HINDMAN'S THEOREM IN ABELIAN GROUPS

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ABSTRACT. Hindman's theorem tells us that for every finite colouring/partition of the natural numbers, there is an infinite subset X for which every finite subset $X_0 \subseteq X$ has the same colour for $\sum X_0$. Certain generalisations of this theorem to abelian groups are known to fail in the uncountable case but an example exists for boolean groups when the desired homogeneous set is finite. In this paper, I will explore some of these questions, such as using different partition criteria, examining nonboolean abelian groups, and lower bounds for the cardinality of groups satisfying the desired Hindman-Ramsey relations.

INTRODUCTION

The generalisations of Hindman's theorem [5] are known for countably infinite groups, notably the Galvin-Glazer theorem [6, Th. 5.8 & Notes], while various versions of the uncountable case have been shown to fail. For instance, for a commutative semigroup, it is always possible to find a colouring with two colours with uncountable FS-homogeneous sets [2]. Stronger failures, involving a negative square-bracket arrow relation, have been proven in [3].

Recently, abelian groups with FS-homogeneity properties have been found with arbitrary uncountable colours and seeking an FS-homogeneous set of size n for a fixed n [1]. In particular, the group obtained by Komjáth is a boolean group, *i.e.* where every non-identity element has order 2. The aim of this paper is to explore Hindman-Ramsey relations for abelian groups in particular, and other well-known algebraic objects in general. Along the way, I explore whether Komjáth's group is optimal, and establish a result for $n = 2$. Also included are some proofs for a sharp, finitary bound on the Erdős-Hajnal-Rado theorem.

Notation. Most of the standard notational conventions of set theory are followed here. For example, every ordinal is a von Neumann ordinal, each cardinal is associated with the least ordinal of that cardinality. If κ is an infinite cardinal, then κ^+ is its successor cardinal. Furthermore, $\beth_0(\kappa) = \kappa$, $\beth_{\alpha+1} = 2^{\beth_\alpha(\kappa)}$ and for α a limit ordinal, one has $\beth_\alpha(\kappa) = \sup\{\beth_\beta(\kappa) : \beta < \alpha\}$. In order to denote disjoint union, we use categorical notation and use $\amalg$ or $\coprod$. For exponentiation, we use prefix notation for general sets *i.e.* ${}^A B = \{f : A \to B\}$ and suffix notation for ordinals and cardinals. For $\alpha < \lambda$, $\pi_\alpha : \prod_{\xi<\lambda} X_\xi \to X_\alpha$ is the projection function, and $\varepsilon_\alpha : X_\alpha \to \coprod_{\xi<\lambda} X_\xi$ denotes the embedding function. More generally, we may define π_I, ε_I for $I \subseteq \lambda$. The set of finite sequences with entries in A

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is denoted by $\mathrm{Seq}(A)$. In the special case that $A = \omega$, then we will omit A, and simply denote $\mathrm{Seq} = \mathrm{Seq}(\omega)$.

To abbreviate, we use notation from category theory, interpreting functions as morphisms in **Set**. Namely, $f : A \to B$ denotes a function, $f : A \rightarrowtail B$ denotes an injective function, $f : A \twoheadrightarrow B$ denotes a surjective function, and $f : A \leftrightarrow B$ denotes a bijective function. When appropriate, the name of the function is omitted.

L will be a first-order language, with signature $\sigma = (\mathcal{F}, \mathcal{R}, \mathcal{C})$ denoting the functions, relations and constants symbols respectively. $\mathfrak{A} = (A, \mathcal{F}^A, \mathcal{R}^A, \mathcal{C}^A)$ denotes an L-structure.

We denote by $\mathbf{B}_\lambda = \mathbf{B}(\lambda) = ([\lambda]^{<\omega}, \triangle)$, the unique Boolean group of cardinality λ, where $\triangle$ denotes symmetric difference. Depending on the circumstance, we may consider $\mathbf{B}_\lambda$ as $\bigoplus_\lambda \mathbf{Z}/2\mathbf{Z}$, the group of functions from λ to $\mathbf{Z}/2\mathbf{Z}$ with finite support. In order to distinguish between sequences and generated subgroups, we use $(a_i : i \in I)$ for sequences and $\langle H \rangle$ for the subgroup generated by H.

For the tetration operation, we use Knuth's up-arrow notation; in particular, $a \uparrow\uparrow 1 = a$ and $a \uparrow\uparrow (n+1) = a^{a \uparrow\uparrow n}$, *i.e.* a tower of exponentials (or tetration) of n-many a..

1. Generalisations in partition criteria

For a more general classification of partition criteria for algebraic objects, we borrow the following concept from model theory:

Definition 1. A *pregeometry* is a pair (X, cl) such that $\mathrm{cl} : (\mathcal{P}(X), \subseteq) \to (\mathcal{P}(X), \subseteq)$ is an idempotent, dominating homomorphism of partially ordered sets, has finite character, and follows the exchange principle. That is:

(i) (idempotent) $\mathrm{cl}^2 = \mathrm{cl}$,
(dominating) $\forall A\, A \subseteq \mathrm{cl}(A)$,
(homomorphism) $\forall AB\, A \subseteq B \to \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;

(ii) (finite character) for every $a \in \mathrm{cl}(A)$, there is some finite $A_0 \subseteq A$ such that $a \in \mathrm{cl}(A_0)$;

(iii) (exchange principle) if $a \in \mathrm{cl}(A \cup \{b\})$, then $a \in \mathrm{cl}(A)$ or $b \in \mathrm{cl}(A \cup \{a\})$.

Furthermore, we define dimension:

Definition 2. Let (X, cl) be a pregeometry. Any subset $A \subseteq X$ is called *independent* if for every $a \in A$, $a \notin \mathrm{cl}(A - \{a\})$. In that case, we call $|A|$ the *dimension* of $\mathrm{cl}(A)$.

