Numerical Ring Invariants and the Alternating Sums of Graded Betti Numbers

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Abstract

Utilizing a computation on the numerical invariants of graded rings and modules, this REU paper will show methods to use the Hilbert function to recover information about the alternating sums of graded Betti numbers. The consistency and structure of this computation will be explored for the coordinate rings of varieties of projective space, and reduced modules over polynomial rings. Next steps in the research will be discussed.

1 Introduction

In the study of commutative algebra and projective algebraic geometry, it is often useful to study numerical invariants on graded S-modules $M = \oplus_{n \in \mathbb{N}} M_i$ where S is a polynomial ring in degree-one variables over an algebraically closed field. We will use the convention that $M_a(d) = M_{a+d}$. Invariants such as the Hilbert function $H_M(n) = \dim_k(M_n)$ and the Hilbert series $h_M(t) = \sum_{k \in \mathbb{N}} H_M(k) t^k$, reveal much about the underlying algebraic geometry and intersection theory of their associated projective varieties.

The deep connection to intersection theory encoded in the Hilbert polynomial here is that the Hirzebruch-Riemann-Roch theorem (a seminal theorem of intersection theory) gives us a way of understanding the Euler characteristic of a projective variety X over an algebraically closed field k with associated ring R. That is, given knowledge of the Chern and Todd classes of $\mathbb{P}^d$, as well as knowledge of the sections $\mathcal{O}(n)$ of a very ample invertible sheaf on X, we can compute the Euler characteristic $\chi(\mathcal{F}(n))$ [4]. Let's see the connection by restating the Hilbert polynomial in terms of the Euler characteristic, χ. That is, we know that given a coherent sheaf $\mathcal{F}$, and a very ample invertible sheaf $\mathcal{O}(1)$ the function $\chi(\mathcal{F}(-)) : \mathbb{N} \longrightarrow \mathbb{N}$ that follows the rule $n \mapsto \chi(\mathcal{F} \otimes \mathcal{O}(1)^{\otimes n})$ is, for $n \gg 0$ equal to the dimension of the cohomology module $h^0(\mathcal{F} \otimes \mathcal{O}(1)^{\otimes n})$. But we have this spectacular theorem that the ring $R' = \oplus_{n \in \mathbb{Z}} (h^0(\mathcal{F} \otimes \mathcal{O}(1)^{\otimes n}))$ is isomorphic to $k[x_0, ..., x_d]/I$, where I is the ideal of the image of the mapping induced by $\mathcal{O}(n)$, which is exactly R. we can now see that $H_R(n) = \chi(\mathcal{F}(n))$ for $n \gg 0$.[1]

Now, of course, the above meanderings are not the most straightforward way to compute the Hilbert polynomial. The most standard way to go about this business is to obtain a free resolution of the ring R in terms of a polynomial ring $k[x_0, ...x_d]$. Then the alternating sums of the syzygy modules' Hilbert polynomials are just the Hilbert polynomial of R.[3]

But there is another way to obtain the Hilbert polynomial. Let's introduce another, third numerical invariant on a graded R-module M, the formal sum $h_{GM}(t) = \sum_{i,j \in \mathbb{N}} (-1)^i \beta_{ij} t^j$. Using the graded free resolution, we may show that $H_M(n) = \sum_{j=0}^{kdim(R)} (\sum_{i=0}^{\infty} (-1)^i \beta_{ij}) H_S(n-j)$, where $kdim$ is the krull dimension of R. We will typically write this as: $H_M(n) = \sum_{j=0}^{kdim(R)} \alpha_i H_S(n-j)$, where the α_i have the obvious meaning. It makes sense to write this as a matrix equation for sufficiently large variables n_i, Represented as $[H_S(n_i - j)][\alpha_i] = [H_M(n_i)]$, where j indexes rows and i indexes columns. From there, we can ask questions about this matrix. Namely, the questions of whether or not this equation is consistent, whether the matrix $[H_S(n_i - j)]$ is invertible, and whether or not the alternating sums of Betti numbers, the α_i, are dependent on our choice of n_i.

