have $in_<(f) = t_1^{a_1} \dots t_{d+1}^{a_{d+1}} \in in_<(I^{(a)}) \subseteq in_<(I)^{(a)}$ so that $a_i + a_{i+1} \le D - a$ for all $1 \le i \le d$.
If d is odd, then $D = (a_1 + a_2) + \dots + (a_d + a_{d+1}) \le \frac{d+1}{2}(D-a)$, so $D \ge \frac{d+1}{d-1}a$.
If d is even, then $D = (a_1 + a_2) + \dots + (a_{d-1} + a_d) + a_{d+1} \le (\frac{d}{2} + 1)(D-a)$, so $D \ge \frac{d+2}{d}a$.

For $a \le \lfloor \frac{d}{2} \rfloor$, since $I = I_2 \begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_d \\ t_2 & t_3 & t_4 & \cdots & t_{d+1} \end{bmatrix} = I_2 \begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_{d-a+1} \\ t_2 & t_3 & t_4 & \cdots & t_{d-a+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{a+1} & t_{a+2} & t_{a+3} & \cdots & t_{d+1} \end{bmatrix}$,

we have $I_{a+1} \begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_{d-a+1} \\ t_2 & t_3 & t_4 & \cdots & t_{d-a+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{a+1} & t_{a+2} & t_{a+3} & \cdots & t_{d+1} \end{bmatrix} \in I^{(a)}$ by lemma 3.3. Therefore,

there is degree $a+1$ element in $I^{(a)}$.
For $k \lfloor \frac{d}{2} \rfloor < a \le (k+1) \lfloor \frac{d}{2} \rfloor, k \in \mathbb{N}$, choose $f \in I^{(\lfloor \frac{d}{2} \rfloor)}$ of degree $\lfloor \frac{d}{2} \rfloor + 1$ and $g \in I^{(a - k \lfloor \frac{d}{2} \rfloor)}$ of degree $a - k \lfloor \frac{d}{2} \rfloor + 1$, then $f^k g$ has degree $a + k + 1$ and $f^k g \in (I^{(\lfloor \frac{d}{2} \rfloor)})^k I^{(a - k \lfloor \frac{d}{2} \rfloor)} \subseteq I^{(a)}$ where $k + 1 = \lceil \frac{a}{\lfloor d/2 \rfloor} \rceil$.
Therefore, if d is odd, then $I^{(a)}$ has element of degree $\lceil a + \frac{2}{d-1}a \rceil$; if d is even, then $I^{(a)}$ has element of degree $\lceil a + \frac{2}{d}a \rceil$. □

Proposition 5.5. Let $I = I_{2,d}$ where $d \ge 4$. If $a \le \lfloor \frac{d}{2} \rfloor$, then $[I^{(a)}]_{a+1} = [J]_{a+1}$

where $J = I_{a+1} \begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_{d-a+1} \\ t_2 & t_3 & t_4 & \cdots & t_{d-a+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{a+1} & t_{a+2} & t_{a+3} & \cdots & t_{d+1} \end{bmatrix}$.

Proof. Since $I = I_2 \begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_d \\ t_2 & t_3 & t_4 & \cdots & t_{d+1} \end{bmatrix} = I_2 \begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_{d-a+1} \\ t_2 & t_3 & t_4 & \cdots & t_{d-a+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{a+1} & t_{a+2} & t_{a+3} & \cdots & t_{d+1} \end{bmatrix}$,

we have $J \subseteq I^{(a)}$ by 3.3.
Let f be a homogeneous polynomial of degree $a+1$ in $I^{(a)}$. Choose graded lexicographic order for monomials. Reduce f mod J so that $f = f' + r$ where $f' \in J$

and either $r = 0$ or $in_<(r) \notin in_<(J)$.
Suppose for contradiction that $in_<(r) \notin in_<(J)$, then $in_<(r) = t_{i_1} \dots t_{i_{a+1}}$ for some integers $1 \leq i_1 \leq \dots \leq i_{a+1} \leq d+1$. If $i_j < i_{j+1} - 1$ for all integers $1 \leq j \leq a$, then

