Small x asymptotics for special function solutions of Painlevé III Equation

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Abstract

In this paper, we apply the method of [Dn18] to Painlevé III equation and study the small x asymptotic behavior of its special function solutions. We first find the simultaneous solutions between Ricatti and generic Painlevé III equations and express it in terms of the solutions of Bessel equation. Via Bäcklund transformations, we construct the solutions for Painlevé III with more general parameters. Since there are no closed formulas for these solutions, we shift our focus to compute the explicit formula for the associated tau function using certain identities from linear algebra. To do asymptotic analysis of the expression, we rewrite the formula in the form suitable for computing $x \to 0$ asymptotics and obtain our main result.

Contents

1 Introduction

Painlevé equations are six **nonlinear second-order ordinary differential equations**. They are written in the form of $y'' = R(y', y, t)$ with R a rational function. Their solutions have the so called **Painlevé property**. That means the locations of singularities that are $\log(z)$ type don't depend on the initial conditions, but the locations of singularities that are $1/z$ type do depend. Most of the solutions of Painlevé equations are **transcendental**. That means their solutions can't be reduced to simpler special functions. However, there are several exceptions in particular for Painlevé III equation. It is easier to study asymptotic behavior of such solutions and that is why we are interested in it. Also, Painlevé equations have wide applications in other areas of mathematics, especially **random matrix theory**, see [FW01, FW02].
In particular, Painlevé III equation is given by:

$$u''(x) = \frac{(u'(x))^2}{u(x)} - \frac{u'(x)}{x} + \frac{\alpha u^2(x) + \beta}{x} + u^3(x) - \frac{1}{u(x)}, \quad \alpha, \beta \in \mathbb{C}.$$

Consider the tau function given by the determinantal form:

$$\tau_n(x) = x^{n(n-1)}(-1)^{\frac{n(n-1)}{2}} \det\left(\left\{f_{\frac{\alpha}{2}-j+k}(x)\right\}_{j,k=0}^{n-1}\right)$$

Then the solution of Painlevé III equation can be written as the following form:

$$q_n(x) = \frac{\tau_{n+1}(x, \alpha-2)\tau_n(x, \alpha)}{\tau_{n+1}(x, \alpha)\tau_n(x, \alpha-2)}$$

After the asymptotic analysis of the expression of $\tau_n(x)$, we rewrite the formula in the form suitable for computing $x \to 0$ asymptotics in terms of the leading term and obtain the following result:

$$\tau_n(x) \sim \frac{2^{-\alpha r_c + 2r_c(n-r_c) + \frac{\alpha n}{2}}}{\pi^n} d_1^{r_c} d_2^{n-r_c} (\sin(\frac{\alpha}{2}\pi))^n \frac{G(r_c+1)G(-\frac{\alpha}{2}+n-r_c+1)G(n-r_c+1)G(r_c+\frac{\alpha}{2}+1)}{G(-\frac{\alpha}{2}+n-2r_c+1)G(\frac{\alpha}{2}-n+2r_c+1)} x^{n(n-1)+\alpha r_c - \frac{n\alpha}{2} - 2r_c(n-r_c)}$$

where

$$\begin{aligned}
&d_1 = c_1(1+\cot(\pi\frac{\alpha}{2})i) + c_2(1-\cot(\pi\frac{\alpha}{2})i),\\
&d_2 = c_2\csc(\pi\frac{\alpha}{2})i - c_1\csc(\pi\frac{\alpha}{2})i,\\
&c_1, c_2 \in \mathbb{C},\\
&r_c(\alpha, n) = \begin{cases} 0 & \text{if } \alpha > 2n-2 \\ n-j & \text{if } -2n+4j-2 < \alpha < -2n+4j+2 \text{ and } j = 1, 2, \dots, n-1 \\ n & \text{if } \alpha < -2n+2 \end{cases},\\
&G(x) \text{ refers to Barnes G-function}
\end{aligned}$$

Eventually, in this paper, we derived the asymptotic of $q_n(x)$ as $x \to 0$ applying the result above to the solution of $q_n(x)$ through tau function as our main result:

$$q_n(x) \sim \begin{cases} \frac{1}{2n+2-\alpha} x & \text{if } \alpha > 2+2n \\ (-1)^n \left(\frac{d_1}{d_2}\right) \left[\left(\frac{\Gamma(-\frac{\alpha}{2}-n+2j+1)}{\Gamma(\frac{\alpha}{2}+n-2j)}\right)^2 \frac{\Gamma(n-j+\frac{\alpha}{2})\Gamma(n-j+1)}{\Gamma(-\frac{\alpha}{2}+j+1)\Gamma(j+1)}\right] (\frac{x}{2})^{\alpha+2n-4j-1} & \text{if } -2n+4j < \alpha < -2n+4j+2 \\ & \text{and } j = 0, 1, \dots, n \\ \left(\frac{d_2}{d_1}\right) \left[\left(\frac{\Gamma(\frac{\alpha}{2}+n-2j+1)}{\Gamma(-\frac{\alpha}{2}-n+2j)}\right)^2 \frac{\Gamma(-\frac{\alpha}{2}+j+1)\Gamma(j)}{\Gamma(n+\frac{\alpha}{2}-j+1)\Gamma(n-j+1)}\right] (\frac{2}{x})^{2n-4j+\alpha+1} & \text{if } -2n+4j-2 < \alpha < -2n+4j \\ & \text{and } j = 1, 2, \dots, n \\ -\frac{\alpha+2n}{x} & \text{if } \alpha < -2n \end{cases}$$

2 Review of important definitions and results

2.1 Bessel equation and contour integral representation of its solution

Definition 2.1.1. *Bessel equation is given by*

$$u''(x) + \frac{u'(x)}{x} + \left(1 - \frac{\nu^2}{x^2}\right) u(x) = 0$$

We can remove $\frac{1}{x^2}$ term changing the unknown function $v(x) = x^{-\nu}u(x)$. The new version of Bessel equation is given by

$$v''(x) + \frac{(2\nu+1)}{x} v'(x) + v(x) = 0 \tag{1}$$

One of the standard solutions in the form of series representation is given by (see [DLMF, (10.9.4)])

$$J_\nu(x) = \frac{x^{\,\nu}}{2^{\,\nu}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{\,2k}}{2^{\,2k} k! \Gamma(\nu + k + 1)} \tag{2}$$

where $\Gamma(x)$ is Gamma function.
There are also contour integral representations for standard solutions which are mainly used in this paper. For $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$, they are given by (see [DLMF, (10.9.17), (10.9.18)])

$$J_\nu(x) = \frac{1}{2\pi i} \int_{\infty - i\pi}^{\infty + i\pi} e^{x\sinh(z) - \nu z} dz, \tag{3}$$

$$H_\nu^{(1)}(x) = \frac{1}{\pi i} \int_{-\infty}^{\infty + i\pi} e^{x\sinh(z) - \nu z} dz, \tag{4}$$

$$H_\nu^{(2)}(x) = -\frac{1}{\pi i} \int_{-\infty}^{\infty - i\pi} e^{x\sinh(z) - \nu z} dz, \tag{5}$$

2.2 Some useful identities between cylinder functions

Define the function

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)} \tag{6}$$

The functions $J_\nu(x)$, $Y_\nu(x)$, $H_\nu^{(1)}(x)$, $H_\nu^{(2)}(x)$ are called **cylinder functions** and are denoted by $C_\nu(x)$. We can show that $C_\nu(x)$ satisfies the following properties (see [DLMF, (10.6.2)]):

$$C'_\nu(x) = \frac{\nu}{x} C_\nu(x) - C_{\nu+1}(x) \tag{7}$$

$$C'_\nu(x) = C_{\nu-1}(x) - \frac{\nu}{x} C_\nu(x) \tag{8}$$

And also, we can obtain the following relations (see [DLMF, (10.4.3)]):

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) \tag{9}$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x) \tag{10}$$

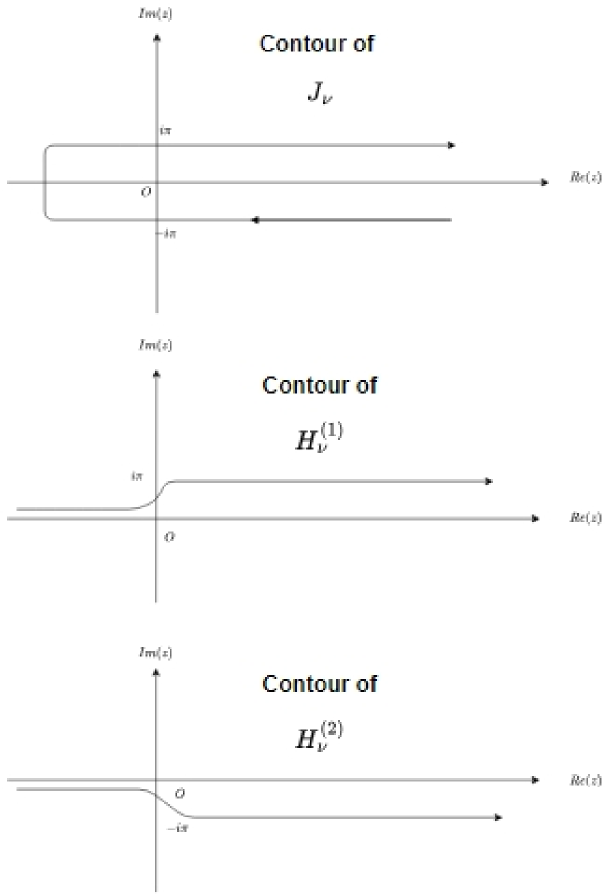

Figure 1: Contours of $J_\nu(x)$, $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$

2.3 The simultaneous solutions of Ricatti and Painlevé III equations

We want to obtain the solutions of Painlevé III, but usually there are no direct method to solve a nonlinear equation. Therefore, in this section, we try to construct simultaneous solutions between Painlevé III and a well-chosen equation which has a close relation with Bessel equation in order to write the solutions of Painlevé III in terms of (4) and (5).