Let L be a language with constants $\mathcal{C}$. Let $\mathfrak{A} \in \mathsf{C}$ where C is a class of L-structures, and let μ, κ be cardinals. Let $\mathcal{P}(\mathsf{C})$ denote the class of subsets of members of C. Let $\mathrm{cl} = \mathcal{P}(\mathsf{C}) \to \mathcal{P}(\mathsf{C})$ be a class function such that $\mathrm{cl} \restriction \mathcal{P}(A)$ for every $A \in \mathsf{C}$ is a pregeometry. For instance, we could take C to be k-**Vect**, the category of k-vector spaces, and take cl to be the pregeometry of affine hull, *i.e.* the smallest affine subspace containing a given subset.

Remark. Unfortunately, it seems that the pregeometry definition precludes the possibility of including FS. Namely, FS fails to be idempotent whenever the group is not boolean, and generally fails to have the exchange principle, although it is a dominating homomorphism of posets, and has finite character. However, it does encompass the other algebraic criteria under concern.

The sentence $\mathfrak{A} \to (\mu)^{\text{cl}}_\kappa$ asserts that for every $c : A - \mathcal{C}^A \to \kappa$, there exists $H \subseteq A$ with $H = \text{cl}(H)$ and $\dim H = \mu$ such that $|c(H - \mathcal{C}^A)| = 1$, called cl-*homogeneous*. Here, we focus on when cl is:

- $\langle \cdot \rangle$ where $\langle H \rangle$ is the smallest subgroup generated by H.
- aff_R, or *affine hull* where for R a principal entire ring[1] and M a free R-module, $M \to (\mu)^{\text{aff}_R}_\kappa$ denotes that for every $c : M - \{0\} \to \kappa$, there exists a μ-dimensional affine subspace which is monochromatic under c;
- Span_R, where R is a principal entire ring and M is free over R, whereby Span : $\mathcal{P}(M) \to \mathcal{P}(M)$ is the span of any subset of M.

The theorems in this section show that for most instances, the arrow-relation for the pregeometries outlined above are negative.

Theorem 1. *Let $R \neq \mathbf{F}_2$ be a principal entire ring, and $0 < n$. Then for every free R-module M, $M \nrightarrow (n)^{\text{Span}}_2$.*

Proof. Observe that in an entire ring, if $a \neq 0$, then $ar = as$ implies $r = s$. Hence, the multiplicative action $f_a : r \mapsto ar$ is a bijection for every $a \in R - \{0\}$. Having chosen an arbitrary a, for every $r \in R$ let $S_r = \{a^i r : i \in \mathbf{N}\}$. We claim that if $r, r' \in S$ such that $S_r \cap S_{r'} \neq \varnothing$, then $S_r \perp S_{r'}$ in that either $S_r \subseteq S_{r'}$ or $S_{r'} \subseteq S_r$. Suppose that $r, r' \in R$ such that $S_r \cap S_{r'} \neq \varnothing$. Let $s = a^n r = a^{n'} r'$ be in the intersection, and without loss of generality suppose that $n \geqslant n'$. Then as R is entire, we may say $a^{n-n'} r = r'$, hence $r' \in S_r$ and $S_{r'} \subseteq S_r$.

As principal entire rings are factorial, we may assume that a is chosen to be irreducible. Then any chain of S_{r_α} must have a maximum. Suppose for the sake of contradiction that $(S_{r_\alpha} : \alpha < \lambda)$ is a strictly increasing sequence. We may choose r_α such that for every $\xi < \alpha$, $r_\alpha \notin S_{r_\xi}$. With the same notation, we observe that $r_\xi = a^i r_\alpha$, and hence it follows that arbitrarily large powers of a divide r_0. This is absurd in a factorial ring, which gives us our contradiction.

By the axiom of choice, let $X \subseteq R$ be such that every $r \in X$, S_r is inclusion-maximal in the set $\{S_s : s \in R\}$ and for $r, r' \in X$ distinct, $S_r \cap S_{r'} = \varnothing$, and $\bigcup_{r \in X} S_r = R - \{0\}$. Then define $g : R - \{0\} \to 2$ as $g(r) = i \mod 2$ where $r = a^i r_0$ for some $r_0 \in X$.

If $\dim M = \lambda$ (where λ need not be infinite), then set the colouring of $M = \bigoplus_\lambda R$ as follows:

$$c(x) = g(x(\min\{\alpha < \lambda : x(\alpha) \neq 0\}))$$

If α is the least index such that $x(\alpha) \neq 0$, then it follows that $F(x) = x(\alpha) \neq ax(\alpha) = F(ax)$, as $g(ax(\alpha)) = g(x(\alpha)) + 1 \mod 2$ and hence M has no Span-homogeneous set of any cardinality. □

Theorem 2. *Let G be an abelian group with no elements of order 2. Then for every $0 < n < \omega$, $G \nrightarrow (n)^{\langle \cdot \rangle}_2$.*

[1] We note that in a principal entire ring, every submodule of a free module is free. Furthermore, dimension is well-defined and increasing with respect to inclusion.

Proof. Let $G \rightarrowtail \bigoplus_{p\in\mathbf{P}} \mathbf{Z}[p^\infty]^{(I_p)} \oplus \mathbf{Q}^{(I)}$ where $\coprod_p I_p \sqcup I = \lambda$. Then let us define the colouring $c : G - \{0\} \to 2$ as follows: if $\min\{\alpha < \lambda : x_\alpha \neq 0 \wedge \alpha \notin I_2\} = \alpha_0$ and $x_{\alpha_0} = \frac{2^i q}{p^k} \in G_{\alpha_0}$ where G_{α_0} is either $\mathbf{Z}[p^\infty]$ or $\mathbf{Q}$, $0 < q < p^k$ and $(2,p) = (2,q) = (p,q) = 1$, then $c(x) = i$.