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2 Consistency in the Easiest Case

What we have really produced in the above section is an algorithm that ultimately finds the alternating sums of graded Betti numbers for a graded module given knowledge of the Hilbert functions of the module and its ambient ring. Let us work in the simplest case, where $S = k[x_0, ..., x_d]$. Let M be a graded, finitely generated S-module, and our equation becomes:

$$\begin{pmatrix} H_S(n_0) & \cdots & H_S(n_0-d) \\ \vdots & \ddots & \vdots \\ H_S(n_d) & \cdots & H_S(n_d-d) \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_d \end{pmatrix} = \begin{pmatrix} H_M(n_0) \\ \vdots \\ H(n_d) \end{pmatrix}$$

where α_i are the alternating sums of the graded Betti numbers of M.

We work in this case for two reasons. The first is that the proofs involved while dealing with this case, and the other cases discussed at the end of the paper are so similar that they do not bear repeating. Instead, lemmas will be introduced that allow direct translation of our inductive arguments. The second reason that this case is the only one that will be fully worked is that it relies on intuition about binomial coefficients. The lemmas introduced later on will just be, in essence, a statement of some version of Pascal's rule for Hilbert polynomials.

2.1 Invertibility

Returning back to our main discussion, we wish to show that the $(d+1)\times(d+1)$ matrix introduced in the above section is invertible.

Theorem 1. *The matrix*

$$A = \begin{pmatrix} H_S(n_0) & \cdots & H_S(n_0-d) \\ \vdots & \ddots & \vdots \\ H_S(n_d) & \cdots & H_S(n_d-d) \end{pmatrix}$$

is invertible if and only if all the n_i are unequal.

Proof. Were some $n_i = n_j$ for $i \neq j$, then two of the columns would be the same, and the determinant 0.

Conversely, proceed by induction on d. When d=1, det(A)=$n_0 - n_1$, which is easily verifiable by hand.

In the inductive case, it is of note that the ijth entry can be written as $H_{S(-j)}(n_i)$. This motivates writing the following:

$$A = \begin{pmatrix} \binom{n_0+d}{d} & \binom{n_0+d-1}{d} & \cdots & \binom{n_0}{d} \\ \vdots & \ddots & \ddots & \vdots \\ \binom{n_d+d}{d} & \binom{n_d+d-1}{d} & \cdots & \binom{n_d}{d} \end{pmatrix}$$

To show that this is invertible, produce a new matrix A' of equal determinant by subtracting each column other than the first from the column to its immediate right, so that the ijth entry of the new matrix A' is $\binom{n_i+d-j}{d} - \binom{n_i+d-j+1}{d}$. After applying Pascal's rule, we write

$$A' = \begin{pmatrix} \binom{n_0+d}{d} & -\binom{n_0+d-1}{d-1} & \cdots & -\binom{n_0}{d-1} \\ \vdots & \ddots & \ddots & \vdots \\ \binom{n_d+d}{d} & -\binom{n_d+d-1}{d-1} & \cdots & -\binom{n_d}{d-1} \end{pmatrix}$$

By our inductive hypothesis, we know that the $d \times d$ minor whose top leftmost entry is $-\binom{n_0+d-1}{d-1}$ is invertible. This implies that columns 2 to d are of full rank. Now consider the columns of A'. Define a linear transformation $T : \mathbb{R}^{d+1} \longrightarrow P_d$, where P_d is the vector space of degree at most d polynomials with coefficients in $\mathbb{R}$. Let $T([x_i])$=$p(k)$, where p is the polynomial produced by performing Lagrange interpolation on $[(x_i, n_i)]$. This function with this particular choice of codomain is an isomorphism of vector spaces, so it sends linearly independent sets to linearly independent sets, and we can see trivially that columns 2 to d will be sent to linearly independent degree $d-1$ polynomials, whereas column 1 will be sent to a degree d polynomial. So the columns of A' are linearly independent, and so A is invertible. □

This fact actually gives as a corollary the ability to understand the polynomial structure of $\det(A)$, up to a scalar factor.