Definition 2.3.1. *Ricatti equation is first order nonlinear ODE given by*

$$u'(x) = a(x)u^2(x) + b(x)u(x) + c(x), \quad a(x) \neq 0. \tag{11}$$

Notice that if $u(x)$ solves Ricatti equation, then $u(x) = -\frac{1}{a(x)}\frac{v'(x)}{v(x)}$ where $v(x)$ solves the following linear ODE:

$$a(x)v''(x) - (a'(x) + a(x)b(x))v'(x) + c(x)a^2(x)v(x) = 0, \quad a(x) \neq 0. \tag{12}$$

Definition 2.3.2. *Painlevé-III equation is second order nonlinear ODE given by*

$$u''(x) = \frac{(u'(x))^2}{u(x)} - \frac{u'(x)}{x} + \frac{\alpha u^2(x) + \beta}{x} + u^3(x) - \frac{1}{u(x)}, \quad \alpha, \beta \in \mathbb{C}. \tag{13}$$

We try to look for the simultaneous solutions of the two equations (11), (13). Taking the first derivative of (11) and plugging in the $u'(x)$, we get:

$$u''(x) = 2a^2(x)u^3(x) + (a'(x) + 3a(x)b(x))u^2(x) + (2a(x)c(x) + b^2(x) + b'(x))u(x) + (b(x)c(x) + c'(x))$$

Meanwhile, plugging (11) into (13), we get:

$$u''(x) = (a^2(x)+1)u^3(x)+(2a(x)b(x)-\frac{a(x)-\alpha}{x})u^2(x)+(b^2(x)+2a(x)c(x)-\frac{b(x)}{x})u(x)+\frac{c(x)-1}{u(x)}+2b(x)c(x)-\frac{c(x)-\beta}{x}$$

By matching and solving for the coefficients, we have totally four cases. We list them below:

$$a(x) = 1, \qquad b(x) = \frac{\alpha-1}{x}, \qquad c(x) = 1, \qquad \beta = 2-\alpha \tag{14}$$

$$a(x) = -1, \qquad b(x) = \frac{-1-\alpha}{x}, \qquad c(x) = -1, \qquad \beta = -2-\alpha \tag{15}$$

$$a(x) = 1, \qquad b(x) = \frac{\alpha-1}{x}, \qquad c(x) = -1, \qquad \beta = \alpha-2 \tag{16}$$

$$a(x) = -1, \qquad b(x) = \frac{-1-\alpha}{x}, \qquad c(x) = 1, \qquad \beta = \alpha+2 \tag{17}$$

Note that for (14), the linearized Ricatti equation (12) becomes the new version of Bessel equation (1) in the following form

$$v''(x) + \frac{(1-\alpha)}{x}v'(x) + v(x) = 0$$

which has a solution

$$v_1(x) = x^{-\nu}(k_1 H_\nu^{(1)}(x) + k_2 H_\nu^{(2)}(x)), \quad \nu = -\frac{\alpha}{2}$$

Alternatively it can be written is

$$v_1(x) = x^{-\nu}(\widetilde{k}_1 H_{-\nu}^{(1)}(x) + \widetilde{k}_2 H_{-\nu}^{(2)}(x)), \quad \nu = -\frac{\alpha}{2}.$$

By changing the function back, the simultaneous solution of (11) and (13) has the form

$$u_1(x) = -\frac{d}{dx}\ln(v_1(x))$$

Similarly, for (15), the linearized Ricatti equation (12) becomes the new version of Bessel equation (1) in the following form

$$v''(x) + \frac{(1+\alpha)}{x}v'(x) + v(x) = 0$$

which has a solution

$$v_2(x) = x^{-\nu}(k_3 H_\nu^{(1)}(x) + k_4 H_\nu^{(2)}(x)), \quad \nu = \frac{\alpha}{2}$$

By changing the function back, the simultaneous solution of (11) and (13) has the form

$$u_2(x) = \frac{d}{dx}\ln(v_2(x))$$

However, we don't consider the rest two cases in this paper since they are corresponding to the so called "modified Bessel equation". The readers can refer to [Cla23, Theorem 3.5] and [DLMF, §32.10(iii)] for details.

2.4 Bäcklund transformation

To construct more solutions for PIII equation with more general parameters, we need to introduce a powerful tool, see [DLMF, §32.7(iii)]

Definition 2.4.1. *Bäcklund transformations for the Painlevé-III equation are given by*

$$B_1 : (u(x), \alpha, \beta) \to \left(\frac{xu'(x) + xu^2(x) - \beta u(x) - u(x) + x}{u(x)(xu'(x) + xu^2(x) + \alpha u(x) + u(x) + x)}, \alpha + 2, \beta + 2 \right)$$

$$B_2 : (u(x), \alpha, \beta) \to \left(\frac{xu'(x) - xu^2(x) + \beta u(x) - u(x) - x}{u(x)(xu'(x) - xu^2(x) - \alpha u(x) + u(x) - x)}, \alpha - 2, \beta - 2 \right)$$

$$B_3 : (u(x), \alpha, \beta) \to \left(-\frac{xu'(x) + xu^2(x) + \beta u(x) - u(x) - x}{u(x)(xu'(x) + xu^2(x) + \alpha u(x) + u(x) - x)}, \alpha + 2, \beta - 2 \right)$$

Assume that $u(x)$ solves Painlevé-III equation. Denote $B_1(u(x), \alpha, \beta) = (v(x), \alpha+2, \beta+2)$. Using Mathematica, we can furtherly show that $v(x)$ solves Painlevé-III equation

$$v''(x) = \frac{(v'(x))^2}{v(x)} - \frac{v'(x)}{x} + \frac{(\alpha+2)v^2(x) + (\beta+2)}{x} + v^3(x) - \frac{1}{v(x)}, \quad \alpha, \beta \in \mathbb{C}.$$

Similarly, denote $B_2(u(x), \alpha, \beta) = (v(x), \alpha - 2, \beta - 2)$. We can show that $v(x)$ solves Painlevé-III equation

$$v''(x) = \frac{(v'(x))^2}{v(x)} - \frac{v'(x)}{x} + \frac{(\alpha-2)v^2(x) + (\beta-2)}{x} + v^3(x) - \frac{1}{v(x)}, \quad \alpha, \beta \in \mathbb{C}.$$

In addition, denote $B_3(u(x), \alpha, \beta) = (v(x), \alpha + 2, \beta - 2)$. We can show that $v(x)$ solves Painlevé-III equation

$$v''(x) = \frac{(v'(x))^2}{v(x)} - \frac{v'(x)}{x} + \frac{(\alpha+2)v^2(x) + (\beta-2)}{x} + v^3(x) - \frac{1}{v(x)}, \quad \alpha, \beta \in \mathbb{C}.$$

More specifically, in our case, we can apply B_1 on $u_1(x)$ to construct all the solutions satisfying $\alpha + \beta = 2 + 4\mathbb{N}$. Moreover, we can also apply B_2 on $u_2(x)$ to construct all the solutions satisfying $\alpha + \beta = -2 - 4\mathbb{N}$.

Remark 1: Notice that one of the factors in the denominator of the new solutions is one side of Ricatti equation. There is a singularity for B_1 when $\alpha + \beta = -2$ and when $u_2(x)$ solves Ricatti eqaution with condition (15) , so it can only be applied on $u_1(x)$. Similarly, there is a singularity for B_2 when $\alpha + \beta = 2$ and when $u_1(x)$ solves Ricatti eqaution with condition (14), so it can only be applied on $u_2(x)$.

Remark 2: We will not use B_3 to construct solutions in this paper since it can only be applied on the solutions corresponding to "modified Bessel equation". But it is useful to formulate the general solution in section 2.7.

Remark 3: Instead of applying B_2 and using $u_2(x)$, we can use the transformation $u(x) \to -u(x)$.

Remark 4: There are no explicit formulae for these solutions constructed by Bäcklund transformations, we have to use other methods to study them.

2.5 Hamiltonian system

We use the formulas presented in [Cla23].

Definition 2.5.1. *Denote $q(x)$ solution of the Painlevé-III equation. Fix $\varepsilon = \pm 1$. The momentum associated to it is given by*

$$p(x) = \frac{1}{2q^2(x)} \left(xq'(x) + xq^2(x) - \varepsilon x + q(x) \left(\frac{\beta - \varepsilon}{\varepsilon} \right) \right)$$

Definition 2.5.2. *Denote $q(x)$ solution of the Painlevé-III equation. Fix $\varepsilon = \pm 1$. The Hamiltonian associated to it is given by*

$$H(x) = p^2(x)q^2(x) - p(x) \left(xq^2(x) - \varepsilon x + q(x) \left(\frac{\beta - \varepsilon}{\varepsilon} \right) \right) + 2xq(x) \left(\frac{\beta - \varepsilon(2+\alpha)}{4\varepsilon} \right)$$

We can show that Painlevé-III equation is equivalent to the **Hamiltonian system**:

$$x\frac{dq}{dx} = \frac{\partial H}{\partial p}$$

$$x\frac{dp}{dx} = -\frac{\partial H}{\partial q}$$

2.6 Tau function and Toda equation

For details of this section see [Oka87] and [FW02].

