THE $(4,3)$ PROPERTY IN FAMILIES OF FAT SETS IN THE PLANE

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Abstract. A family of sets satisfies the (p,q) property if among any p members of it some q intersect. A set $S \subset \mathbb{R}^2$ is r-fat for some $0 < r \leq 1$, if there exists a point $c \in S$ such that $B(c,r) \subseteq S \subseteq B(c,1)$, where $B(c,r)$ is a disk of radius r with center-point c. For $\sqrt{8}-2 \leq r \leq 1$ we prove that the piercing number of every family of r-fat sets in $\mathbb{R}^2$ that satisfies the $(4,3)$ property is at most 4. This generalizes the bound of 3 on the piercing numbers of 1-fat sets satisfying the $(4,3)$ property, which was proved by Kynčl and Tancer [9]. This research was done as part of an REU project at the University of Michigan, Summer 2017.

1. introduction

1.1. **The (p,q) problem.** The classical theorem of Helly [5] asserts that if $\mathcal{F}$ is a family of convex sets in $\mathbb{R}^d$, such that every $d+1$ members of $\mathcal{F}$ intersect, then all the members of $\mathcal{F}$ intersect, namely, there exists a point in $\mathbb{R}^d$ *piercing* every set in $\mathcal{F}$. Helly's theorem initiated the broad area of research in discrete geometry, dealing with questions regarding the number of points needed to pierce families of convex sets in $\mathbb{R}^d$ satisfying certain intersection properties.

Given integers $p \geq q > 1$, a family $\mathcal{F}$ of sets is said to satisfy the (p,q) *property* if among any p elements in $\mathcal{F}$ there exist q elements with a non-empty intersection. We denote by $\tau(\mathcal{F})$ the *piercing number* (also called in the literature *covering number*, *stubbing number*, or *hitting number*) of $\mathcal{F}$, namely the minimal size of a set of points in $\mathbb{R}^d$ intersecting every element in $\mathcal{F}$. The *matching number* of $\mathcal{F}$, namely the maximum number of pairwise disjoint sets in $\mathcal{F}$, is denoted by $\nu(F)$. Clearly, $\nu(\mathcal{F}) \leq \tau(\mathcal{F})$. If $\nu(\mathcal{F}) = 1$ then we say that $\mathcal{F}$ is an *intersecting*

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family. Note that $\nu(\mathcal{F}) \leq p-1$ if and only if $\mathcal{F}$ satisfies the $(p,2)$ property.

Helly's theorem says that if a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ satisfies the $(d+1,d+1)$ property, then $\tau(\mathcal{F})=1$. Finding the piercing numbers of families of sets in $\mathbb{R}^d$ satisfying the (p,q) property has been known in the literature as the (p,q) *problem*.

In 1992 Alon and Kleitman [1] resolved a long standing conjecture of Hadwiger and Debrunner [4], proving that for every $p \geq q \geq d+1$ there exists a constant $c=c(d;p,q)$ depending only on d,p,q, such that if a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ satisfies (p,q) property then $\tau(\mathcal{F}) \leq c$.

In general, the upper bounds given by Alon and Kleitman's proof for $c(d;p,q)$ are far from being optimal. For example, the Alon-Kleitman proof gives $c(2;4,3) \leq 253$; however, in [8] Kleitman, Gyárfás and Tóth proved that at most 13 points are needed to pierce a family of convex sets in $\mathbb{R}^2$ that satisfies the $(4,3)$ property. Over the last few decades extensive research has been done to improve the Alon-Kleitman bounds, see e.g., [8, 9, 7, 10]. For an excellent survey on the (p,q) problem we refer the reader to [3].

Of course, there does not exist a general bound on $\tau(\mathcal{F})$ when $\mathcal{F}$ is an intersecting family of convex sets in $\mathbb{R}^2$, as is exemplified by a family of lines in general position. However in some cases, when $\mathcal{F}$ consists of certain "nice" sets, a constant bound on the piercing number can be proved. One such example is a result by Danzer [2], who proved that an intersecting family of disks in $\mathbb{R}^2$ has $\tau(\mathcal{F}) \leq 4$. A generalization of this result for certain families of homothets in the plane was proved by Karasev [6].