$$t_{i_1} \dots t_{i_{a+1}} = in_<(det \begin{bmatrix} t_{i_1} & t_{i_2-1} & \cdots & t_{i_a-a+1} & t_{i_{a+1}-a} \\ t_{i_1+1} & t_{i_2} & \cdots & t_{i_a-a+2} & t_{i_{a+1}-a+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{i_1+a-1} & t_{i_2+a-2} & \cdots & t_{i_a} & t_{i_{a+1}-1} \\ t_{i_1+a} & t_{i_2+a-1} & \cdots & t_{i_a+1} & t_{i_{a+1}} \end{bmatrix}) \in in_<(J), \text{ so}$$

there is $1 \leq j \leq a$ such that $i_{j+1} \geq i_j \geq i_{j+1} - 1$.
Since $J \subseteq I^{(a)}$, we have $r \in I^{(a)}$ so that $\frac{\partial}{\partial t_{i_1}} \cdots \frac{\partial}{\partial t_{i_{j-1}}} \frac{\partial}{\partial t_{i_{j+2}}} \cdots \frac{\partial r}{\partial t_{i_{a+1}}} \in I$ by Zariski-Nagata theorem. However, in order for $t_{i_j} t_{i_{j+1}}$ where $i_{j+1} \geq i_j \geq i_{j+1} - 1$ to appear in $\frac{\partial}{\partial t_{i_1}} \cdots \frac{\partial}{\partial t_{i_{j-1}}} \frac{\partial}{\partial t_{i_{j+1}}} \cdots \frac{\partial r}{\partial t_{i_{a+1}}}$, we must have $t_{i_j-1} t_{i_{j+1}+1}$ appear in $\frac{\partial}{\partial t_{i_1}} \cdots \frac{\partial}{\partial t_{i_{j-1}}} \frac{\partial}{\partial t_{i_{j+1}}} \cdots \frac{\partial r}{\partial t_{i_{a+1}}}$ where $2 \leq i_j \leq i_{j+1} \leq d$, which implies that the term $t_{i_1} \dots t_{i_{j-1}} t_{i_j-1} t_{i_{j+1}+1} t_{i_{j+2}} \dots t_{i_{a+1}}$ will appear in r, contradicting $in_<(r) = t_{i_1} \dots t_{i_{a+1}}$. Therefore, $r = 0$ and $f \in J$. □

In particular, from proposition 5.5, we know for all $d \geq 4$, the degree 3 piece of the second symbolic power is $[I_{2,d}^{(2)}]_3 = [I_3 \begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_{d-1} \\ t_2 & t_3 & t_4 & \cdots & t_d \\ t_3 & t_4 & t_5 & \cdots & t_{d+1} \end{bmatrix}]_3$. This, together with proposition 5.2, shows that

$$I_{2,d}^{(2)} = I_{2,d}^2 + I_3 \begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_{d-1} \\ t_2 & t_3 & t_4 & \cdots & t_d \\ t_3 & t_4 & t_5 & \cdots & t_{d+1} \end{bmatrix}$$

for all $d \geq 4$.

Lemma 5.6. Let $I = I_{2,d}$ where $d \geq 4$ and $2 \leq m < a$ where $a \geq 3$. Then in graded lexicographic order, $[in_<(I^{(a)})]_{a+m} \subseteq in_<(I^{(q)})^r in_<(I^{(q-1)})^{m-r}$ where $q = \lfloor \frac{a+m-1}{m} \rfloor$ and $r = a + m - qm$.