Suppose for sake of contradiction that $H \leqslant G$ is a nonempty $\langle\cdot\rangle$-homogeneous subgroup for c. Let $x \in H$; if $x_{\alpha_0} \in \mathbf{Q}$ where α_0 is defined as above, then we are done, since $c(2x) = i+1 \neq i$. On the other hand, if $x_{\alpha_0} \in \mathbf{Z}[p^\infty]$, then the set $\langle x_{\alpha_0}\rangle \subseteq H \restriction \{\alpha_0\}$ is the unique subgroup of the form $\mathbf{Z}/p^k\mathbf{Z} \rightarrowtail \mathbf{Z}[p^\infty]$. Hence, $1/p^k, 2/p^k \in \langle x_{\alpha_0}\rangle$ *i.e.* there exists $m \in \mathbf{Z}$ such that $c(mx) = 0$, $c(2mx) = 1$ so H again fails to be $\langle\cdot\rangle$-homogeneous. □

The following is a negative analogue of van der Waerden's theorem for abelian groups. Van der Waerden's theorem gives an arbitrarily large homogeneous arithmetic sequence in the natural numbers, given finitely many colours. In [4], a variant called the *Hindman-van der Waerden* theorem was proved, in which the criterion was an uncountable homogeneous set under $\mathrm{FS}^{\{a,a+b,\ldots,a+db\}}$. Here, we show that there is no arithmetic progression of length greater than 3 in boolean-free abelian groups.

Theorem 3. *Let G be an infinite abelian group with no elements of order 2. Then there exists* $c : G - \{0\} \to \omega$ *such that there does not exist* $a, b \in G$ *with* $b \neq 0$ *such that* $|c(\{a, a+b, a+2b\})| = 1$.

Proof. Here we introduce the following notation: let $D = \bigoplus_{p\in\mathbf{P}} \mathbf{Z}[p^\infty]^{(I_p)} \oplus \mathbf{Q}^{(I)}$, where $\lambda = \coprod_p I_p \sqcup I$. Then for q a prime number, $\mathrm{supp}_q : D \to \lambda$ is defined by

$$\mathrm{supp}_q(x) = \{\alpha < \lambda : x(\alpha) \neq 0 \wedge \alpha \notin I_q\}$$

Let $G \rightarrowtail D$. Then let $c : G \to \omega$ be defined as follows:

$$c(x) = (x(\alpha_0), \ldots, x(\alpha_{n-1}))$$

where $|\mathrm{supp}_2(x)| = n$ and $\alpha_0 = \min(\mathrm{supp}_2(x))$ and $\alpha_{i+1} = \min\{\alpha \in \mathrm{supp}_2(x) : \alpha > \alpha_i\}$.

Suppose that $a, b \in G$ are such that $b \neq 0$. Let $\beta = \min\{\alpha < \lambda : b(\alpha) \neq 0 \wedge b(\alpha) \notin \mathbf{Z}[2^\infty]\}$ and let $m = |\mathrm{supp}_2(a \restriction \beta)|$. If $a(\beta) = 0$, then the mth nonzero and non-$\mathbf{Z}[2^\infty]$ element of $a+b$ and $a+2b$ are $b(\beta)$ and $2b(\beta)$ respectively. As we assume G is boolean-free, these are distinct and nonzero, and will have nonzero entries which are not elements of $\mathbf{Z}[2^\infty]$. If $a(\beta) \neq 0$, then we again consider the mth nonzero and non-$\mathbf{Z}[2^\infty]$ elements. Either $a(\beta) + b(\beta) \neq 0$, in which case $a(\beta) \neq a(\beta) + b(\beta)$, which is absurd; or $a(\beta) + b(\beta) = 0$, in which case $a(\beta) + 2b(\beta) \neq a(\beta)$ since G is boolean free, which is again absurd. This completes the proof by contradiction. □

The following are immediate corollaries:

Corollary 1. *Let G be an abelian group, let* $0 < n < \omega$. *Then there exists* $c : G \to \omega$ *such that there does not exist a nontrivial subgroup* $H \leqslant G$ *and* $a \in G$ *such that* $|c(a+H)| = 1$.

Corollary 2. *Let R be a principal entire ring, M be a free R-module, and let* $0 < n < \omega$. *Then* $M \nrightarrow (n)^{\mathrm{aff}_R}_\omega$.

2. COROLLARIES OF KOMJÁTH'S RESULT [1]

Komjáth's proof of [1] involves an application of a finite, weak version of the Erdős-Rado theorem, stated as follows[2]:

Lemma 1. *Let κ be an infinite cardinal. Then for every $0 < n < \omega$,*

$$\beth_{2^n-1}(\kappa)^+ \to (2^{n+1}-1)^{2^n}_\kappa.$$

Thus, a possible line of inquiry is whether or not this is an optimal use of Erdős-Rado, in the sense of finding the least λ such that $\lambda \to (2^{n+1}-1)^{2^n}_\kappa$. We do not answer this, but a related question which may be of interest.

Definition 3. A *tree* T is a partially ordered set such that for every $a \in \Gamma$, the initial segment $\{x \in T : x < a\}$ forms a well-ordered set under the induced order. Here, we assume every tree has a unique minimum.

A tree is called *binary* if for every $a \in T$, $|\inf\{x \in T : x > a\}| \leqslant 2$ (*i.e.* the set of immediate succesors of a has cardinality at most 2). It is called *strictly binary* if for every $a \in T$, $|\inf\{x \in T : x > a\}|$ is 0 or 2, (*i.e.* every node bifurcates or is terminal/maximal).

Let T be a tree. The set of *branches*, defined as maximal linearly ordered subsets of T, is denoted as $B(T)$. The *height* of T is denoted $h(T) = \sup\{o(x)+1 : x \in T\}$ where $o(x)$ denotes the order type of $\{y \in T : y < x\}$.

Lemma 2. *Let T be a finite binary tree. Then*

$$|B(T)| \leqslant 2^{h(T)}.$$

Proof. The proof is by induction on the height of the binary tree. The singleton tree is trivial. Assume that Lemma 2 has been proven for every binary tree S such that $h(S) \leqslant n$. Let T be any binary tree whose height is $n+1$. Let $a = \min(T)$. As T is binary, there at most two maximal proper subtrees, T_0, T_1, whose roots are a_0 and a_1 respectively. If there is only one proper subtree, then we are done, as the number of branches remains the same but the height is one greater. By induction, $2^n \geqslant 2^{h(T_i)} \geqslant |B(T_i)|$ for $i < 2$. As $B(T) = \bigcup\{\bigcup\{\{a\} \cup C : C \in B(T_i)\} : i < 2\}$, it follows that $|B(T)| = |B(T_0)| + |B(T_1)|$. Hence, $B(T) \leqslant 2^{h(T_0)} + 2^{h(T_1)} \leqslant 2^{h(T)}$, as desired. ☐

The following states that not only does Erdős-Hajnal-Rado establish a sharp bound, the sharpness is finitary.