1.2. **The $(4,3)$ property in $\mathbb{R}^2$.** Here we investigate the piercing numbers of families of sets in $\mathbb{R}^2$ satisfying the $(4,3)$ property. Let As mentioned above, in [8] it was proved that the piercing numbers in families of sets in $\mathbb{R}^2$ satisfying the $(4,3)$ property is at most 13. However, there is no known example os such a family with $\tau > 3$.

It seems that improving the bound on $c(2;4,3)$ for general families of convex sets is a hard task. However, bounds on the piercing number $\tau(\mathcal{F})$ can be significantly improved if one considers only certain restricted families $\mathcal{F}$ of sets in the plane which satisfy $(4,3)$-property. For example, Kynĉl and Tancer proved in [9] that if $\mathcal{F}$ is a family of unit disks that satisfies the $(4,3)$ property, then $\tau(\mathcal{F}) \leq 3$, and this bound is tight.

Other types of set families $\mathcal{F}$ satisfying the $(4,3)$ property that were proved in [9] to achieve $\tau(\mathcal{F}) \leq 3$ are families of translations of a triangle in $\mathbb{R}^2$ and families of segments in $\mathbb{R}^d$.

Given a centrally symmetric body B in $\mathbb{R}^2$ and $0 < r \leq 1$, a r-homothet of B is a set $tB + u$ for some $r \leq t \leq 1$ and $u \in \mathbb{R}^2$. Danzer [2] proved that a family of disks in $\mathbb{R}^2$ that satisfies the $(2,2)$ property has $\tau(\mathcal{F}) \leq 4$. Karasev [6] showed that if $\mathcal{F}$ is a family of $\frac{1}{2}$-homothets of a centrally symmetric body in $\mathbb{R}^2$ that satisfy the $(2,2)$ property then $\tau(\mathcal{F}) \leq 3$. These results imply:

Theorem 1.1. *If $\mathcal{F}$ is a family of disks in $\mathbb{R}^2$ satisfying the $(4,3)$ property then $\tau(\mathcal{F}) \leq 5$.*

Theorem 1.2. *If $\mathcal{F}$ is a family of $\frac{1}{2}$-homothets of a centrally symmetric body in $\mathbb{R}^2$ and $\mathcal{F}$ satisfies the $(4,3)$ property then $\tau(\mathcal{F}) \leq 4$.*

Both theorems follow by applying the following simple observation to Danzer's and Karasev's results.

Observation 1.3. *Let $\mathcal{C}$ be a collection of sets in $\mathbb{R}^2$. If for every finite family $\mathcal{F} \subset \mathcal{C}$ that satisfy the $(2,2)$ property we have $\tau(\mathcal{F}) \leq c$ for some $c \geq 3$ then for every finite family $\mathcal{F} \subset \mathcal{C}$ that satisfy the $(4,3)$ property we have $\tau(\mathcal{F}) \leq c + 1$.*

Proof. Let $\mathcal{F} \subset \mathcal{C}$ be a finite collection of sets satisfying the $(4,3)$ property. If $|F| < 4$ the observation is trivial. If $\mathcal{F}$ contains at least 4 sets then $\nu(\mathcal{F}) \leq 2$, for otherwise a matching of size 3 together with any other set in $\mathcal{F}$ is a collection of 4 sets violating the $(4,3)$ property. If $\nu(\mathcal{F}) = 1$ then $\mathcal{F}$ satisfies the $(2,2)$ property and thus $\tau(\mathcal{F}) \leq c$. Suppose $\nu(\mathcal{F}) = 2$ and let A, B be two disjoint sets in $\mathcal{F}$. Then either every set in $\mathcal{F} \setminus \{A, B\}$ intersect A or every set in $\mathcal{F} \setminus \{A, B\}$ intersect B, for otherwise, if there exist $D, E \in \mathcal{F} \setminus \{A, B\}$ such that $D \cap A = E \cap B = \emptyset$, then A, B, D, E violate the $(4,3)$ property. Assume without loss of generality that every set in $\mathcal{F} \setminus \{A, B\}$ intersect A. Thus $\mathcal{F} = \mathcal{F}_A \cup \mathcal{F}_{AB} \cup \{B\}$, where $\mathcal{F}_A$ is the family of sets in $\mathcal{F}$ intersecting A and not intersecting B, and $\mathcal{F}_{AB}$ of sets in $\mathcal{F}$ intersecting both A and B. Observe that $\mathcal{F}_A$ must satisfy the $(3,3)$ property, since otherwise a non-intersecting triple of sets in $\mathcal{F}_A$ together with B violate the $(4,3)$ property. Thus by Helly's theorem $\tau(\mathcal{F}_A) = 1$. Furthermore, $F_{AB} \cup \{B\}$ satisfy the $(2,2)$ property since if $E, D \in \mathcal{F}_{AB}$ are disjoint then A, B, E, D violate the $(4,3)$ property. Thus $\tau(\mathcal{F}) \leq \tau(F_{AB} \cup \{B\}) + \tau(\mathcal{F}_A) \leq c + 1$, proving the observation. $\square$