Definition 2.6.1. *Consider Painlevé-III equation. Fix $\varepsilon = \pm 1$. Define auxiliary Hamiltonian using formula*

$$h(x) = \frac{1}{2}\left(H(x) + q(x)p(x) - \varepsilon x^2 + \frac{1}{4}(\beta - 4\varepsilon)(\beta + \varepsilon(\alpha - 2))\right)$$

Consider Painlevé-III equation. Choose $\varepsilon = 1$. Since momentum and Hamiltonian are expressed in terms of $q(x)$, the action of Bäcklund transformation B_1 can be extended from $(q(x), \alpha, \beta)$ to $p(x)$, $H(x)$ and $h(x)$. We denote

$$(q_n(x), \alpha + 2n, \beta + 2n) = B_1^n(q(x), \alpha, \beta) \tag{18}$$

$$p_n(x) = p(x)|_{q(x)\to q_n(x), \beta\to\beta+2n} \tag{19}$$

$$H_n(x) = H(x)|_{q(x)\to q_n(x), p(x)\to p_n(x), \alpha\to\alpha+2n, \beta\to\beta+2n} \tag{20}$$

$$h_n(x) = h(x)|_{H(x)\to H_n(x), q(x)\to q_n(x), p(x)\to p_n(x), \alpha\to\alpha+2n, \beta\to\beta+2n} \tag{21}$$

To study the behavior of PIII equation more easily, we introduce the tau function associated with the solution.

Definition 2.6.2. *Define tau function for Painlevé-III equation using formula*

$$x\frac{d}{dx}\ln(\tau_n(x)) = h_n(x)$$

It is defined up to a constant.

Using the following identity:

$$h_{n+1}(x) = h_n(x) - p_n q_n(x) - \frac{3}{2} + \frac{\alpha}{4} - \frac{3\beta}{4} + 2n$$

We can show that the **Toda equation** holds:

$$x\frac{d}{dx}x\frac{d}{dx}\ln(\tau_n(x)) = C_n\frac{\tau_{n+1}(x)\tau_{n-1}(x)}{\tau_n^2(x)}$$

for some constant C_n.

Actually, denote $f(x) = x\frac{d}{dx}x\frac{d}{dx}\ln(\tau_n(x))$ and $g(x) = \frac{\tau_{n+1}(x)\tau_{n-1}(x)}{\tau_n^2(x)}$. Basically, we want to show $f(x) = C_n g(x)$. Taking natural log on both sides, we get $\ln f(x) = \ln g(x) + \ln C_n$. Therefore, it is equivalent to show $\frac{d}{dx}(\ln f(x) - \ln g(x)) = 0$.

Well, by using Definition 2.6.2 and the identity above, we have:

$$\frac{d}{dx}\ln(g(x)) = \frac{p_{n-1}(x)q_{n-1}(x) - p_n(x)q_n(x) + 2}{x} \tag{22}$$

$$\frac{d}{dx}\ln(f(x)) = \frac{h_n'(x) + xh_n''(x)}{xh_n'(x)} \tag{23}$$

By using Definition 2.6.1 and (18)–(21), we rewrite (23),(22) in terms of $q_n(x)$. With the aid of Mathematica, we can verify the difference of them is zero.

Actually, using transformation $\tau_n(x) \to a_n\tau_n(x)$ one can change constant C_n in the Toda equation to be 1. a_n can be obtained by solving a difference equation

$$C_n a_n^2 = a_{n+1}a_{n-1}$$

The general solution is:

$$a_n = \frac{a_1^n}{a_0^{n-1}}\prod_{j=1}^{n-1}\prod_{i=1}^{j} C_i, \quad n \in \mathbb{N}$$

By picking the initial conditions $a_0 = a_1 = 1$, we get:

$$a_n = \prod_{j=1}^{n-1}\prod_{i=1}^{j} C_i, \quad n \in \mathbb{N}$$

2.7 Some linear algebra

Toda equation determines the tau function recursively. If we want to derive some nice formula for it, we need some properties of determinants.

Proposition 2.7.1. *The Leibniz formula for the determinant of $n \times n$ matrix $A = \{a_{ij}\}_{i,j=1}^{n}$ is given by*

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} a_{k,\sigma(k)}$$

where S_n is the set of permutations of n elements and $\operatorname{sgn}(\sigma)$ is sign of permutation σ.

Directly using the proposition above, we can show the following formula for derivative of a determinant

$$\frac{d}{dx}\det(A) = \sum_{j=1}^{n} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(\frac{d}{dx} a_{j,\sigma(j)}\right) \prod_{\substack{k=1 \\ k \neq j}}^{n} a_{k,\sigma(k)} \tag{24}$$

Proposition 2.7.2. *Denote $A_{i|j}$ the matrix obtained from A by deleting of ith row and jth column. The Laplace expansion for the determinant along the jth row is given by*

$$\det(A) = \sum_{k=1}^{n} (-1)^{k+j} a_{jk} \det(A_{j|k})$$

Proposition 2.7.3 (see [VV23])**.** *Denote $A_{ij|kl}$ the matrix obtained from A by deleting of ith and jth row anf kth and lth column. The Deshanot-Jacobi identity is given by*

$$\det(A)\det(A_{ij|ij}) = \det(A_{i|i})\det(A_{j|j}) - \det(A_{i|j})\det(A_{j|i}), \quad 1 \le i, j \le n$$

Corollary 2.7.1. *Using the formula for the derivative of determinant and Deshanot-Jacobi identity, we can show that functions in the form of*

$$f_n(x) = \det\left(\left\{\left(x\frac{d}{dx}\right)^{i+j} f_0(x)\right\}_{i,j=0}^{n}\right) \tag{25}$$

solve Toda equation corresponding to Painlevé-III equation

$$\left(x\frac{d}{dx}\right)^2 \ln(f_n(x)) = \frac{f_{n+1}(x) f_{n-1}(x)}{f_n^2(x)}, \quad n \ge 1 \tag{26}$$

Proof. Specifically, to match the expression in Proposition 2.7.3, we rewrite (25) as:

$$f_{n-1}(x) f_{n+1}(x) = f_n(x) \left(x\frac{d}{dx}\right)^2 f_n(x) - \left(x\frac{d}{dx} f_n(x)\right)^2, \quad n \ge 1$$

Let $f_{n+1}(x) = \det(A)$. Obviously, it follows that $f_n(x) = \det(A_{n+2|n+2})$ and $f_{n-1}(x) = \det(A_{n+1,n+2|n+1,n+2})$. Then we take the first derivative of the determinant in (25) by multi-linearity with respect to rows. Since a determinant with two identical rows is zero, the simplified result we end up with is $x\frac{d}{dx} f_n(x) = \det(A_{n+1|n+2})$. Since $A_{n+1|n+2} = (A_{n+2|n+1})^T$, it implies that $x\frac{d}{dx} f_n(x) = \det(A_{n+1|n+2}) = \det(A_{n+2|n+1})$. Then we take the second derivative of the determinant in (25) successively by multi-linearity with respect to columns. Similarly, since a determinant with two identical columns is zero, the simplified result is ended with $\left(x\frac{d}{dx}\right)^2 f_n(x) = \det(A_{n+1|n+1})$. It shows that the statement holds. □

Now we already have the fundamental form for τ_n. We compute $h_0(x)$, $h_1(x)$, and $h_2(x)$ in the case of Painlevé-III equation and $\varepsilon = 1$, $\alpha + \beta = 2$ in terms of functions $H^{(1)}_{\frac{\alpha}{2}}(x)$, $H^{(2)}_{\frac{\alpha}{2}}(x)$. If we pick the base cases carefully in the following way

$$\tau_0(x) = 1$$

$$\tau_1(x) = c_1 H^{(1)}_{\frac{\alpha}{2}}(x) + c_2 H^{(2)}_{\frac{\alpha}{2}}(x)$$

$$\tau_2(x) = \det \begin{pmatrix} \tau_1(x) & \tau_1'(x) \\ \tau_1'(x) & \tau_1''(x) \end{pmatrix}$$

Then we will get the inductive formula for τ_n

$$\tau_n(x) = \det\left(\left\{\left(x\frac{d}{dx}\right)^{i+j-2} \tau_1(x)\right\}_{i,j=1}^{n}\right)$$

Remark: One can show numerically that the explicit formula for q_n is

$$q_n(x) = \frac{\tau_{n+1}(x, \alpha-2)\tau_n(x,\alpha)}{\tau_{n+1}(x,\alpha)\tau_n(x,\alpha-2)} \tag{27}$$

Obviously, there is a close relation between τ_n and q_n from this formula. We can also use this to verify our main result.

2.8 Andréief identity

To improve our result based on the inductive formula, we introduce a more advanced identity.