Proof. In graded lexicographic order, we have $t_i t_j \in in_<(I)$ if and only if $|i - j| > 1$. Without loss of generality, assume $[I^{(a)}]_{a+m} \neq \emptyset$, so $a \leq m \lfloor \frac{d}{2} \rfloor$ and then $q \leq \lfloor \frac{d}{2} \rfloor$.
Suppose $f \in [I^{(a)}]_{a+m}$, and $in_<(f) = t_{i_1} t_{i_2} t_{i_3} \dots t_{i_{a+m-1}} t_{i_{a+m}} \in [in_<(I^{(a)})]_{a+m}$ where $1 \leq i_1 \leq i_2 \leq i_3 \leq \dots \leq i_{a+m-1} \leq i_{a+m} \leq d+1$.
By Zariski-Nagata theorem, we have $\frac{\partial}{\partial t_{i_{a+m}}} \frac{\partial}{\partial t_{i_{a+m-1}}} \cdots \frac{\partial}{\partial t_{i_{a+m-(a-2)}}} f \in I$ so that $t_{i_1} t_{i_2} \dots t_{i_{m+1}} = in_<(\frac{\partial}{\partial t_{i_{a+m}}} \frac{\partial}{\partial t_{i_{a+m-1}}} \cdots \frac{\partial}{\partial t_{i_{m+2}}} f) \in in_<(I)$, which implies $t_{i_1} t_{i_{m+1}} \in in_<(I)$. For the same reason, we have $t_{i_l} t_{i_{m+l}} \in in_<(I)$ for all $1 \leq l \leq a$.

Then $t_{i_l} t_{i_{m+l}} t_{i_{2m+l}} \dots t_{i_{qm+l}} \in in_<(I_{q+1} \begin{bmatrix} t_1 & t_2 & \cdots & t_{d-q+1} \\ t_2 & t_3 & \cdots & t_{d-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ t_{q+1} & t_{q+2} & \cdots & t_{d+1} \end{bmatrix}) \subseteq in_<(I^{(q)})$

for all $1 \leq l \leq r$ and $t_{i_l} t_{i_{m+l}} t_{i_{2m+l}} \dots t_{i_{(q-1)m+l}} \in in_<(I_q \begin{bmatrix} t_1 & t_2 & \cdots & t_{d-q+2} \\ t_2 & t_3 & \cdots & t_{d-q+3} \\ \vdots & \vdots & \ddots & \vdots \\ t_q & t_{q+1} & \cdots & t_{d+1} \end{bmatrix}) \subseteq$

$in_<(I^{(q-1)})$ for all $r+1 \le l \le m$.
Therefore, $t_{i_1}t_{i_2}t_{i_3}\dots t_{i_{a+m-1}}t_{i_{a+m}} \in in_<(I^{(q)})^r in_<(I^{(q-1)})^{m-r}$. □

Proposition 5.7. Let $I = I_{2,d}$ where $d \ge 4$ and $2 \le m < a \le m\lfloor \frac{d}{2} \rfloor$ where $a \ge 3$. Then $[I^{(a)}]_{a+m} = [(I^{(q)})^r (I^{(q-1)})^{m-r}]_{a+m}$ where $q = \lfloor \frac{a+m-1}{m} \rfloor$ and $r = a+m-qm$.

Proof. By lemma 3.5, we have $(I^{(q)})^r (I^{(q-1)})^{m-r} \subseteq I^{(qr+(q-1)(m-r))} = I^{(qm+r-m)} = I^{(a)}$. Then it follows from $[in_<(I^{(a)})]_{a+m} \subseteq [in_<(I^{(q)})^r in_<(I^{(q-1)})^{m-r}]_{a+m} \subseteq [in_<((I^{(q)})^r) in_<((I^{(q-1)})^{m-r})]_{a+m} \subseteq [in_<((I^{(q)})^r (I^{(q-1)})^{m-r})]_{a+m}$ that $[I^{(a)}]_{a+m} \subseteq (I^{(q)})^r (I^{(q-1)})^{m-r}$. □

Proposition 5.2, 5.5, and 5.7 help us completely determine the elements in the symbolic power of $I_{2,d}$.

Example 5.8. Let $I = I_{2,d}$ where $d \ge 4$.