1.3. **Our result.** In this work we further investigate the $(4,3)$ problem in $\mathbb{R}^2$. To this end we define the notion of fat sets. A set $S \subset \mathbb{R}^2$ will be called r-fat for some number $0 < r \leq 1$ if there exists a point $c \in S$ such that $B(c,r) \subseteq S \subseteq B(c,1)$, where $B(c,r)$ is the ball in $\mathbb{R}^2$ of radius r with center-point c. Thus a 1-fat set is a unit disk. Note that for $r < 1$ an r-fat set is not necessarily convex. Let $c_{fat}(r)$ denote the maximal piercing number in families of r-fat sets in $\mathbb{R}^2$ that satisfy the $(4,3)$ property. In this terminology, Kynĉl and Tancer's result is the following:

Theorem 1.4 ([9])**.** *We have* $c_{fat}(1) = 3$.

In this REU project we extend Theorem 1.4 by proving bounds on the $c_{fat}(r)$ for $\sqrt{8} - 2 \leq r < 1$. We prove:

Theorem 1.5. *We have* $c_{fat}(\sqrt{8} - 2) \leq 4$.

In Section 2 we establish some preliminaries needed for the proof of this theorem, and the proof is then given in Section 3.

2. Preliminaries

For an r-fat set $S \subset \mathbb{R}^2$ let $c_S \in S$ be such that $B(c_S, r) \subseteq S \subseteq B(c_S, 1)$. Let $\mathcal{F}$ be a family of r-fat sets in $\mathbb{R}^2$ satisfying the $(4,3)$-property. We may assume that $|\mathcal{F}| \geq 4$, for other wise Theorem 1.5 is trivial.

Let $A, B \in \mathcal{F}$ be such that $d := \text{dist}(c_A, c_B) = \max_{D,E \in \mathcal{F}} \text{dist}(c_D, c_E)$, where dist stands for the Euclidean distance. By rotating and translating $\mathcal{F}$ we may assume that c_A is the origin and c_B is to the right of c_A, namely c_B is the point $(d, 0)$.

We will need the following three simple lemmas.

Lemma 2.1. *For every* $D, E \in \mathcal{F} \setminus \{A, B\}$ *we have* $\mathit{dist}(c_D, c_E) \leq 2$.

Proof. The lemma is trivial if $d \leq 2$. If not, then $A \cap B = \emptyset$. If in addition $D \cap E = \emptyset$, then in the collection $\{A, B, D, E\} \subset \mathcal{F}$ no three of the sets intersect, violating the $(4,3)$ property of $\mathcal{F}$. Thus D, E intersect, implying $\text{dist}(c_D, c_E) \leq 2$. □

By the same arguments as in Observation 1.3 we have:

Lemma 2.2. *If* $A, B \in \mathcal{F}$ *are disjoint then* $\nu(\mathcal{F}) = 2$. *Moreover, either* A *intersects every disk in* $\mathcal{F} \setminus \{B\}$ *or* B *intersects every disk in* $\mathcal{F} \setminus \{A\}$.