Proposition 2.8.1 (see [For18]). *Andréief identity is given by the following formula*

$$\int_\Gamma \dots \int_\Gamma \det\left(\{f_j(x_k)\}_{j,k=1}^n\right) \det\left(\{g_j(x_k)\}_{j,k=1}^n\right) \prod_{k=1}^n h(x_k)dx_k = n! \det\left(\left\{\int_\Gamma f_j(x)g_k(x)h(x)dx\right\}_{j,k=1}^n\right)$$

Proposition 2.8.2. *The formula for the Vandermonde determinant is given by*

$$\det(\{x_j^k\}_{j,k=0}^n) = \prod_{0\le j<k\le n}(x_k - x_j)$$

Lemma 2.8.1. *Using identities (7), (8) from section 2.2, we show that in the case of Painlevé-III and $\varepsilon = 1$ we have*

$$\tau_n(x) = x^{n(n-1)}(-1)^{\frac{n(n-1)}{2}} \det\left(\left\{f_{\frac{\alpha}{2}-j+k}(x)\right\}_{j,k=0}^{n-1}\right)$$

where

$$f_\nu(x) = c_1 H^{(1)}_\nu(x) + c_2 H^{(2)}_\nu(x)$$

Proof. From the inductive formula of $\tau_n(x)$, we know:

$$\tau_n(x) = \det\left(\left\{\left(x\frac{d}{dx}\right)^{k+j} f_{\frac{\alpha}{2}}\right\}_{k,j=0}^{n-1}\right)$$

By identities (7), (8) from section 2.2, we can obtain the following relations:

$$x\frac{d}{dx}f_{\frac{\alpha}{2}}(x) = xf_{\frac{\alpha}{2}-1}(x) - \frac{\alpha}{2}f_{\frac{\alpha}{2}}(x) \tag{28}$$

$$x\frac{d}{dx}f_{\frac{\alpha}{2}}(x) = -xf_{\frac{\alpha}{2}+1}(x) + \frac{\alpha}{2}f_{\frac{\alpha}{2}}(x) \tag{29}$$

For relation (28), by induction, we can show:

$$\left(x\frac{d}{dx}\right)^j f_{\frac{\alpha}{2}}(x) = x^j f_{\frac{\alpha}{2}-j}(x) + \sum_{k=0}^{j-1} c_{kj}x^k f_{\frac{\alpha}{2}-k}(x) \tag{30}$$

for some coefficients c_{kj}. Well, we furtherly simplify the determinant using (30):

$$\det\left(\left\{\left(x\frac{d}{dx}\right)^{k+j} f_{\frac{\alpha}{2}}\right\}_{k,j=0}^{n-1}\right) = \begin{vmatrix} f_{\frac{\alpha}{2}}(x) & x\frac{d}{dx}f_{\frac{\alpha}{2}}(x) & \cdots \\ xf_{\frac{\alpha}{2}-1}(x)+c_{01}f_{\frac{\alpha}{2}}(x) & x\frac{d}{dx}(xf_{\frac{\alpha}{2}-1}(x))+c_{01}x\frac{d}{dx}f_{\frac{\alpha}{2}} & \cdots \\ x^2f_{\frac{\alpha}{2}-2}(x)+c_{12}xf_{\frac{\alpha}{2}-1}(x)+c_{02}f_{\frac{\alpha}{2}} & x\frac{d}{dx}(x^2f_{\frac{\alpha}{2}-2}(x))+c_{12}x\frac{d}{dx}xf_{\frac{\alpha}{2}-1}+c_{02}x\frac{d}{dx}f_{\frac{\alpha}{2}} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}$$

Observe that by elementary row operations, we can always use the previous rows to eliminate the $\sum_{k=0}^{j-1} c_{kj}x^k f_{\frac{\alpha}{2}-k}(x)$ part in a fixed row and the value of the determinant doesn't change. Finally, we will end up with:

$$\det\left(\left\{\left(x\frac{d}{dx}\right)^{k+j} f_{\frac{\alpha}{2}}\right\}_{k,j=0}^{n-1}\right) = \begin{vmatrix} f_{\frac{\alpha}{2}}(x) & x\frac{d}{dx}f_{\frac{\alpha}{2}}(x) & \cdots \\ xf_{\frac{\alpha}{2}-1}(x) & x\frac{d}{dx}(xf_{\frac{\alpha}{2}-1}(x)) & \cdots \\ x^2f_{\frac{\alpha}{2}-2}(x) & x\frac{d}{dx}(x^2f_{\frac{\alpha}{2}-2}(x)) & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} = \det\left(\left\{\left(x\frac{d}{dx}\right)^{k} x^j f_{\frac{\alpha}{2}-j}\right\}_{k,j=0}^{n-1}\right)$$

By relation (29), by induction, we can show:

$$\left(x\frac{d}{dx}\right)^k x^j f_{\frac{\alpha}{2}-j}(x) = (-1)^k x^{j+k} f_{\frac{\alpha}{2}-j+k}(x) + \sum_{n=0}^{k-1} d_{nkj}x^{j+n}f_{\frac{\alpha}{2}-j+n}(x) \tag{31}$$

To prove (31) by (29), we first fix $k = 1$ and induct on j. We have

$$x\frac{d}{dx}x^j f_{\frac{\alpha}{2}-j} = -x^{j+1}f_{\frac{\alpha}{2}-j+1}(x) + x^j\left(\frac{\alpha}{2}-j\right)f_{\frac{\alpha}{2}-j}(x) + jx^jf_{\frac{\alpha}{2}-j}(x) \tag{32}$$

$$= -x^{j+1}f_{\frac{\alpha}{2}-j+1}(x) + x^j\frac{\alpha}{2}f_{\frac{\alpha}{2}-j}(x) \tag{33}$$

After showing (32) we induct on k. We can noticed that d_{nkj} actually doesn't depend on j. So (31) can be written as

$$\left(x\frac{d}{dx}\right)^k x^j f_{\frac{\alpha}{2}-j} = (-1)^k x^{j+k} f_{\frac{\alpha}{2}-j+k}(x) + \sum_{n=0}^{k-1} d_{nk}x^{j+n}f_{\frac{\alpha}{2}-j+n}(x) \tag{34}$$

Again, we futherly simplify the determinant by (34):

$$\det\left(\left\{\left(x\frac{d}{dx}\right)^{k+j} f_{\frac{\alpha}{2}}\right\}_{k,j=0}^{n-1}\right) = \begin{vmatrix} f_{\frac{\alpha}{2}}(x) & -xf_{\frac{\alpha}{2}+1}+d_{01}f_{\frac{\alpha}{2}} & \cdots \\ xf_{\frac{\alpha}{2}-1}(x) & -x^2f_{\frac{\alpha}{2}}+d_{01}xf_{\frac{\alpha}{2}-1} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}$$

Similarly, by applying elementary row operation on columns, we can always use the previous columns to eliminate the $\sum_{n=0}^{k-1} d_{nk}x^{j+n}f_{\frac{\alpha}{2}-j+n}(x)$ part in a fixed column and the value of the determinant doesn't change. Finally, we will end up with:

$$\det\left(\left\{\left(x\frac{d}{dx}\right)^{k+j} f_{\frac{\alpha}{2}}\right\}_{k,j=0}^{n-1}\right) = \det\left((-1)^k x^{j+k} f_{\frac{\alpha}{2}-j+k}(x)\right)$$

By multi-linearity of determinant, we can factor out $(-1)^k x^{j+k}$ and reach the conclusion:

$$\tau_n(x) = \det\left((-1)^k x^{j+k} f_{\frac{\alpha}{2}-j+k}(x)\right) = x^{n(n-1)}(-1)^{\frac{n(n-1)}{2}}\det\left(\left\{f_{\frac{\alpha}{2}-j+k}(x)\right\}_{j,k=0}^{n-1}\right)$$

where

$$f_\nu(x) = c_1H_\nu^{(1)}(x) + c_2H_\nu^{(2)}(x)$$

That completes the proof. ∎

We apply Andréief identity to show that for the case of Painlevé-III and $\varepsilon = 1$ we get the following result.

Theorem 2.8.1. *The explicit formula for* $\tau_n(x)$ *is given by*

$$\tau_n(x) = \frac{x^{n(n-1)}(-1)^{\frac{n(n-1)}{2}}}{n!} \int\limits_{\Gamma_3\cup\Gamma_4} \cdots \int\limits_{\Gamma_3\cup\Gamma_4} \prod_{0\le j<k\le n-1} (t_k - t_j)\left(\frac{1}{t_k} - \frac{1}{t_j}\right) \prod_{k=0}^{n-1} h(t_k)dt_k \tag{35}$$

where

$$h(t) = \frac{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{\pi i t^{1+\frac{\alpha}{2}}}(c_1\chi_{\Gamma_3}(t) - c_2\chi_{\Gamma_4}(t))$$

Γ_3 *is contour of integration for* $H_\nu^{(1)}(x)$ *and* Γ_4 *is contour of integration for* $H_\nu^{(2)}(x)$ *after change of variable* $e^z = t$.

Proof. By (4),(5) we get:

$$\begin{aligned} f_{\frac{\alpha}{2}}(x) &= c_1 H^{(1)}_{\frac{\alpha}{2}}(x) + c_2 H^{(2)}_{\frac{\alpha}{2}}(x) \\ &= c_1 \frac{1}{\pi i} \int\limits_{-\infty}^{\infty+i\pi} e^{x\sinh(z) - \frac{\alpha}{2}z}dz - c_2\frac{1}{\pi i}\int\limits_{-\infty}^{\infty-i\pi} e^{x\sinh(z)-\frac{\alpha}{2}z}dz \\ &= \frac{1}{\pi i}\left(c_1 \int\limits_{-\infty}^{\infty+i\pi} e^{\frac{x}{2}(e^z-e^{-z})-\frac{\alpha}{2}z}dz - c_2 \int\limits_{-\infty}^{\infty-i\pi} e^{\frac{x}{2}(e^z-e^{-z})-\frac{\alpha}{2}z}dz\right) \end{aligned}$$