(1) When $a = 3$, for $d = 4, 5$, $I^{(3)} = I^{(2)} I$ and for all $d \ge 6$,

$$I^{(3)} = I^{(2)} I + I_4 \begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_{d-2} \\ t_2 & t_3 & t_4 & \cdots & t_{d-1} \\ t_3 & t_4 & t_5 & \cdots & t_d \\ t_4 & t_5 & t_6 & \cdots & t_{d+1} \end{bmatrix}$$

(2) When $a = 4$, since $I^{(2)} I^2 \subseteq (I^{(2)})^2$, for $d = 4, 5, 6, 7$, $I^{(4)} = (I^{(2)})^2$ and for all $d \ge 8$,

$$I^{(4)} = I^{(4)} = (I^{(2)})^2 + I_5 \begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_{d-3} \\ t_2 & t_3 & t_4 & \cdots & t_{d-2} \\ t_3 & t_4 & t_5 & \cdots & t_{d-1} \\ t_4 & t_5 & t_6 & \cdots & t_d \\ t_5 & t_6 & t_7 & \cdots & t_{d+1} \end{bmatrix}$$

Proposition 5.9. Let $I = I_{2,4}$, $f = t_3^3 - 2t_2t_3t_4 + t_1t_4^2 + t_2t_5 - t_3t_4t_5$.

(1) $I^{(2)} = I^2 + (f)$.
(2) $I^{(2k+1)} = I^{(2k)} I$ for all $k \geqslant 1$.
(3) $I^{(2k+2)} = (I^{(2)})^{k+1}$ for all $k \geqslant 1$.

Proof. (1) As is proved earlier, $I^{(2)} = I^2 + I_3 \begin{bmatrix} t_1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \\ t_3 & t_4 & t_5 \end{bmatrix}$.

(2) We have $I^{(2k)} I \subseteq I^{(2k+1)}$ by lemma 3.5.
The least degree of an element in $I^{(2k+1)}$ is $\lceil \frac{6(2k+1)}{4} \rceil = 3k+2$.
Consider $[I^{(2k+1)}]_{2k+1+m}$ where $k+1 \le m \le 2k$. Then $\frac{2k+1+2k-1}{2k} \le \frac{2k+1+m-1}{m} \le \frac{2k+1+k+1}{k+1}$, so $\lceil \frac{2k+1+m-1}{m} \rceil = 2$. By proposition 5.7, we have $[I^{(2k+1)}]_{2k+1+m} \subseteq (I^{(2)})^{2k+1-m} I^{2m-2k-1} \subseteq (I^{(4k+2-2m)}) I^{2m-2k-1} \subseteq I^{(2k)} I$. By proposition 5.2, $[I^{(2k+1)}]_{\ge 4k+2} \subseteq I^{2k+1} = I^{2k} I \subseteq I^{(2k)} I$, so $I^{(2k+1)} \subseteq I^{(2k)} I$.
Therefore, $I^{(2k+1)} = I^{(2k)} I$.

(3) We have $(I^{(k+1)})^2 \subseteq I^{(2k+2)}$.
The least degree of an element in $I^{(2k+2)}$ is $\lceil \frac{6(2k+2)}{4} \rceil = 3k+3$.
Consider $[I^{(2k+2)}]_{2k+2+m}$ where $k+1 \le m \le 2k+1$. Then $\lceil \frac{2k+1+m-1}{m} \rceil = 2$.
By proposition 5.7, we have $[I^{(2k+2)}]_{2k+2+m} \subseteq (I^{(2)})^{2k+2-m} I^{2m-2k-2} =$

$(I^{(2)})^{2k+2-m}(I^2)^{m-k-1} \subseteq (I^{(2)})^{k+1}$. By proposition 5.2, $[I^{(2k+2)}]_{\geq 4k+4} \subseteq I^{2k+2} \subseteq (I^{(2)})^{k+1}$, so $I^{(2k+2)} \subseteq (I^{(2)})^{k+1}$.
Therefore, $I^{(2k+2)} = (I^{(2)})^{k+1}$.

□

Proposition 5.10. Let $I = I_{3,3}$, the least degree of an element in $I^{(a)}$ is $\lceil \frac{4a}{3} \rceil$.