Lemma 2.3. *Let* $F_i = B(c_i, r_i), 1 \leq i \leq n$ *be disks in* $\mathbb{R}^2$ *with* $r_i \leq r_{i+1}$ *for all* $1 \leq i \leq n - 1$.

1. *If there exists* $c \in \mathbb{R}_2$ *such that such that* $c_i \in B(c, r_1)$ *for all* $1 \leq i \leq n$, *then* $\bigcap_{i=1}^n F_i \neq \emptyset$.
2. *If* $\bigcap_{i=1}^n F_i \neq \emptyset$ *then there exists* $c \in \mathbb{R}^2$ *such that* $c_i \in B(c, r_n)$ *for all* $1 \leq i \leq n$.

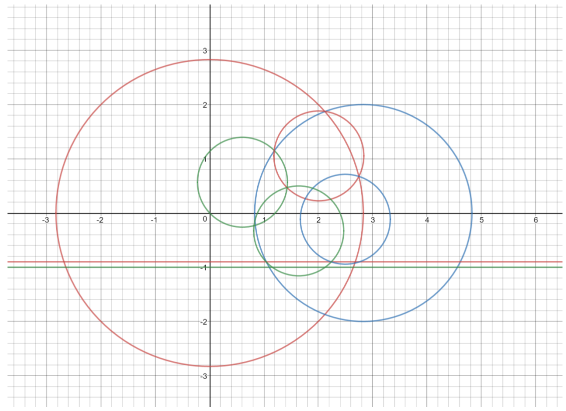

FIGURE 1. R_1 is contained in the union of 4 disks of radii $\sqrt{8}-2$ in Case 1.

Proof. (1) We have $\operatorname{dist}(c, c_i) \le r \le r_i$ for every $1 \le i \le n$, implying $c \in \bigcap_{i=1}^n F_i$. (2) Let $p \in \bigcap_{i=1}^n F_i$. Then for every $1 \le i \le n$ we have $\operatorname{dist}(p, c_i) \le r_i \le r_n$, implying $c_i \in B(p, r_n)$ for every i. □

For $a \in \mathbb{R}$ let $H^+(a)$ and $H^-(a)$ denote the closed half planes above and below the line $y = a$, respectively. For $u, v \in \mathbb{R}^2$ let $[u, v]$ denote the line segment connecting u and v.

3. Proof of Theorem 1.5

Define $C = \{c_F \mid F \in \mathcal{F}\}$. By Lemma 2.3, the proof of the first assertion in Theorem 1.5 will follow if we show that C is contained in the union of at most 4 disks of radii $\sqrt{8}-2$.

Let $A, B \in \mathcal{F}$ and d be as in the previous section. If A, B intersect then $d \le 2$, and thus $C \subset B(c_A, 2) \cap B(c_B, 2)$. If A, B are disjoint, then by Lemma 2.2 we may assume that B intersects every set in $\mathcal{F} \setminus \{A, B\}$, and thus $C \setminus \{c_A, c_B\} \subset B(c_A, d) \cap B(c_B, 2)$. We distinct three cases.

Case 1. $d \le \sqrt{8}$ and there exists $F \in \mathcal{F}$ such that $c_F \in H^+(1.1)$. In this case, by Lemma 2.2 we must have $c_E \subset H^+(-0.9)$ for every $E \in \mathcal{F} \setminus \{A, B\}$. Therefore we have $C \subseteq R_1$, where

$$R_1 = \Big(\big(B(c_A, \sqrt{8}) \cap B(c_B, 2)\big) \cup [c_A, c_B]\Big) \setminus H^-(-0.9).$$

The theorem then follows since $R_1 \subset \bigcup_{i=1}^4 B(p_i, \sqrt{8}-2)$, where $p_1 = ((\sqrt{8}-2)\cos(0.24\pi), (\sqrt{8}-2)\sin(0.24\pi))$, $p_2 = (2.01, 1.053)$, $p_3 = (2.4972, -0.115)$ and $p_4 = (1.64, -0.33)$ (see Figure 1).