Corollary 1. *The determinant of the matrix A of Theorem 1 is, for some scalar c equal to* $\frac{1}{c}\prod_{i\neq j}(n_i - n_j)$

From the above proof we know that A is invertible if and only if the sequence $n_0 \cdots n_d$ contains only distinct integers (forwards is free, if $n_i = n_j$ was equal, the $i+1$st and the $j+1$st row would be the same). From this we can infer that the determinant of A must divide the following polynomial in the n_i:

$$(*) = \frac{1}{c}\prod_{i\neq j}(n_i - n_j)$$

So the degree of $\det(A)$ as a polynomial in the n_i is at least $\binom{d}{2}$. Now, by performing elementary column operations on A in a similar manner as described in the proof of theorem 1, we can obtain the following matrix:

$$A'' = \begin{pmatrix} \binom{n_0+d}{d} & -\binom{n_0+d-1}{d-1} & -\binom{n_0+d-2}{d-2} & \cdots & -\binom{n_0}{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \binom{n_d+d}{d} & -\binom{n_d+d-1}{d-1} & -\binom{n_d+d-2}{d-2} & \cdots & -\binom{n_d}{0} \end{pmatrix}$$

In this way, we know that $\det(A'')$=$\det(A)$ can be written as a linear combination of polynomials of degree $0+1+2+....+d = \frac{1}{2}d(d-1) = \binom{d}{2}$. So $\deg(\det(A))$ is at most $\binom{d}{2}$. So it must be that $\det(A)$ is $(*)$, for some scalar c.

2.2 n_i dependency

The above sections only proved that solutions to $[H_S(n_i - j)][\alpha_i] = [H_M(n_i]$ exist, not whether they are independent on choices of $n_0 \cdots n_d$. This is an important consistency check. To motivate this, we note that these equations are often too large to reasonably solve by hand, and that a program such as Mathematica may be unable to give a reasonable closed form. In even a simple example, if you try to solve this equation for $d = 3$ and $M = S/(f)$ where f has degree $i > 2$, you will most likely end up with:

$$\begin{pmatrix} 1/6(11i - 6i^2 + i^4) \\ 1/2(-6 + 5i^2 - i^3) \\ 1/2(-6i - 4i^2 - i^3) \\ r(n_0, n_1, n_2, n_3, i) \end{pmatrix}$$

Where $r(n_0, n_1, n_2, n_3, i)$ is a rational function whose closed form takes up about 3 pages. This is not a useful solution. However, after a few hours of frustration, you might be lucky enough to obtain the more reasonable closed form:

$$\begin{pmatrix} 1/6i(11+5i+i) \\ -1/2i(6-5i+i^2) \\ 1/2i(3-4i+i^2) \\ -1/6i(2-3i+i^2) \end{pmatrix}$$

Although this case can be resolved in a reasonable amount of time with computing software, larger, more complicated cases may not be tractable. Here we prove a result motivated by examples as above:

Theorem 2. *For $k[x_0, ..., x_d] = S$, the alternating sums of graded Betti numbers of a finitely generated S-module M are expressible as polynomials of degree $\leqslant d$ in the twists of S appearing in the graded free resolution of M. It is possible to find these polynomials from the matrix equation $[H_S(n_i - j)][\alpha_i] = [H_M(n_i]$ for any choices of sufficiently large n_i.*

Proof. M is Noetherian, so Hilbert's syzygy theorem gives that it has a finite graded free resolution [3]. This implies that its Hilbert polynomial may be written as a finite linear combination of shifted binomial coefficients, so it suffices to show that, for

$$\begin{pmatrix} H_S(n_0) & \cdots & H_S(n_0-d) \\ \vdots & \ddots & \vdots \\ H_S(n_d) & \cdots & H_S(n_d-d) \end{pmatrix} \begin{pmatrix} f_0 \\ \vdots \\ f_d \end{pmatrix} = \begin{pmatrix} \binom{n_0+d-k}{d} \\ \vdots \\ \binom{n_d+d-k}{d} \end{pmatrix}$$

The f_i are polynomials in k. In fact, we only need to prove this for a single row. We can prove this by induction on d (the base step here is omitted, but easy to verify). Now suppose there exist polynomials $p_i(k)$ of degrees $\leqslant d-1$ that satisfy, for all $n \in \mathbb{Z}$:

$p_0(k)\binom{n+d-1}{d-1} + \cdots + p_d(k)\binom{n}{d-1} = \binom{n+d-1-k}{d-1}$

Then, by elementary properties of binomial coefficients:

$\frac{n+d-k}{r-1}(p_0(k)\binom{n+d-1}{d-1} + \cdots + p_d(k)\binom{n}{d-1}) = \binom{n+d-k}{d}$