Proof. $I = I_2 \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\ t_2 & t_4 & t_5 & t_7 & t_8 & t_9 \\ t_3 & t_5 & t_6 & t_8 & t_9 & t_{10} \end{bmatrix}$.
Choose graded lexicographic order for monomials so that the initial ideal

$$in_<(I) = (t_1t_4, t_1t_5, t_1t_6, t_1t_7, t_1t_8, t_1t_9, t_1t_{10}, t_2t_5, t_2t_6, t_2, t_7, t_2t_8, t_2t_9, t_2t_{10}, t_3t_7, t_3t_8, t_3t_9, t_3t_{10}, t_4t_6, t_4t_8, t_4t_9, t_4t_{10}, t_5t_8, t_5t_9, t_5t_{10}, t_7t_9, t_7t_{10}, t_8t_{10})$$

is radical. The primary decomposition of $in_<(I)$ is

$$in_<(I) = (t_1, t_2, t_3, t_4, t_5, t_7, t_8) \cap (t_1, t_2, t_3, t_4, t_5, t_7, t_{10}) \cap (t_1, t_2, t_3, t_4, t_5, t_9, t_{10}) \cap (t_1, t_2, t_3, t_4, t_8, t_9, t_{10}) \cap (t_1, t_2, t_3, t_6, t_8, t_9, t_{10}) \cap (t_1, t_2, t_4, t_7, t_8, t_9, t_{10}) \cap (t_1, t_2, t_6, t_7, t_8, t_9, t_{10}) \cap (t_1, t_5, t_6, t_7, t_8, t_9, t_{10}) \cap (t_4, t_5, t_6, t_7, t_8, t_9, t_{10})$$

so its symbolic power is

$$in_<(I)^{(a)} = (t_1, t_2, t_3, t_4, t_5, t_7, t_8)^a \cap (t_1, t_2, t_3, t_4, t_5, t_7, t_{10})^a \cap (t_1, t_2, t_3, t_4, t_5, t_9, t_{10})^a \cap (t_1, t_2, t_3, t_4, t_8, t_9, t_{10})^a \cap (t_1, t_2, t_3, t_6, t_8, t_9, t_{10})^a \cap (t_1, t_2, t_4, t_7, t_8, t_9, t_{10})^a \cap (t_1, t_2, t_6, t_7, t_8, t_9, t_{10})^a \cap (t_1, t_5, t_6, t_7, t_8, t_9, t_{10})^a \cap (t_4, t_5, t_6, t_7, t_8, t_9, t_{10})^a$$

Suppose $f \in I^{(a)}$ is a homogeneous polynomial of degree D.
Then $in_<(f) = \prod_{i=1}^{10} t_i^{a_i} \in in_<(I^{(a)}) \subseteq in_<(I)^{(a)}$, so as is in the proof of 5.2, we have $\begin{cases} a_1 + a_2 + a_3 + a_4 + a_5 + a_7 + a_8 \geq a \\ a_1 + a_2 + a_3 + a_4 + a_5 + a_9 + a_{10} \geq a \\ a_1 + a_2 + a_6 + a_7 + a_8 + a_9 + a_{10} \geq a \\ a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} \geq a \end{cases}$, i.e. $\begin{cases} a_6 + a_9 + a_{10} \leq D - a \\ a_6 + a_7 + a_8 \leq D - a \\ a_3 + a_4 + a_5 \leq D - a \\ a_1 + a_2 + a_3 \leq D - a \end{cases}$.
Thus, $D = \sum_{i=1}^{10} a_i \leq 4D - 4a$ and then $D \geq \lceil \frac{4a}{3} \rceil$.
We have $f = t_5^4 - 2t_4t_5^2t_6 + t_4^2t_6^2 + t_3t_5t_6t_7 - t_2t_6^2t_7 - 2t_3t_5^2t_8 - t_3t_4t_6t_8 + 3t_2t_5t_6t_8 + t_3^2t_8^2 - t_1t_6t_8^2 + 3t_3t_4t_5t_9 - 2t_2t_5^2t_9 - t_2t_4t_6t_9 - t_3^2t_7t_9 + t_1t_6t_7t_9 - t_2t_3t_8t_9 + t_1t_5t_8t_9 + t_2^2t_9^2 - t_1t_4t_9^2 + t_3t_4^2t_{10} + t_2t_4t_5t_{10} + t_2t_3t_7t_{10} - t_1t_5t_7t_{10} - t_2^2t_8t_{10} + t_1t_4t_8t_{10} \in I^{(3)}$.