Theorem 4. *For $0 < n < \omega$ and κ an infinite cardinal, $\beth_n(\kappa) \nrightarrow (2\uparrow\uparrow n+1)^{n+1}_\kappa$*

Proof. The case $n = 1$ is well-known result by Gödel and by Erdős and Kakutani [7]. $2^\kappa \nrightarrow (3)^2_\kappa$ is given by the mapping $F : [{}^\kappa 2]^2 \to \kappa$ by $\{f, g\} \mapsto \min\{\alpha < \kappa : f(\alpha) \neq g(\alpha)\}$. The main part of the proof involves the induction.

Let $\lambda = \beth_{n-1}(\kappa)$. Denote $N = 2\uparrow\uparrow(n-1)+1$. Assume we have shown that $\lambda \nrightarrow (N)^n_\kappa$, and let $F_{n-1} : [\lambda]^n \to \kappa$ be a function which witnesses this. Define $\delta : [{}^\lambda 2]^2 \to \lambda$ as $\delta(f, g) = \min\{\alpha < \lambda : f(\alpha) \neq g(\alpha)\}$. Fix some $\gamma < \kappa$. Then for $a \in [{}^\lambda 2]^{n+1}$, $F_n : [{}^\lambda 2]^{n+1} \to \kappa$

[2] Komjáth uses a homogeneous set of size 2^{n+1}, but $2^{n+1}-1$ is sufficient.

is defined as

$$F_n(a) = \begin{cases} F_{n-1}(\delta([a]^2)) & \text{if } |\delta([a]^2)| = n; \\ \gamma & \text{otherwise.} \end{cases}$$

Let $H \subseteq {}^{\lambda}2$ of size $2 \upuparrows n+1$. H can be thought of as a finite binary tree T by considering the elements of H as branches. Formally, the poset T has as its elements $f \restriction \alpha$ where $f \in H$ and $\alpha \in \delta(\{x \in [H]^2 : f \in x\}) \cup \{\lambda\}$. In this case, for $f', g' \in T$, $f' < g'$ whenever $f' \subseteq g'$, *i.e.* $\operatorname{dom} f' \subseteq \operatorname{dom} g'$ and $f' = g' \restriction \operatorname{dom} f'$. Hence, $B(T) = 2 \upuparrows n+1$, and by Lemma 2, it follows that

$$h(T) \geqslant \log_2(2 \upuparrows n+1)$$

and hence $h(T) \geqslant \lceil \log_2(2 \upuparrows n+1) \rceil = N$. As $\delta([H]^2)$ correspond to the nonterminal nodes of the tree, it follows that $|\delta([H]^2)| \geqslant h(T)$.

We claim that there is some $\Delta \in [\delta[H]^2]^N$ such that for every $d \in [\Delta]^n$, there is some $a \in [H]^{n+1}$ such that $d = \delta([a]^2)$. Note that this would complete the proof, as this implies $|\delta([a]^2)| \geqslant N$. Δ is constructed by induction: let $\Delta_0 = \{\min \delta[H]^2\}$, and call $\delta_0 = \min \delta[H]^2$. In the tree representation T, δ_0 corresponds to the root. Note that by construction, T must have two distinct maximal proper subtrees T_0, T_1, since otherwise δ_0 is no longer minimal in $\delta([H]^2)$. In fact, by similar reasoning we observe that T must be strictly binary. By pigeonhole, there must be $i_0 < 2$ such that $|B(T_{i_0})| \geqslant \lceil |B(T)|/2 \rceil = \lceil (2^{N-1}+1)/2 \rceil = 2^{N-2}+1$. Here, we choose an element f_0 corresponding to a branch in T_{1-i_0}, and let $a_0 = \{f_0\}$.

The induction goes as follows: assume we have chosen $i^{(t)} = i \in {}^t2$ for $t < N$ *i.e.* a finite sequence of 0s and 1s, such that for every $i' = i \restriction t'$ ($t' < t$), $T_{i' \frown i(t')}$ is the maximal subtree of $T_{i'}$ such that $|B(T_{i' \frown i(t')})| \geqslant \lceil |B(T_{i'})|/2 \rceil$ whenever $t < N$. As $|B(T)| = 2 \upuparrows n+1$, by induction it follows that $|B(T_i)| \geqslant 2^{N-1-t}+1$ as each inductive step results in an integer lower bound. Hence, T_i must have two maximal proper subtrees. Furthermore, assume that $a_t = \{f_0, \ldots, f_{t-1}\}$ has been built such that $f_{t'}$ corresponds to a branch in $T_{i \restriction (t'-1) \frown (1-i(t'))}$. Let $\Delta_t = \delta[a_t]^2 = \{\delta_0, \ldots, \delta_{t-1}\}$, indexed by increasing order on λ. We note that for $t' < t'' < t$, that $\delta_{t'} = \delta(\{f_{t'}, f_{t''}\})$ by construction. We build Δ_{t+1}, a_{t+1} and $j \in {}^{t+1}2$ in the following manner: from the previous observation, T_i must have two maximal proper subtrees whenever $t < N$ and hence T_i must have two distinct maximal proper nonempty subtrees. By pigeonholing, there must be $i_t < 2$ such that $|B(T_{i \frown i_t})| \geqslant |B(T_i)|/2$. We choose an arbitrary f_t corresponding to a branch in $T_{i \frown (1-i(t))}$ and set $a_{t+1} = a_t \cup \{f_t\}$. Define $j = i \cup \{(t, i_t)\}$, and let $\Delta_{t+1} = \Delta_t \cup \{\delta_t\}$ where $\delta_t = \delta(\{f_t, f_{t-1}\})$.

To conclude the induction, we let $\Delta = \Delta_N$ and $A = a_N \cup \{f_N\}$ where f_N corresponds to any branch in $T_{i(N)}$. Then for every $d \in [\Delta]^n$, we let $a = \{f_j : \delta_j \in d\} \cup \{f_N\}$, and by the previous remark, $\delta[a]^2 = d$ as desired. As $|\Delta| = N$, it follows that H is not homogeneous, as desired. □

Remark. As an example, $\beth_2(\kappa) \nrightarrow (5)^3_\kappa$. It follows easily from this that $\beth_2(\kappa) \nrightarrow (6)^4_\kappa$.