To show that the resulting polynomials are not dependent on choice of n, consider the ith term in the above linear combination:

$\frac{n+d-k}{d-1}p_i(k)\binom{n+d-i}{d-1}$

$=(\frac{n+d-i}{d-1} + \frac{i-k}{d-1})p_i(k)\binom{n+d-i}{d-1}$

$=(\binom{n+d-i+1}{d} + \frac{i-k}{d-1}(\binom{n+d-i}{d} - \binom{n+d-i+1}{d}))p_i(k)$

This gives that $f_{i+1}(k) = (1 + \frac{i-k}{d-1})p_i(k) + p_{i+1}(k)$, which is a polynomial of degree $\leqslant d$. □

Another useful observation: If we have $M = S/I$, our polynomial ring modded out by the ideal of a regular sequence, $s_1 \cdots s_n \in S$ whose degrees are r_i and $n < d$, a quick modification to the above proof gives that the polynomials α_i will have degrees exactly d in the r_i. If $(s_1 \cdots s_n) = (1)$, then some r_i=0 and $S/(1) = 0$. All of the Betti numbers of $S/(1)$ are 0. This implies that the α_i are all zero, so that $\alpha_i(r_1, \cdots, r_n) = \prod_{k=1}^{n} r_k \gamma(r_1, \cdots, r_n)$, where γ is a degree $n-d$ polynomial.

3 The case of S/I

The above proofs were painfully reliant on results of the Hilbert syzygy theorem, that is, our ability to write Hilbert functions of a module over a ring $k[x_0, ..., x_r] = S$ as finite linear combinations of S's Hilbert function. This finite-length free resolution is what gives us an easy way to write the Hilbert polynomials of finitely generated S-modules as finite linear combinations of H_S. When we change our ambient ring, so that instead of S-modules we consider R-modules where $R = S/I$ is a factor of S by some homogeneous ideal I, we very quickly lose our ability to write finite free resolutions in terms of R. The simplest example of this is that if $R = k[x]/(x^n)$, then there is no $m \neq 0$ that $R/(x^m)$ has a finite free resolution in terms of R [2]. There are two ways to cope with this. The first of which is to use properties of rings and modules to reduce the statement into something easily understandable, the second of which is to consider the R-module as a

S-module and obtain the Hilbert polynomial by using a finite syzygy complex. Here is an example of the former, before we move on to a longer discussion of the latter:

Example It is well known that for a homogeneous ideal I not generated by linear elements and ring $R = S/I$, the quotient R-module R/aR with a being the irrelevant ideal has no finite free resolution over R [2]. However, we know that $R/aR \cong S/a \otimes_S R$. Then we have that the ith graded part of $S/a \otimes_S R$ is:

$$(S/a \otimes_S R)_i = \oplus_{m+n=i}((S/a)_m \otimes_S R_n)$$

We know that since S/a is a residue field, if $m > 0$, then $(S/a)_m \otimes_S R_n = 0$. This vastly simplifies our problem since now we know that $H_{R/aR}(i)$ =dim$_k$ $(S/a \otimes_S R_i)$, where dim$_k$ is the dimension of the graded part of a module over a k-algebra as a k-vector space. This allows us to conclude that $H_{R/aR} = H_R$. This also gives that when we consider the system of linear equations $A\alpha = H_M$, our α is the standard basis vector e_1, for any I.

Even in this relatively simple example, calculation of the Hilbert polynomial becomes very much non-obvious. This motivates the next section.

3.1 Regular Sequences, or How I Learned to Stop Worrying and Love the Koszul Complex

The simplest interesting case of considering our ambient ring being $R = S/I$ is when $I = (f)$, for some homogeneous element f who is at least quadratic. The main question is whether or not the theorems of section 2 hold in this capacity. This exclusively involves coming up with a method of generalizing the property of binomial coefficients used in section 2 to Hilbert polynomials. Namely, Pascal's rule.