Let $e^z = t$, then we have:

$$\begin{aligned} f_{\frac{\alpha}{2}}(x) &= \frac{1}{\pi i}\left(c_1 \int\limits_{\Gamma_3} \frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{\frac{\alpha}{2}+1}}dt - c_2 \int\limits_{\Gamma_4} \frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{\frac{\alpha}{2}+1}}dt\right) \\ &= \int\limits_{\Gamma_3\cup\Gamma_4} \frac{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{\pi i t^{1+\frac{\alpha}{2}}}(c_1\chi_{\Gamma_3}(t) - c_2\chi_{\Gamma_4}(t))dt \end{aligned}$$

where $\chi(t)$ is the characteristic function.
Let $h(t) = f_{\frac{\alpha}{2}}(x)$, then it implies that:

$$f_{\frac{\alpha}{2}-j+k}(x) = \int\limits_{\Gamma_3\cup\Gamma_4} t^{-k}t^j h(t)dt$$

From Lemma 2.8.1, we obtain:

$$\begin{aligned} \tau_n(x) &= x^{n(n-1)}(-1)^{\frac{n(n-1)}{2}} \det\left(\left\{f_{\frac{\alpha}{2}-j+k}(x)\right\}_{j,k=0}^{n-1}\right) \\ &= x^{n(n-1)}(-1)^{\frac{n(n-1)}{2}} \det\left(\left\{\int\limits_{\Gamma_3\cup\Gamma_4} t^{-k}t^j h(t)dt\right\}_{j,k=0}^{n-1}\right) \end{aligned}$$

Let $g_k(t) = t^{-k}$ and $f_j(t) = t^j$, by Proposition 2.8.1, we get:

$$\begin{aligned} \tau_n(x) &= \frac{x^{n(n-1)}(-1)^{\frac{n(n-1)}{2}}}{n!} \int\limits_{\Gamma_3\cup\Gamma_4} \cdots \int\limits_{\Gamma_3\cup\Gamma_4} \det\left(\{f_j(t_k)\}_{j,k=1}^n\right) \det\left(\{g_j(t_k)\}_{j,k=1}^n\right) \prod_{k=1}^n h(t_k)dt_k \\ &= \frac{x^{n(n-1)}(-1)^{\frac{n(n-1)}{2}}}{n!} \int\limits_{\Gamma_3\cup\Gamma_4} \cdots \int\limits_{\Gamma_3\cup\Gamma_4} \det\left(\left\{t_k^j\right\}_{j,k=1}^n\right) \det\left(\left\{t_k^{-j}\right\}_{j,k=1}^n\right) \prod_{k=1}^n h(t_k)dt_k \end{aligned}$$

By Theorem 2.8.2, we can compute the two determinants in the integrand:

$$\det\left(\left\{t_k^j\right\}_{j,k=1}^n\right) = \prod_{0\le j<k\le n-1} (t_k - t_j)$$

$$\det\left(\left\{t_k^{-j}\right\}_{j,k=1}^n\right) = \prod_{0\le j<k\le n-1} \left(\frac{1}{t_k} - \frac{1}{t_j}\right)$$

Thus, the explicit formula for $\tau_n(x)$ is given by:

$$\tau_n(x) = \frac{x^{n(n-1)}(-1)^{\frac{n(n-1)}{2}}}{n!} \int\limits_{\Gamma_3\cup\Gamma_4} \cdots \int\limits_{\Gamma_3\cup\Gamma_4} \prod_{0\le j<k\le n-1} (t_k - t_j)\left(\frac{1}{t_k} - \frac{1}{t_j}\right) \prod_{k=0}^{n-1} h(t_k)dt_k$$

□

2.9 Orthogonal polynomial

For details about this section see [Ism05].

Definition 2.9.1. *The monic polynomial of degree n*

$$p_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$$

is called orthogonal polynomial with respect to weight $w(z)$ on contour Γ if it satisfies conditions

$$\int\limits_\Gamma p_n(z)z^j w(z)dz = 0, \quad 0 \le j \le n-1.$$

Definition 2.9.2. *Moments of the weight $w(z)$ are given by*

$$\mu_j = \int\limits_\Gamma z^j w(z)dz$$

Definition 2.9.3. *Hankel determinant associated to the orthogonal polynomials $p_n(z)$ is given by*

$$H_n = \det\left(\{\mu_{j+k}\}_{j,k=0}^n\right)$$

Proposition 2.9.1. *Denote by $M_n(z)$ the matrix $\{\mu_{j+k}\}_{j,k=0}^n$ with last row replaced by $(1, z, z^2, \dots, z^n)$. The orthogonal polynomials $p_n(z)$ are given by*

$$p_n(z) = \frac{\det(M_n(z))}{H_{n-1}}$$

We can use multi-linearity property of the determinant to verify the explicit formula for p_n given above is a monic polynomial and satisfies the orthogonality condition by expanding the last row.

Definition 2.9.4. *The normalizing constant for orthogonal polynomials $p_n(z)$ is given by*

$$h_n = \int\limits_\Gamma p_n(z)z^n w(z)dz$$

Proposition 2.9.2. *The Hankel determinant is given by*

$$H_n = \prod_{j=0}^n h_j$$

It is easy to check using multi-linearity property of determinant that we can compute h_n recursively in terms of H_n.

Remark: The normalizing constants for classical orthogonal polynomials can be found in [DLMF, Table 18.3.1].

3 Main result

3.1 Basic strategies

Up to this point, we've got enough preparation to compute the asymptotics at zero. Our goal is to rewrite $\tau_n(x) \sim b(n)x^{a(n)}$ when $x \to 0$ in the case of $\varepsilon = 1$, $\frac{\alpha}{2} \notin \mathbb{Z}$, $|Re(\alpha)| < 2n$ In this part, I will summarize several key ideas to achieve this goal as following:

- The contours Γ_3 and Γ_4 spread out to zero and infinity in the formula (35). We can't put $x = 0$ here without losing convergence of the integral.
- We rewrite $H_\nu^{(1)}$ and $H_\nu^{(2)}$ in terms of J_ν and $J_{-\nu}$. Contour Γ_1 for J_ν in variable t is located around infinity, it does not approach zero. Contour Γ_2 for $J_{-\nu}$ in variable t is located around zero, it does not approach infinity.
- Expanding the product in the integrand of (35) we get the expressions, where some of the variables t_k will belong to contour Γ_1 and others will belong to Γ_2.
- We apply change of variable technique $t = \frac{2s}{x}$ to variables on contours Γ_1. The integrand will preserve exponential decay at infinity when we put $x = 0$. On the other hand, we apply change of variable technique $t = \frac{1}{2}xs$ to variables on contours Γ_2. In this case the integrand will preserve exponential decay at zero when we put $x = 0$.

3.2 Alternative formula of $\tau_n(x)$

According to the blueprints above, we can start to derive the alternative formula for τ_n. Using (6), (9), (10) we have

$$H_\nu^{(1)}(x) = (1 + i\cot(\pi\nu))J_\nu(x) - i\csc(\pi\nu)J_{-\nu}(x)$$
$$H_\nu^{(2)}(x) = (1 - i\cot(\pi\nu))J_\nu(x) + i\csc(\pi\nu)J_{-\nu}(x)$$

Recall that: $f_\nu(x) = c_1 H_\nu^{(1)}(x) + c_2 H_\nu^{(2)}(x)$, then we have:

$$f_\nu(x) = d_1 J_\nu(x) + d_2 J_{-\nu}(x) \tag{36}$$

where

$$d_1 = c_1(1 + i\cot(\pi\nu)) + c_2(1 - i\cot(\pi\nu))$$
$$d_2 = c_2\csc(\pi\nu)i - c_1\csc(\pi\nu)i$$

By formula (3) of $J_\nu(x)$ it follows that

$$f_\nu(x) = d_1 \frac{1}{2\pi i}\int_{\infty - i\pi}^{\infty + i\pi} e^{x\sinh(z) - \nu z}dz + d_2 \frac{1}{2\pi i}\int_{\infty - i\pi}^{\infty + i\pi} e^{x\sinh(z) + \nu z}dz$$

Making the change of variable $z \to i\pi - z$ in the second integral we get

$$f_\nu(x) = d_1 \frac{1}{2\pi i}\int_{\infty - i\pi}^{\infty + i\pi} e^{x\sinh(z) - \nu z}dz - d_2 \frac{e^{i\pi\nu}}{2\pi i}\int_{-\infty + 2i\pi}^{-\infty} e^{x\sinh(z) - \nu z}dz$$

Let $e^z = t$, we have

$$f_\nu(x) = \int_{\Gamma_1 \cup \Gamma_2} \frac{e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}}{2\pi i t^{1+\nu}}(d_1\chi_{\Gamma_1}(t) - d_2 e^{i\pi\nu}\chi_{\Gamma_2}(t))dt$$

Γ_1 is contour of integration for $J_\nu(x)$ and Γ_2 is contour of integration for $J_{-\nu}(x)$ after change of variable $e^z = t$ shown in the Figure 4. Now, redefine $h(t) = \frac{e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}}{2\pi i t^{1+\frac{\alpha}{2}}}(d_1\chi_{\Gamma_1}(t) - d_2 e^{i\pi\frac{\alpha}{2}}\chi_{\Gamma_2}(t))$ in the explicit formula of τ_n. It is worthy to point out that the way to define $h(t)$ is valid. It seems that the coefficients d_1 and $d_2 e^{i\pi\nu}$ are non-constants depending on j and k. However, the periodicity of cotangent function