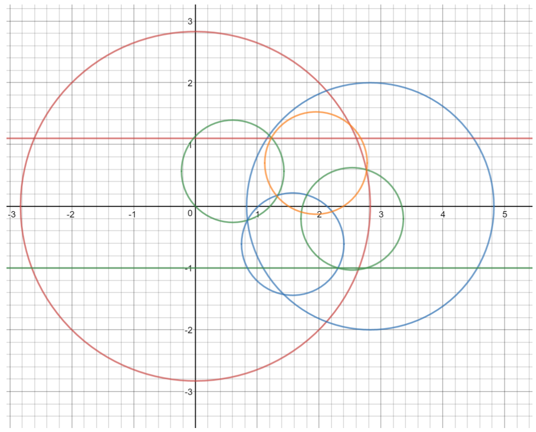

FIGURE 2. R_2 is contained in the union of 4 disks of radii $\sqrt{8}-2$ in Case 2.

Case 2. $d \leq \sqrt{8}$ and for every $F \in \mathcal{F}$ we have $c_F \in H^-(1.1)$. In this case $C \subset R_2$, where

$$R_2 = \Big(B(c_A, \sqrt{8}) \cap B(c_B, 2) \cap H^-(1.1)\Big) \cup [c_A, c_B].$$

In this case our theorem follows from $R_2 \subset \bigcup_{i=1}^4 B(p_i, \sqrt{8}-2)$, where $p_1 = ((\sqrt{8}-2)\cos(0.24\pi), (\sqrt{8}-2)\sin(0.24\pi))$, $p_2 = (1.5739, -0.6133)$, $p_3 = (2.5357, -0.204)$, and $p_4 = (1.95, 0.7)$ (see Figure 2).

Case 3. $d > \sqrt{8}$. Here A, B are disjoint, and as before we assume without loss of generality that B intersect every set in $\mathcal{F} \setminus \{A, B\}$.

Let $\mathcal{F}_B \subset \mathcal{F}$ be the subfamily of sets in $\mathcal{F}$ that do not intersect A, and let $\mathcal{F}_{AB} \subset \mathcal{F}$ be the subfamily of elements in $\mathcal{F}$ intersecting both A and B. Then we have $\mathcal{F} = \mathcal{F}_B \cup \mathcal{F}_{AB} \cup \{A\}$. We further observe that since $\mathcal{F}$ satisfies the $(4,3)$ property, $\mathcal{F}_B$ must satisfy the $(3,3)$ property, and thus by Helly's theorem we have $\tau(\mathcal{F}_B) = 1$, implying $\tau(\mathcal{F}_B \cup \{A\}) = 2$.

Finally, note that for every $E \in \mathcal{F}_{AB}$ we have $c_E \in R_3$, where

$$R_3 = B(c_A, 2) \cap B(c_B, 2),$$

and $R_3 \subset B((\sqrt{2}, 2-\sqrt{2}), \sqrt{8}-2) \cup B((\sqrt{2}, \sqrt{2}-2), \sqrt{8}-2)$ (see Figure 3). Therefore, by Lemma 2.3, $\tau(\mathcal{F}_{AB}) \leq 2$. This completes the proof of the theorem. $\square$

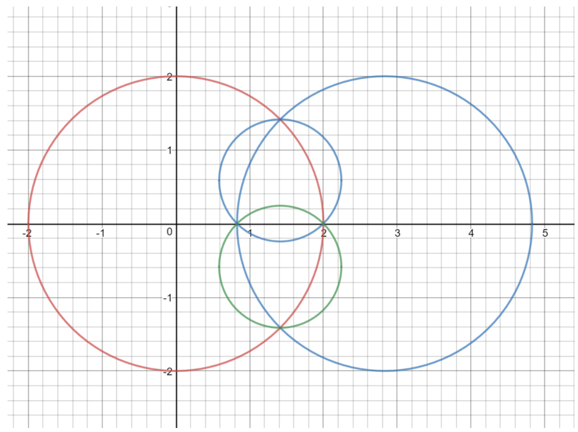

FIGURE 3. R_3 is contained in the union of 2 disks of radii $\sqrt{8}-2$ in Case 3.

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