We restate Komjáth's result, and establish some easy corollaries:

Theorem 5 (Komjáth, [1]). *Let κ be infinite, $n < \omega$ and $\lambda = \beth_{2^{n-1}-1}(\kappa)^+$. Then $\mathbf{B}_\lambda \to (n)^{\mathrm{FS}}_\kappa$.*

Corollary 3. *Let λ be any strong limit cardinal, $\kappa < \lambda$ and $n < \omega$. Then $\mathbf{B}_\lambda \to (n)^{\mathrm{FS}}_\kappa$.*

Proof. If λ is a strong limit, then it follows by simple induction that $\lambda > \beth_{2^{n-1}-1}(\kappa)^+$. By [1, Theorem 1], the arrow relation is established. □

Corollary 4. *Let κ be an infinite cardinal, $\lambda = \beth_\omega(\kappa)$ and let $n < \omega$. Then $\mathbf{B}_\lambda \to (n)^{\mathrm{FS}}_\kappa$.*

A natural question is if the converse of Corollary 3 holds:

Conjecture 1. *Let λ be an uncountable cardinal. If for every $\kappa < \lambda$ and $n < \omega$, the relation $\mathbf{B}_\lambda \to (n)^{\mathrm{FS}}_\kappa$ holds, then λ is a strong limit.*

3. Countable versions of Hindman-Komjáth

Theorem 6. *If G is an infinite abelian group, then $G \nrightarrow (\omega)^{\mathrm{FS}}_\omega$.*

Proof. Let $\lambda = |G|$. By the injectivity of divisible groups, let $G \rightarrowtail \bigoplus_{p\in\mathbf{P}} \mathbf{Z}[p^\infty]^{(I_p)} \oplus \mathbf{Q}^{(I)}$ where $\coprod_p I_p \amalg I = \lambda$. Then define $c : G - \{0\} \to \mathrm{Seq}$ be defined by

$$x \overset{c}{\mapsto} (x(\alpha_1), \dots x(\alpha_{m-1}))$$

where $m = |\mathrm{supp}(x)|$, $\alpha_0 = \min\{\alpha < \lambda : x_\alpha \neq 0\}$ and $\alpha_{i+1} = \min\{\alpha_i < \alpha < \lambda : x_\alpha \neq 0\}$. In particular, we define the enumeration of the summands by using a bijection $\omega \leftrightarrow \coprod_p \mathbf{Z}[p^\infty] \amalg \mathbf{Q}$.

Suppose for sake of contradiction that $H \subseteq G$ is infinite andis FS-homogeneous with respect to c; in particular, let $c(\mathrm{FS}(H)) = \{\bar{a}\}$ and let $|\bar{a}| = m$. Name $H_{-1} = H$. Let $y_0 \in H$; by pigeonhole, there exists some $s_0 \subseteq \mathrm{supp}(y_0)$ such that the set $H_0 = \{x \in H_{-1} : \mathrm{supp}(x) \cap \mathrm{supp}(y_0) = s_0\}$ is infinite. We may assume s_0 is inclusion-maximal with that property. In particular, we may assume s_0 is nonempty, since otherwise, it would imply $H_0' = \{x \in H_{-1} : \mathrm{supp}(x) \cap \mathrm{supp}(y_0) = \varnothing\}$ is cofinite in H, and in particular, nonempty. Then for any $x \in H_0'$, we have $\mathrm{supp}(x + y_0) = 2m$, which contradicts FS-homogeneity.

Assume that $y_i, s_i, H_i = \{x \in H_{i-1} : \mathrm{supp}(x) \cap \mathrm{supp}(y_i) = s_i\}$ have been built, that $|s_i| < m$ and that s_i is maximal such that H_i is infinite. We claim there exists $y_{i+1} \in H_i$ and $s_{i+1} \subseteq \lambda$ such that $s_i \subsetneq s_{i+1}$ and $H_{i+1} = \{x \in H_i : \mathrm{supp}(x) \cap \mathrm{supp}(y_{i+1}) = s_{i+1}\}$ is infinite. Suppose not. This implies for every $y \in H_i$, the set $H_{i+1}'(y) = \{x \in H_i : \mathrm{supp}(x) \cap \mathrm{supp}(y) = s_i\}$ is cofinite in H_i. As cofinite subsets are closed under finite intersections, it follows that we may build $z_0, \dots, z_{m-1}$ such that $z_0 \in H_i$, and for every $j < m$, choosing $z_j \in \bigcap_{l<j} H_{i+1}'(z_l)$. By construction, we observe that for $j, j' < m$, $\mathrm{supp}(z_j) \cap \mathrm{supp}(z_{j'}) = s_i$. As $|s_i| < m$ by assumption, it follows that $|\mathrm{supp}(z_0 + \cdots + z_{m-1})| \geqslant |s_i| + m(m - |s_i|) > m$, which is a contradiction.

As $|s_i| < |s_{i+1}|$, it follows that there is some m' such that $|s_{m'}| > m$. This is absurd, and concludes our proof. □

4. Lower bounds for Hindman's result [1]

Theorem 7. *For every infinite κ, $\mathbf{B}(2^\kappa) \nrightarrow (2)^{\mathrm{FS}}_\kappa$.*

Proof. The desired map which witnesses $\mathbf{B}(2^\kappa) \nrightarrow (2)^{\mathrm{FS}}_\kappa$ is given as follows: define $c : [{}^\kappa 2]^{<\omega} \to \kappa$ by

$$c : x \mapsto (\delta(\{x_i, x_j\}) : i \neq j \in |x|)$$

where $x = \{x_0, \ldots, x_{|x|-1}\}$ is indexed in lexicographical order, and $[x]^2$ is ordered in some induced way: for example, when $x_i < x_j$ and $x_{i'} < x_{j'}$, we have $\{x_i, x_j\} < \{x_{i'}, x_{j'}\}$ if and only if either $i + j < i' + j'$ or $i + j = i' + j'$ and $i < i'$. $\delta : [{}^{\kappa}2]^2 \to \kappa$ is defined as $\delta(x_i, x_j) = \min\{\xi < \kappa : x_i(\xi) \neq x_j(\xi)\}$. For every $a \in [{}^{\kappa}2]^{<\omega}$, we may consider $\text{rk}(a) = |\delta([a]^2)|$. As every such a is finite, $\text{rk}(a)$ is finite. Furthermore, rk is an c-invariant, in the sense that if $c(a) = c(b)$, then $\text{rk}(a) = \text{rk}(b)$.