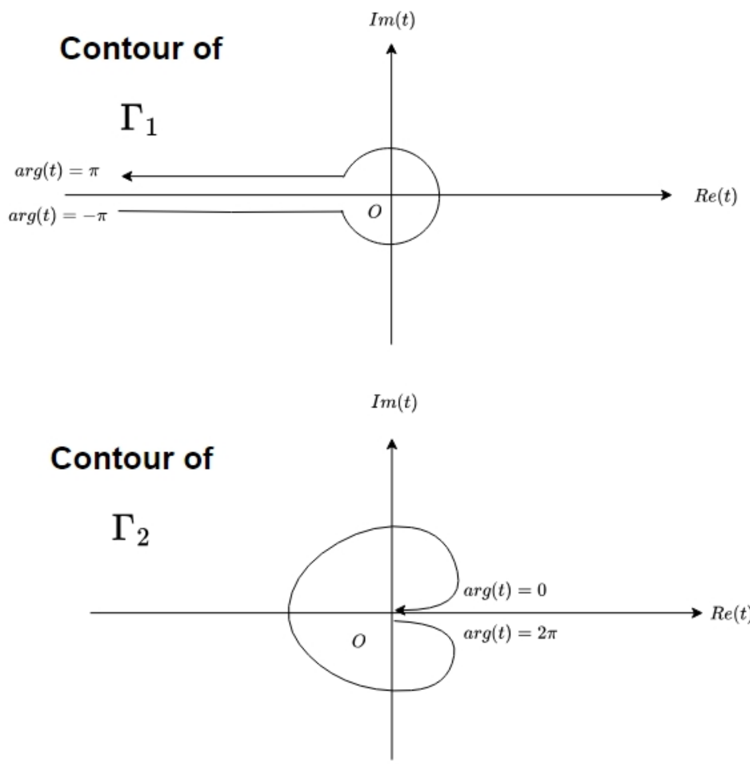

Figure 2: Contours Γ_1 and Γ_2

will make $d_1 = c_1(1+\cot(\pi\frac{\alpha}{2})i)+c_2(1-\cot(\pi\frac{\alpha}{2})i)$. On the other hand, the periodicity of secant function and exponential function will make $d_2 e^{i\pi\nu} = \left(c_2\csc(\pi\frac{\alpha}{2})i - c_1\csc(\pi\frac{\alpha}{2})i\right)e^{i\pi\frac{\alpha}{2}}$. They are actually both constants.

We should make another remark about power function $\frac{1}{t^{1+\frac{\alpha}{2}}}$. When we use it, we assume $-i\pi < \arg(t) < i\pi$ on contour Γ_1 and $0 < \arg(t) < 2\pi i$ on contour Γ_2.

Lemma 3.2.1. *Let I denote a subset of the set of indices and r denote its cardinal. One can observe the following algebraic identity:*

$$\prod_{k=1}^{n}(c_1\chi_{\Gamma_1}(t_k)+c_2\chi_{\Gamma_2}(t_k)) = \sum_{\substack{I\subset\{1,\dots,n\}\\|I|=r}} c_1^r c_2^{n-r}\prod_{k\in I}\chi_{\Gamma_1}(t_k)\prod_{j\in I^c}\chi_{\Gamma_2}(t_j)$$

Now we apply the lemma above to Theorem 2.8.1 with the redefined $h(t)$ in order to convert the formula into a big summation form and decouple the contours as following

$$\begin{aligned}
\tau_n(x) =& \frac{x^{n(n-1)}}{n!}\int_{\Gamma_1\cup\Gamma_2}\cdots\int_{\Gamma_1\cup\Gamma_2}\prod_{0\le j<k\le n-1}\frac{(t_j-t_k)^2}{t_jt_k}\prod_{l=1}^{n}\frac{e^{\frac{x}{2}(t_l-\frac{1}{t_l})}}{2\pi i t_l^{\frac{\alpha}{2}+1}}\prod_{m=1}^{n}(d_1\chi_{\Gamma_1}(t_m)-d_2e^{\frac{i\pi\alpha}{2}}\chi_{\Gamma_2}(t_m))\prod_{q=1}^{n}dt_q\\
=& \frac{x^{n(n-1)}}{n!}\int_{\Gamma_1\cup\Gamma_2}\cdots\int_{\Gamma_1\cup\Gamma_2}\prod_{0\le j<k\le n-1}\frac{(t_j-t_k)^2}{t_jt_k}\prod_{l=1}^{n}\frac{e^{\frac{x}{2}(t_l-\frac{1}{t_l})}}{2\pi i t_l^{\frac{\alpha}{2}+1}}\left(\sum_{\substack{I\subset\{1,\dots,n\}\\|I|=r}}d_1^rd_2^{n-r}(-e^{\frac{i\pi\alpha}{2}})^{n-r}\prod_{i\in I}\chi_{\Gamma_1}(t_i)\prod_{j\in I^c}\chi_{\Gamma_2}(t_j)\right)\prod_{q=1}^{n}dt_q\\
=& \frac{1}{(2\pi i)^n}\sum_{\substack{I\subset\{1,\dots,n\}\\|I|=r}}d_1^rd_2^{n-r}(-e^{\frac{i\pi\alpha}{2}})^{n-r}\frac{x^{n(n-1)}}{n!}\int_{\Gamma_1}\cdots\int_{\Gamma_1}\int_{\Gamma_2}\cdots\int_{\Gamma_2}\sqrt{\prod_{\substack{j\neq k\\j,k\in I}}\frac{(t_j-t_k)^2}{t_jt_k}}\prod_{l\in I}\frac{e^{\frac{x}{2}(t_l-\frac{1}{t_l})}}{t_l^{\frac{\alpha}{2}+1}}\sqrt{\prod_{\substack{j\neq k\\j,k\in I^c}}\frac{(t_j-t_k)^2}{t_jt_k}}\\
&\prod_{l\in I^c}\frac{e^{\frac{x}{2}(t_l-\frac{1}{t_l})}}{t_l^{\frac{\alpha}{2}+1}}\sqrt{\prod_{\substack{j\in I\\k\in I^c}}\frac{(t_j-t_k)^2}{t_jt_k}}\sqrt{\prod_{\substack{j\in I^c\\k\in I}}\frac{(t_j-t_k)^2}{t_jt_k}}\prod_{q\in I}dt_q\prod_{q\in I^c}dt_q
\end{aligned}$$

For $t_k \in I$, we use change of variable $t_k = \frac{2s_k}{x}$. On the other hand, for $t_j \in I^c$, we use change of variable $t_j = \frac{1}{2}s_j x$. The formula above becomes

$$\tau_n(x) = \frac{1}{(2\pi i)^n} \sum_{\substack{I\subset\{1,\dots,n\}\\|I|=r}} d_1^r d_2^{n-r} (-e^{\frac{i\pi\alpha}{2}})^{n-r} \frac{x^{n(n-1)}}{n!} \int_{\Gamma_1}\cdots\int_{\Gamma_1}\int_{\Gamma_2}\cdots\int_{\Gamma_2} \prod_{\substack{j<k\\j,k\in I}} \frac{(s_j-s_k)^2}{s_j s_k} \prod_{l\in I} \frac{e^{s_l}(1+O(x^2))}{2^{\frac{\alpha}{2}+1} s_l^{\frac{\alpha}{2}+1} x^{-\frac{\alpha}{2}-1}}$$

$$\prod_{\substack{j<k\\j,k\in I^c}} \frac{(s_j-s_k)^2}{s_j s_k} \prod_{l\in I^c} \frac{e^{-\frac{1}{s_l}}(1+O(x^2))}{(\frac{1}{2})^{\frac{\alpha}{2}+1} s_l^{\frac{\alpha}{2}+1} x^{\frac{\alpha}{2}+1}} \left(2^{2r(n-r)} \prod_{\substack{j\in I\\k\in I^c}} \frac{(s_j(1+O(x^2))^2}{x^2 s_j s_k} \right) \left(2^r (\frac{1}{2})^{n-r} x^{n-2r} \prod_{q\in I} ds_q \prod_{q\in I^c} ds_q \right)$$

Group all the 2-factors together and pull them out of the summation,

$$\tau_n(x) = \frac{2^{-\alpha r+2r(n-r)+\frac{\alpha n}{2}}}{(2\pi i)^n} \sum_{\substack{I\subset\{1,\dots,n\}\\|I|=r}} d_1^r d_2^{n-r} (-e^{\frac{i\pi\alpha}{2}})^{n-r} \frac{x^{n(n-1)}}{n!} \int_{\Gamma_1}\cdots\int_{\Gamma_1}\int_{\Gamma_2}\cdots\int_{\Gamma_2} \prod_{\substack{j<k\\j,k\in I}} \frac{(s_j-s_k)^2}{s_j s_k}$$

$$\prod_{l\in I} \frac{e^{s_l}(1+O(x^2))}{s_l^{\frac{\alpha}{2}+1} x^{-\frac{\alpha}{2}-1}} \prod_{\substack{j<k\\j,k\in I^c}} \frac{(s_j-s_k)^2}{s_j s_k} \prod_{l\in I^c} \frac{e^{-\frac{1}{s_l}}(1+O(x^2))}{s_l^{\frac{\alpha}{2}+1} x^{\frac{\alpha}{2}+1}} \prod_{\substack{j\in I\\k\in I^c}} \frac{(s_j(1+O(x^2))^2}{x^2 s_j s_k} \prod_{q\in I} ds_q \prod_{q\in I^c} ds_q$$