(1) For $a = 3k - 2, k \in \mathbb{N}$, choose any $g \in [I]_2$. Then gf^{k-1} has degree $2 + 4(k-1) = 4k - 2$ and $gf^{k-1} \in I^{(1)}(I^{(3)})^{k-1} \subseteq I^{(1+3k-3)} = I^{(a)}$.
(2) For $a = 3k - 1, k \in \mathbb{N}$, choose any $h \in [I_3 \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\ t_2 & t_4 & t_5 & t_7 & t_8 & t_9 \\ t_3 & t_5 & t_6 & t_8 & t_9 & t_{10} \end{bmatrix}]_3 \subseteq I^{(2)}$. Then hf^{k-1} has degree $3+4(k-1) = 4k-1$ and $hf^{k-1} \in I^{(2)}(I^{(3)})^{k-1} \subseteq I^{(2+3k-3)} = I^{(a)}$.
(3) For $a = 3k, k \in \mathbb{N}$, f^k has degree $4k$ and $f^k \in (I^{(3)})^k \subseteq I^{(3k)} = I^{(a)}$.

Therefore, the least degree of an element in $I^{(a)}$ is $\lceil \frac{4a}{3} \rceil$. □

Using the same method as in this proof, we can show that in graded lexicographic order the least degree of an element in $in_<(I_{3,4})^{(a)}$ is $\lceil \frac{6a}{5} \rceil$ with $t_1t_4t_6t_{11}t_{13}t_{15} \in in_<(I_{3,4})^{(5)}$.

Proposition 5.11. $in_<(I_{n,d}^{(2b-1)}) \subseteq in_<(I_{n,d}^b)$ for all $b \in \mathbb{N}$.

Proof. Since $in_<(I_{n,d})^b \subseteq in_<(I_{n,d}^b)$ by lemma 3.7, it is enough to show $in_<(I_{n,d}^{(2b-1)}) \subseteq in_<(I_{n,d})^b$ and we prove this by induction.
For $b = 1$, we have $in_<(I_{n,d}^{(1)}) = in_<(I_{n,d})$.
Suppose $in_<(I_{n,d}^{(2b-1)}) \subseteq in_<(I_{n,d})^b$ for some $b \in \mathbb{N}$. Let $g = in_<(f)$ for some $f \in I_{n,d}^{(2b+1)}$. By Zariski-Nagata theorem, $f \in I_{n,d}$ and then $g \in in_<(I_{n,d})$. Since $in_<(I_{n,d})$ is a monomial ideal generated by some quadratic forms, we have $g \in (t_\alpha t_\beta)$ for some $t_\alpha t_\beta \in in_<(I_{n,d})$. By Zariski-Nagata theorem, we have $\frac{\partial^2 f}{\partial t_\alpha \partial t_\beta} \in I^{(2b-1)}$, so $\frac{g}{t_\alpha t_\beta} = in_<(\frac{\partial^2 f}{\partial t_\alpha \partial t_\beta}) \in in_<(I^{(2b-1)})$. Then $\frac{g}{t_\alpha t_\beta} \in in_<(I_{n,d})^b$ by induction hypothesis, which implies $g \in in_<(I_{n,d})^{b+1}$. □

Proposition 5.12. $I_{2,d}^{(2b-1)} \subseteq I_{2,d}^b$ for all $b \in \mathbb{N}$.

Proof. By proposition 5.11, we know the minimal degree of elements in $I_{2,d}^{(2b-1)}$ is $\geq 2b$, so $I_{2,d}^{(2b-1)} \subseteq \mathfrak{m}^{2b}$. Also, $I_{2,d}^{(2b-1)} \subseteq I_{2,d}^{(b)}$ since $2b - 1 \geq b$.
Therefore, we have $I_{2,d}^{(2b-1)} \subseteq I_{2,d}^{(b)} \cap \mathfrak{m}^{2b} = I_{2,d}^b$ for all $b \in \mathbb{N}$. □