Suppose for sake of contradiction that $a, b \in [{}^{\kappa}2]^{<\omega}$ are distinct such that $c(a) = c(b) = c(a + b)$. Given the previous remark, we may assume that the ranks of a, b are minimal, and as remarked, finite. Then consider $\alpha = \min \delta([a]^2) = \min \delta([b]^2)$, where the ordering of ${}^{\kappa}2$ is lexicographic induced by κ. Let $\iota_x : [|x|]^2 \to \kappa$ be defined as

$$\iota_x(\{i, j\}) = \delta(\{x_i, x_j\}).$$

Then as $c(a) = c(b) = c(a + b)$, it follows that $\iota_a^{-1}(\alpha) = \iota_b^{-1}(\alpha) = \iota_{a+b}^{-1}(\alpha) \neq \varnothing$. We shall refer by A this subset of $[|a|]^2$.

Every $x \in [{}^{\kappa}2]^{<\omega}$ can be identified with a function $f_x : |x| \to {}^{\kappa}2$ by setting $f_x(0) = \min\{g \in {}^{\kappa}2 : g \in x\}$ and $f_x(i + 1) = \min\{g \in {}^{\kappa}2 : g > f_x(i) \wedge g \in x\}$. Then let $A_0 = \{\min t : t \in A\}$ and $A_1 = \{\max t : t \in A\}$. We observe that as α forms the root of the binary tree corresponding to $\delta[a]^2, \delta[b]^2$, we have that $A_0 \cup A_1 = |a|$ and $A_0 < A_1$, by which we mean for every $x_0 \in A_0, x_1 \in A_1$, one has $x_0 < x_1$. This follows because $a = \{a_0, \ldots, a_{|a|-1}\}$ is ordered lexicographically, so in the tree representation, the root divides the branches into an initial segment and a coinitial segment.

Now consider $a' = a \upharpoonright A_0$ $(= f_a \upharpoonright A_0)$ and $b' = b \upharpoonright A_0$. Since $A_0 < A_1$ and $A = \iota_{a+b}^{-1}(\alpha)$, it follows that $a' + b' = (a + b) \upharpoonright A_0$. As A_0 is an initial segment of A and by our previous remark, we conclude that $c(a') = c(b') = c(a' + b')$. We notice for $A \subseteq B \subseteq |x|$, that $\delta([x \upharpoonright A]^2) \subseteq \delta([x \upharpoonright B]^2)$. For every $a_i, a_j \in a'$ we have $\alpha \neq \delta(\{a_i, a_j\})$. Therefore, $\text{rk}(a') < \text{rk}(a)$, contradicting the minimality of $\text{rk}(a)$. □

For generalisation of the negative arrow relation, we introduce some notation:

Definition 4. For κ an infinite cardinal, let

$$\beta(n, \kappa) = \min\{\lambda : \mathbf{B}_\lambda \to (n)^{\text{FS}}_\kappa\},$$

and

$$\beth^*(n, \kappa) = \max\{N < \omega : \mathbf{B}(\beth_N(\kappa)) \nrightarrow (n)^{\text{FS}}_\kappa\}.$$

Here is an immediate lemma:

Lemma 3. *Let κ be an infinite cardinal. The following are equivalent:*

(i) $\lim_{n\to\omega} \beta(n, \kappa) = \beth_\omega(\kappa)$;
(ii) $\lim_{n\to\omega} \beth^*(n, \kappa) = \omega$.

Proof. Immediate from Corollary 4. □

Proposition 1. *Let λ be an uncountable cardinal. The following are equivalent:*

(i) For every $\kappa < \lambda$, $\lim_{n\to\omega} \beta(n, \kappa) = \beth_\omega(\kappa)$;
(ii) For every $\kappa < \lambda$, $\lim_{n\to\omega} \beth^*(n, \kappa) = \omega$;
(iii) If for every $\kappa < \lambda$ and $n < \omega$, the relation $\mathbf{B}_\lambda \to (n)^{\text{FS}}_\kappa$ holds, then λ is a strong limit.

5. Partition calculus for abelian groups

Theorem 8. *Let λ, κ be infinite cardinals and let $|G| = \lambda$. If $\lambda \to (3)^2_\kappa$, then $G \to (2)^{\mathrm{FS}}_\kappa$.*

The proof given is using the methods of proof in the Hindman-Schur theorem.

Definition 5. Let G be a group. Then $A = \{g_\beta : \beta < \alpha\}$ is *difference-independent with respect to $\alpha \leftrightarrow A$* if whenever $\alpha_i < \beta_i$ $(i < 2)$ such that $g_{\alpha_0} - g_{\beta_0} = g_{\alpha_1} - g_{\beta_1}$, then $\alpha_0 = \alpha_1$ and $\beta_0 = \beta_1$.

Alternatively, we may call $\alpha \rightarrowtail G$ difference-independent.

Lemma 4. *Let λ be an infinite cardinal and let $|G| = \lambda$. Then there exists $\lambda \rightarrowtail G$ which is difference-independent.*

Proof. (Lemma 4.) We will construct $f : \lambda \rightarrowtail G$ inductively as follows:

Let $f(0) = \varnothing$. Assume for all $\beta < \alpha$, $f(\beta)$ has been built such that $f \restriction \alpha$ is difference-independent. Let $S_\alpha = \{f(\beta) + f(\gamma) - f(\delta) : \beta < \alpha, \gamma < \delta < \alpha\}$. Then $|S_\alpha| \leqslant |\alpha|^3 < \lambda$, so there is $g \in G - \{0\}$ such that $g \notin S_\alpha$. Let $f(\alpha) = g$. Then $f \restriction (\alpha + 1)$ is clearly difference-independent.