Next group all the powers of x together,

$$\tau_n(x) = \frac{2^{-\alpha r+2r(n-r)+\frac{\alpha n}{2}}}{(2\pi i)^n} \sum_{\substack{I\subset\{1,\dots,n\}\\|I|=r}} d_1^r d_2^{n-r} (-e^{\frac{i\pi\alpha}{2}})^{n-r} \frac{x^{n(n-1)+\alpha r-\frac{n\alpha}{2}-2r(n-r)}}{n!} \int_{\Gamma_1}\cdots\int_{\Gamma_1}\int_{\Gamma_2}\cdots\int_{\Gamma_2} \prod_{\substack{j<k\\j,k\in I}} \frac{(s_j-s_k)^2}{s_j s_k}$$

$$\prod_{l\in I} e^{s_l} s_l^{-\frac{\alpha}{2}-1} \prod_{\substack{j<k\\j,k\in I^c}} \frac{(s_j-s_k)^2}{s_j s_k} \prod_{l\in I^c} e^{-\frac{1}{s_l}} s_l^{-\frac{\alpha}{2}-1} \prod_{\substack{j\in I\\k\in I^c}} \frac{s_j}{s_k} \prod_{q\in I} ds_q \prod_{q\in I^c} ds_q$$

We also want to group all the products of variable s together and separate the integrals based on different contours. We rewrite the following three parts,

$$\prod_{\substack{j<k\\j,k\in I}} \frac{(s_j-s_k)^2}{s_j s_k} = \prod_{\substack{j<k\\j,k\in I}} (s_j-s_k)^2 \prod_{\substack{j\neq k\\j,k\in I}} \frac{1}{\sqrt{s_j s_k}} = \prod_{\substack{j<k\\j,k\in I}} (s_j-s_k)^2 \prod_{l\in I} s_l^{\gamma_1}$$

$$\prod_{\substack{j<k\\j,k\in I^c}} \frac{(s_j-s_k)^2}{s_j s_k} = \prod_{\substack{j<k\\j,k\in I^c}} (s_j-s_k)^2 \prod_{\substack{j\neq k\\j,k\in I^c}} \frac{1}{\sqrt{s_j s_k}} = \prod_{\substack{j<k\\j,k\in I^c}} (s_j-s_k)^2 \prod_{l\in I^c} s_l^{\gamma_2}$$

$$\prod_{\substack{j\in I\\k\in I^c}} \frac{s_j}{s_k} = \prod_{l\in I} s_l^{\gamma_3} \prod_{l\in I^c} s_l^{\gamma_4}$$

Then we try to find the powers γ_1, γ_2, γ_3 and γ_4. It is observed that

$$\begin{aligned}
\gamma_1 &= \frac{-2(r-1)}{2} = 1-r \\
\gamma_2 &= \frac{-2(n-r-1)}{2} = -n+r+1 \\
\gamma_3 &= n-r \\
\gamma_4 &= -r
\end{aligned}$$

It is worthy to illustrate how can get γ_3 and γ_4 here, which is tricky. Since we are splitting the product with indices belonging to different sets, there is certain dependency between index k and j in terms of the order like doing a double integral. The number of the s-factors can be visualized in the following matrix

$$\left\{ \frac{s_{j_\alpha}}{s_{k_\beta}} \right\}_{\substack{1\le\alpha\le r\\1\le\beta\le n-r}} = \begin{pmatrix} \frac{s_{j_1}}{s_{k_1}} & \frac{s_{j_1}}{s_{k_2}} & \frac{s_{j_1}}{s_{k_3}} & \cdots \\ \frac{s_{j_2}}{s_{k_1}} & \frac{s_{j_2}}{s_{k_2}} & \frac{s_{j_2}}{s_{k_3}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It is obvious that horizontally, for each s_{j_α},there are $(n-r)$ factors. Vertically, for each s_{k_β} there are $-r$ factors.

Combine all the powers of s_l together,

$$\tau_n(x) = \frac{2^{-\alpha r+2r(n-r)+\frac{\alpha n}{2}}}{(2\pi i)^n} \sum_{\substack{I\subset\{1,\dots,n\}\\|I|=r}} d_1^r d_2^{n-r} (-e^{\frac{i\pi\alpha}{2}})^{n-r} \frac{x^{n(n-1)+\alpha r-\frac{n\alpha}{2}-2r(n-r)}}{n!} \int_{\Gamma_1}\cdots\int_{\Gamma_1}\int_{\Gamma_2}\cdots\int_{\Gamma_2} \prod_{\substack{j<k\\j,k\in I}} (s_j-s_k)^2$$

$$\prod_{l\in I} e^{s_l} s_l^{-\frac{\alpha}{2}-2r+n} \prod_{\substack{j<k\\j,k\in I^c}} (s_j-s_k)^2 \prod_{l\in I^c} e^{-\frac{1}{s_l}} s_l^{-\frac{\alpha}{2}-n} \prod_{q\in I} ds_q \prod_{q\in I^c} ds_q$$

After some further algebraic manipulations, we eventually obtain the alternative form of τ_n.

Proposition 3.2.1. *The alternative formula of $\tau_n(x)$ is given by*

$$\tau_n(x) = \frac{2^{-\alpha r+2r(n-r)+\frac{\alpha n}{2}}}{(2\pi i)^n} \sum_{\substack{I\subset\{1,\dots,n\}\\|I|=r}} d_1^r d_2^{n-r}(-e^{\frac{i\pi\alpha}{2}})^{n-r}\frac{x^{n(n-1)+\alpha r-\frac{n\alpha}{2}-2r(n-r)}}{n!}\int_{\Gamma_1}\dots\int_{\Gamma_1}\prod_{\substack{j<k\\j,k\in I}}(s_j-s_k)^2$$
$$\prod_{l\in I} e^{s_l} s_l^{-\frac{\alpha}{2}-2r+n}\prod_{q\in I} ds_q \int_{\Gamma_2}\dots\int_{\Gamma_2}\prod_{\substack{j<k\\j,k\in I^c}}(s_j-s_k)^2\prod_{l\in I^c} e^{-\frac{1}{s_l}} s_l^{-\frac{\alpha}{2}-n}\prod_{q\in I^c} ds_q$$

Γ_1 is contour of integration for $J_\nu(x)$ and Γ_2 is contour of integration for $J_{-\nu}(x)$ after change of variable $e^z = t$

3.3 Asymptotic of $\tau_n(x)$

The asymptotic of τ_n is the leading term of the alternative formula. To obtain that, we need to find the smallest power in terms of r. Denote the power of the leading term of τ_n as $p(r)$, then

$$p(r) = n(n-1) + \alpha r - \frac{n\alpha}{2} - 2r(n-r)$$

Take the derivative to find the critical point, we get

$$\frac{dp}{dr} = \alpha - 2n + 4r = 0$$

and

$$r_c(n,\alpha) = \frac{2n - Re(\alpha)}{4}$$

Denote $||x||$ as the closest integer near x. Choose $r_c = ||\frac{2n-Re(\alpha)}{4}||$ roughly, then the leading term is

$$\tau_n(x) \sim \frac{2^{-\alpha r_c+2r_c(n-r_c)+\frac{\alpha n}{2}}}{(2\pi i)^n}\frac{n!}{(n-r_c)!r_c!} d_1^{r_c} d_2^{n-r_c}(-e^{\frac{i\pi\alpha}{2}})^{n-r_c}\frac{x^{n(n-1)+\alpha r_c-\frac{n\alpha}{2}-2r_c(n-r_c)}}{n!}\int_{\Gamma_1}\dots\int_{\Gamma_1}\prod_{\substack{j<k\\j,k\in I}}(s_j-s_k)^2$$
$$\prod_{l\in I} e^{s_l} s_l^{-\frac{\alpha}{2}-2r_c+n}\prod_{q\in I} ds_q \int_{\Gamma_2}\dots\int_{\Gamma_2}\prod_{\substack{j<k\\j,k\in I^c}}(s_j-s_k)^2\prod_{l\in I^c} e^{-\frac{1}{s_l}} s_l^{-\frac{\alpha}{2}-n}\prod_{q\in I^c} ds_q$$

when $x\to 0$. oBy Andréief identity again, we have

$$\tau_n(x) \sim \frac{2^{-\alpha r_c+2r_c(n-r_c)+\frac{\alpha n}{2}}}{(2\pi i)^n} d_1^{r_c} d_2^{n-r_c}(-e^{\frac{i\pi\alpha}{2}})^{n-r_c} x^{n(n-1)+\alpha r_c-\frac{n\alpha}{2}-2r_c(n-r_c)}$$
$$\det\left(\left\{\int_{\Gamma_1} s^{k+j}e^s s^{-\frac{\alpha}{2}-2r_c+n}ds\right\}_{j,k=0}^{r_c-1}\right)\det\left(\left\{\int_{\Gamma_2} s^{k+j}e^{-\frac{1}{s}} s^{-\frac{\alpha}{2}-n}ds\right\}_{j,k=0}^{n-r_c-1}\right)$$

when $x\to 0$.
Notice that the two determinants are Hankel determinants with $w_1(s) = e^s s^{-\frac{\alpha}{2}-2r_c+n}$ and $w_2(s) = e^{-\frac{1}{s}} s^{-\frac{\alpha}{2}-n}$ respectively. To evaluate them, we need to refer to the explicit formula of H_n and the result in the literature. In Table 18.3.1 of [DLMF], we have the information about the normalizing constant of Laguerre Polynomial with the weight $w(x)$ in the form of $e^{-x}x^\alpha$, where $\alpha > -1$. We try to use this information, so it motivates us to do some change of variables again in our case to make the weight in the same form as the literature. Denote

$$H_{r_c-1} = \int_{\Gamma_1}\dots\int_{\Gamma_1}\prod_{\substack{j<k\\j,k\in I}}(s_j-s_k)^2\prod_{l\in I} e^{s_l} s_l^{-\frac{\alpha}{2}-2r_c+n}\prod_{q\in I} ds_q$$

and

$$H_{n-r_c-1} = \int\limits_{\Gamma_2} \cdots \int\limits_{\Gamma_2} \prod_{\substack{j<k \\ j,k\in I^c}} (s_j - s_k)^2 \prod_{l\in I^c} e^{-\frac{1}{s_l}} s_l^{-\frac{\alpha}{2}-n} \prod_{q\in I^c} ds_q$$

In H_{r_c-1}, we let $s = \widetilde{s}e^{-i\pi}$, then the modulus and argument of the variable respectively becomes

$$|s| = |\widetilde{s}|$$
$$\arg(s) = \arg(\widetilde{s}) - \pi, 0 < \arg(\widetilde{s}) < 2\pi$$

Correspondingly, the contour Γ_1 becomes $\widetilde{\Gamma_1}$ as shown in the picture below.