Lemma 1. *Let $f \in S = k[x_0, ..., x_d]$ and $f' \in S' = k[x_0, ..., x_{d-1}]$ have that deg(f)=deg(f')=$i \geqslant 2$. Then $H_{S/(f)}(n) - H_{S/(f)}(n+1) = -H_{S'/(f')}(n)$. Moreover, $H_{S/(f)}$ has degree $r-1$ and $H_{S'/(f')}$ degree $r-2$.*

Proof. This is a pretty straightforward binomial coefficient argument. We note that $S/(f)$ has free resolution $\mathbf{0} \longrightarrow S(-i) \longrightarrow S$ (this is simply the Kozsul complex of the regular sequence f). We can now explicitly calculate its Hilbert polynomial:

$H_{S/(f)}(n) = \binom{n+d}{d} - \binom{n+d-i}{d}$ By Pascal's rule:

$= \binom{n+d-i}{d} + \sum_{k=1}^{i} \binom{n+d-k}{d-1} - \binom{n+d-i}{d}$

$= \sum_{k=1}^{i} \binom{n+d-k}{d-1}$. This shows that $H_{S/(f)}$ has degree $d-1$.

Now we can see a way to proceed with the remainder of the proof:

$H_{S/(f)}(n) - H_{S/(f)}(n+1)$

$=\sum_{k=1}^{i} \binom{n+d-k}{r-1} - \sum_{k=1}^{i} \binom{n+1+d-k}{d-1}$

$=\binom{n+d-i}{r-1} - \binom{n+d}{d-1} + \sum_{k=1}^{i}(\binom{n+d-k}{d-1} - \binom{n+d-k}{d-1})$

$=\binom{n+d-i}{d-1} - \binom{n+d}{d-1}$

Noting that the free resolution of $S'/(f')$ is $\mathbf{0} \longrightarrow S'(-i) \longrightarrow S'$ completes the proof. □

Actually, as it turns out these results generalize further to any ambient ring $R = S/I$ with I generated by a regular sequence. The underlying feature here is that if you have a regular sequence of length n whose elements have degree r_i in $S = k[x_0, ..., x_d]$, and $R = S'/J$ with J generated by a regular sequence of length n whose elements have degree r_i in $S' = k[x_0, ..., x_{d-1}]$ (if it exists) will have the same free resolution with the same twists, that is, the Koszul complex on n elements, only exchanging S for S' in the second case. So, when you write the Hilbert polynomial H_R as a linear combination of $H_{S(-j)}$'s, they have the same coefficients, just substituting $H_{S(-j)}$ for $H_{S'(-j)}$. Moreover, we have the property that deg(H_R)=$r - n$, so that we know additionally that degree is preserved correctly over these generalizations. I will state without proof the necessary generalizations of binomial identities to Hilbert polynomials of factor rings of regular sequences, whose proofs are easier than the above lemma. We will see that the algorithm explored in section 2 is consistent for all regular sequences.

Lemma 2. *Let $n < d-1$ and $f_1, ...f_n \in S = k[x_0, ..., x_d]$ and $f'_1...f'_n \in S' = k[x_0, ..., x_{d-1}]$ be regular sequences who have that deg(f_i)=deg(f'_i)=$j \geqslant 2$. Then $H_{S/(f_1,...f_n)}(k) - H_{S/(f)}(k+1) = -H_{S'/(f'_1...f'_n)}(k)$. Moreover, $H_{S/(f_1,...f_n)}$ has degree $d-n$ and $H_{S'/(f'_1...f'_n)}$ degree $d-(n+1)$.*

4 Future Directions and Conclusions

This paper presents proofs for a wide variety of modules over polynomial rings and factors thereof for the consistency of the equation $[H_S(n_i - j)][\alpha_i] = [H_M(n_i)]$. Consistency is checked by first proving that the matrix $[H_S(n_i-j)]$ is invertible, then showing that solutions to this equation can be shown to be independent of the choice of the values of the variables n_i This consistency is checked for rings $S = k[x_0, ..., x_d]$ and factors of S by ideals generated by regular sequences, and finitely generated S- and R- modules. It would be interesting to look at classes of rings and modules where this equation is not consistent, if they exist. In order to find more classes of rings and modules for which this equation either works or doesn't, it will be necessary to characterize in greater detail the Krull dimension and the degree of the Hilbert polynomial of the ring. The former issue is not too complex, and for modules that are factor rings, can be done by looking at localizations at maximal ideals. The latter however, is a harder question. It can be done by looking directly at the syzygies of a module and computing examples directly. In [1], a sheaf-theoretic approach to talking about the degree of Hilbert polynomials is presented, which may have more power than direct computation.

References

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