On the other hand, if α is a limit ordinal, then we simply let $f \restriction \alpha = \bigcup_{\beta<\alpha} f \restriction \beta$. This concludes the induction. □

Now we proceed with the proof of Theorem 8:

Proof. (Theorem 8.) Let $f : \lambda \rightarrowtail G$ be difference-independent. Given any $c : G - \{0\} \to \kappa$, define $d : [\lambda]^2 \to \kappa$ by the following: whenever $\alpha < \beta$, we define:

$$d(\{\alpha, \beta\}) = c(f(\alpha) - f(\beta)).$$

By the assumption that $\lambda \to (3)^2_\kappa$, there is $H = \{\alpha < \beta < \gamma\}$ such that $d([H]^2) = \{\delta\}$. That is, $c(f(\alpha) - f(\beta)) = c(f(\beta) - f(\gamma)) = c(f(\alpha) - f(\gamma))$.

Let $a = f(\alpha) - f(\beta)$ and $b = f(\beta) - f(\gamma)$. Clearly, $c(a) = c(b) = c(a + b)$, and by difference-independence, $|\{a, b, a + b\}| = 3$, which gives us our desired FS-homogeneous set. □

It would be desirable, given the context of [1], to have a result for a larger finite FS-homogeneous set given a large enough group. Proposition 2 will give a characterisation of said problem. First, we prove the following lemma:

Lemma 5. *Let G be an abelian group with $|G| = \lambda$ an uncountable regular cardinal. Then there is some cyclic group Z for which $\bigoplus_\lambda Z \rightarrowtail G$.*

Proof. By injectivity of divisible groups in abelian groups, we have $G \rightarrowtail \bigoplus_{\alpha<\lambda} G_\alpha$ where G_α are $\mathbf{Z}[p^\infty]$ or $\mathbf{Q}$. Let us denote the torsion subgroup of G by G_{tor}, and the torsion-free elements of G (which need not form a subgroup) by G_{tf}. Then $G = G_{\mathrm{tor}} \cup G_{\mathrm{tf}}$, so either the torsion group or set of torsion-free elements must have cardinality λ.

If $|G_{\mathrm{tor}}| = \lambda$, then we may build a sequence $\{x_\alpha : \alpha < \kappa\}$ whose supports $\{\mathrm{supp}(x_\alpha) : \alpha < \kappa\}$ form a Δ-system with root r and tails s_α. Furthermore, since $\mathrm{cf}(\lambda) > \omega$, we may assume that for all $\alpha < \lambda$, $x_\alpha \restriction r = x$ where $o(x) = m$ as there are countably many possible $x_\alpha \restriction r$.

Let $\{a_\alpha \in [\lambda]^m : \alpha < \lambda\}$ be pairwise disjoint in $[\lambda]^m$ and define

$$y_\alpha = \sum_{\xi \in a_\alpha} x_\xi.$$

Then $\mathrm{supp}(y_\alpha) \subseteq \bigcup_{\xi \in a_\alpha} s_\xi$, and since $|a_\alpha| = m$, it follows that $y_\alpha \restriction r = 0$. Now $B = \{y_\alpha : \alpha < \lambda\}$ consists of λ-many elements of finite order with disjoint supports. By pigeonhole, there is $B' \subseteq B$ such that every element of B' has order lp where p is prime; let us consider $f : B' \to G$ by $z \mapsto lz$. For every $z \in B'$, since $o(z) = lp$, we have $lz \neq 0$, and since for $z_i \in B'$ $(i < 2)$ we have $\mathrm{supp}(z_0) \cap \mathrm{supp}(z_1) = \varnothing$ whenever $z_0 \neq z_1$, it follows that f is injective, so fB' is a set of cardinality λ where every element has order p. As $\langle fB' \rangle$ is an elementary abelian p-group, it is isomorphic to $\bigoplus_\lambda \mathbf{Z}/p\mathbf{Z}$.

If $|G_{\mathrm{tf}}| = \lambda$, the Δ-system lemma also gives us a similar proof. By the Δ-system lemma, let $B = \{x_\alpha \in G_{\mathrm{tf}} : \alpha < \lambda\}$ be such that the supports $\{\mathrm{supp}(x_\alpha) : \alpha < \lambda\}$ form a Δ-system of cardinality λ with root r and tails s_α. There are two cases:

If for every x_α, $x_\alpha \restriction s_\alpha$ is torsion-free, then we have that every $x \in \langle B \rangle$ is uniquely representable as $\sum_{i<t} m_i x_{\alpha_i}$ for some $m_i \in \mathbf{Z}$, $n > 0$, and α_i increasing. This follows from the fact that if $\sum_{i<t'} m'_i x_{\alpha'_i} = \sum_{i<t} m_i x_{\alpha_i}$, then inspecting $x_{\alpha_0^{(\prime)}} \restriction s_{\alpha_0^{(\prime)}}$ gives us uniqueness by induction.

Otherwise, if we have only $< \lambda$ many such α such that $x_\alpha \restriction s_\alpha$ is torsion-free, then there exist λ-many $\alpha < \lambda$ such that $x_\alpha \restriction s_\alpha$ is torsion. Hence, we may assume that our Δ-system of cardinality λ has been chosen such that $x_\alpha \restriction s_\alpha$ is torsion. As before, we may assume that $x_\alpha \restriction r$ is constant. Hence, for any $\alpha, \beta < \lambda$ distinct ordinals, we have that $x_\alpha - x_\beta$ is torsion since its support is a subset of $s_\alpha \cup s_\beta$ and unique in the following sense: if $\alpha_i < \beta_i$ such that $x_{\alpha_0} - x_{\beta_0} = x_{\alpha_1} - x_{\beta_1}$, then $\alpha_0 = \alpha_1$ and $\beta_0 = \beta_1$. Hence, there are λ-many torsion elements, and hence we are done. □

The following consist of two implications between the abelian Ramsey theorem and a statement about finite combinatorics of abelian groups.

Denote $D_p = \mathbf{Z}[p^\infty]$ for p a prime and $D_0 = \mathbf{Q}$. Let $\mathbf{P}' = \mathbf{P} \cup \{0\}$. Let $A_p = \bigoplus_\omega \mathbf{Z}/p\mathbf{Z}$ for $p \in \mathbf{P}'$. Generally, for $P \subseteq \mathbf{P}'$, let $A_P = \bigoplus_{p \in P} A_p$.

Proposition 2. *For $p \in \mathbf{P}'$, let $c_p : A_p \to Seq(\mathbf{Z}/p\mathbf{Z})$ be defined by*

$$c_p(x) = (x(\alpha_0), \ldots, x(\alpha_{|\mathrm{supp}\, x|-1}))$$

where $\alpha_0 = \min\{\alpha : x(\alpha) \neq 0\}$ and $\alpha_{i+1} = \min\{\alpha > \alpha_i : x(\alpha) \neq 0\}$. Let G be an abelian group with $|G| = \lambda$.

(i) Assume there exists $q \in \mathbf{P}'$ such that $\bigoplus_\lambda \mathbf{Z}/q\mathbf{Z} \rightarrowtail G$,[3] and that there exist $X_q = \{x_0, \ldots, x_{n-1}\} \subseteq A_q^*$ such that $|c_q(\mathrm{FS}(X_q))| = 1$, . Let $|\mathrm{supp}(x_0)| = m$.
If $\lambda \to (\omega)^m_{\kappa^\omega}$, then $G \to (n)^{\mathrm{FS}}_\kappa$.

(ii) Assume that for every $p \in \mathbf{P}'$ such that $A_p \rightarrowtail G$, and that for every $X_p = \{x_{0,p}, \ldots, x_{n-1,p}\} \subseteq A_p^*$, $|c_p(\mathrm{FS}(X_p))| > 1$.
Then $G \nrightarrow (n)^{\mathrm{FS}}_\omega$.

[3]We remark here that if λ is regular, then this hypothesis is true by Lemma 5.

Proof. (i) We may assume by our hypothesis that $G = \bigoplus_\lambda \mathbf{Z}/q\mathbf{Z}$. Let $d : G - \{0\} \to \kappa$ be an arbitrary colouring. Define $e_0 : [\lambda]^{<\omega} \to \kappa^\omega$ by

$$e_0(x) = (d(g) : g \in G \wedge \operatorname{supp}(g) = x),$$

where we fix some ordering of G, say by the lexicographical order induced by λ. Let $e = e_0 \restriction [\lambda]^m$. As we have assumed $\lambda \to (\omega)^m_{\kappa^\omega}$, let $H \subseteq \lambda$ be such that $\operatorname{tp}(H) = \omega$ and H is homogeneous for e.

Let us consider $K = \bigoplus_{i \in H} \mathbf{Z}/q\mathbf{Z} \leqslant G$. We note that as $|H| = \omega$, $K \cong A_q$; let $f : A_q \rightarrowtail G$ be the embedding whose induced map $\tilde{f} : A_q \leftrightarrow K$ is a canonical isomorphism, respecting the induced well-order on H. Hence if for $i < n$ we let $y_i \in G$ be defined by

$$y_i = f(x_i).$$

By our assumption regarding X_q and the homogeneity of H, it follows that $d(y_i)$ is constant.

(ii) Let $G \rightarrowtail \bigoplus_{p \in \mathbf{P}} \mathbf{Z}[p^\infty]^{(I_p)} \oplus \mathbf{Q}^{(I)} = D$, and assume $|I_p|, |I|$ are minimal. Define $c : D - \{0\} \to \operatorname{Seq}(\kappa)$ as before: $c(x) = (x(\alpha_0), \dots, x(\alpha_{m-1}))$. We may assume that D_p are pairwise disjoint as sets, and hence if any monochromatic set H exists, then

$$H_p = \{y \in D : (\exists x \in H)(\forall \alpha < \kappa)[(x(\alpha) \in D_p \to y(\alpha) = x(\alpha)) \wedge (x(\alpha) \notin D_p \to y(\alpha) = 0)]\}$$

are all monochromatic for $p \in \mathbf{P}'$.[4]

We claim that c as defined is the desired colouring which witnesses $G \nrightarrow (n)^{\mathrm{FS}}_\kappa$. We note that $|\operatorname{Seq}(\kappa)| = \kappa$. Suppose for the sake of contradiction that $H \subseteq G - \{0\}$ is FS-homogeneous with respect to c. Then by our previous remark, each H_p for $p \in \mathbf{P}'$ is homogeneous. Furthermore, by the nature of our colouring, if H_p is FS-homogeneous, for any $z \in D_p$, one has zH_p is FS-homogeneous under c. Therefore, as $|H| = n$, by finding $z = \max\{e_i : i < |x|\} - 1$ where for arbitrary $x \in H_p$, we have $x(\alpha_i) = \frac{n_i}{p^{e_i}}$ where $p \nmid n_i < p^{e_i}$. Then zH_p is nondegenerate, in the sense that for each $x \in H_p$, $zx \neq 0$. Hence, every nonzero element of zx can be considered in A_p. But as $c_p(zx) = c(zx)$, we have $|c_p(zH_p)| = 1$, contradicting our assumption. Thus concludes the proof. □

References

[1] P. Komjáth. A Ramsey statement for infinite groups. To appear in *Combinatorica*, 2016.

[2] D. Fernández-Bretón. Hindman's theorem is only a countable phenomenon. To appear in *Order* [arXiv:1506.05834v3], 2016.

[3] D. Fernández-Bretón, A. Rinot. Strong failures of higher analogs of Hindman's theorem. To appear in *Transactions of the American Mathematical Society* [arXiv:1608.01512], 2016.

[4] L. Carlucci. A note on Hindman-type theorems for uncountable cardinals. preprint [arXiv:1703.06706], 2017.

[5] N. Hindman. Finite sums from sequences within cells of a partition of **N**. *Journal of Combinatorial Theory* Series A, 17:1-11, 1974.

[6] N. Hindman and D. Strauss. *Algebra in the Stone-Čech compactification*, 2nd. ed. Walter de Gruyter, Berlin 2012.

[7] Erdős, P., Kakutani, S. On non-denumerable graphs. BAMS **49** (1943), 457-461.

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[4] In words: H_p consists of elements of D which are elements of H with any entry not in D_p replaced with 0. Or alternatively, the image of π_{I_p} mapped back via ε_{I_p}.