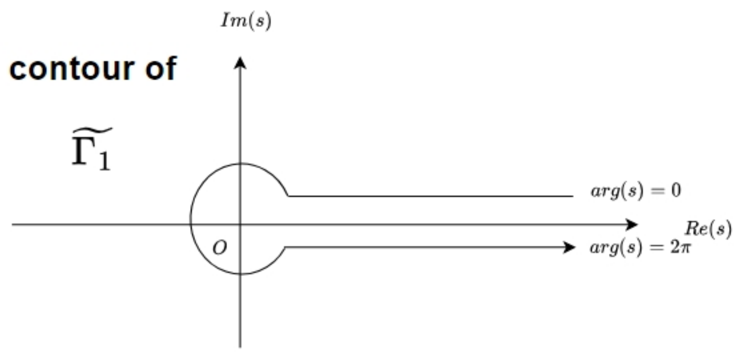

Figure 3: Contour of $\widetilde{\Gamma_1}$

Also notice that

$$\begin{aligned} s_l^{-\frac{\alpha}{2}-2r_c+n} &= e^{(-\frac{\alpha}{2}-2r_c+n)\ln|s_l| + (-\frac{\alpha}{2}-2r_c+n)i\arg(s_l)} \\ &= e^{(-\frac{\alpha}{2}-2r_c+n)\ln|\widetilde{s}_l| + (-\frac{\alpha}{2}-2r_c+n)i(\arg(\widetilde{s}_l)-\pi)} \\ &= \widetilde{s}_l^{-\frac{\alpha}{2}-2r_c+n} e^{-i\pi(-\frac{\alpha}{2}-2r_c+n)} \end{aligned}$$

Here we use a simple identity for exponential function of complex variables

$$z^\alpha = e^{\alpha \ln|z| + i\alpha\arg(z)}, \ \forall z \in \mathbb{C}$$

Then the first multi-integral eventually becomes

$$\begin{aligned} H_{r_c-1} &= \int\limits_{\widetilde{\Gamma_1}} \cdots \int\limits_{\widetilde{\Gamma_1}} \prod_{\substack{j<k \\ j,k\in I}} (\widetilde{s}_j - \widetilde{s}_k)^2 \prod_{l\in I} e^{-\widetilde{s}_l} \widetilde{s}_l^{-\frac{\alpha}{2}-2r_c+n} e^{-i\pi(-\frac{\alpha}{2}-2r_c+n)} \left((-1)^{r_c} \prod_{q\in I} d\widetilde{s}_q \right) \\ &= (-1)^{r_c} (e^{-i\pi(-\frac{\alpha}{2}-2r_c+n)})^{r_c} \int\limits_{\widetilde{\Gamma_1}} \cdots \int\limits_{\widetilde{\Gamma_1}} \prod_{\substack{j<k \\ j,k\in I}} (\widetilde{s}_j - \widetilde{s}_k)^2 \prod_{l\in I} e^{-\widetilde{s}_l} \widetilde{s}_l^{-\frac{\alpha}{2}-2r_c+n} \prod_{q\in I} d\widetilde{s}_q \end{aligned}$$

Similarly, in H_{n-r_c-1}, we let $s = \frac{1}{\widetilde{s}}$,then the modulus and argument of the variable respectively becomes

$$|s| = \frac{1}{|\widetilde{s}|}$$
$$\arg(\widetilde{s}) = -\arg(s), -2\pi < \arg(\widetilde{s}) < 0$$

Correspondingly, the contour Γ_2 becomes $\widetilde{\Gamma_2}$ as shown in the picture below.

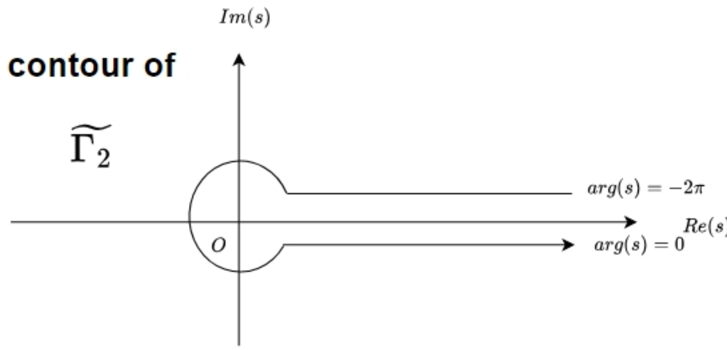

Figure 4: Contour of $\widetilde{\Gamma_2}$

Then the second multi-integral eventually becomes

$$\begin{aligned} H_{n-r_c-1} &= \int_{\widetilde{\Gamma_2}} \dots \int_{\widetilde{\Gamma_2}} \prod_{\substack{j<k \\ j,k\in I^c}} \left(\frac{1}{\widetilde{s_j}} - \frac{1}{\widetilde{s_k}}\right)^2 \prod_{l\in I^c} e^{-\widetilde{s_l}} \widetilde{s_l}^{\frac{\alpha}{2}+n} \left((-1)^{n-r_c} \widetilde{s_q}^{-2(n-r_c)} \prod_{q\in I^c} d\widetilde{s_q} \right) \\ &= (-1)^{n-r_c} \int_{\widetilde{\Gamma_2}} \dots \int_{\widetilde{\Gamma_2}} \prod_{\substack{j<k \\ j,k\in I^c}} \left(\frac{\widetilde{s_j} - \widetilde{s_k}}{\widetilde{s_j}\widetilde{s_k}} \right)^2 \prod_{l\in I^c} e^{-\widetilde{s_l}} \widetilde{s_l}^{\frac{\alpha}{2}+n-2} \prod_{q\in I^c} d\widetilde{s_q} \\ &= (-1)^{n-r_c} \int_{\widetilde{\Gamma_2}} \dots \int_{\widetilde{\Gamma_2}} \prod_{\substack{j<k \\ j,k\in I^c}} (\widetilde{s_j} - \widetilde{s_k})^2 \prod_{l\in I^c} \widetilde{s_l}^{-2(n-r_c-1)} \prod_{l\in I^c} e^{-\widetilde{s_l}} \widetilde{s_l}^{\frac{\alpha}{2}+n-2} \prod_{q\in I^c} d\widetilde{s_q} \\ &= (-1)^{n-r_c} \int_{\widetilde{\Gamma_2}} \dots \int_{\widetilde{\Gamma_2}} \prod_{\substack{j<k \\ j,k\in I^c}} (\widetilde{s_j} - \widetilde{s_k})^2 \prod_{l\in I^c} e^{-\widetilde{s_l}} \widetilde{s_l}^{\frac{\alpha}{2}+2r_c-n} \prod_{q\in I^c} d\widetilde{s_q} \end{aligned}$$

Here we use the same technique as the derivation in section 3.2

$$\prod_{\substack{j<k \\ j,k\in I^c}} \left(\frac{\widetilde{s_j} - \widetilde{s_k}}{\widetilde{s_j}\widetilde{s_k}} \right)^2 = \prod_{\substack{j<k \\ j,k\in I^c}} (\widetilde{s_j} - \widetilde{s_k})^2 \prod_{\substack{j\neq k \\ j,k\in I^c}} \frac{1}{\widetilde{s_j}\widetilde{s_k}} = \prod_{\substack{j<k \\ j,k\in I^c}} (\widetilde{s_j} - \widetilde{s_k})^2 \prod_{l\in I^c} \widetilde{s_l}^{-2(n-r_c-1)}$$

Now, the weights turn into the same form as the literature as our expect: $w_1(\widetilde{s}) = e^{-\widetilde{s_l}} \widetilde{s_l}^{-\frac{\alpha}{2}-2r_c+n}$ and $w_2(\widetilde{s}) = e^{-\widetilde{s_l}} \widetilde{s_l}^{\frac{\alpha}{2}+2r_c-n}$. Furthermore, notice that the contours $\widetilde{\Gamma_1}$ and $\widetilde{\Gamma_2}$ in our cases are loops, but in the literature, the contour is the positive real line. To make them match, we need to introduce the following theorems.

Theorem 3.3.1. *For the generalized Laguerre polynomials with weight $w(s) = e^{-s}s^\gamma$, the orthogonality condition is*

$$\int_0^\infty P_n(s) s^j e^{-s} s^\gamma ds = 0$$

with $0 \le j \le n-1$, $Re(\gamma) > -1$, $P_n(x)$ is monic.
The corresponding normalizing constant is given by

$$h_n(\gamma) = \int_0^\infty P_n(s) s^n e^{-s} s^\gamma ds = \Gamma(n+\gamma+1)n!$$

Claim 3.3.1. *By theorem of uniqueness of analytic continuation,we can extend the contour in previous theorem to a loop* $\widetilde{\Gamma_1}$ *on the complex plane. For the generalized Laguerre polynomials with weight* $w(s) = e^{-s}s^{\gamma}$*, the orthogonality condition after the extension is*

$$\int_0^\infty P_n(s)s^j e^{-s}s^\gamma ds = \frac{1}{1-e^{2\pi i\gamma}} \int_{\widetilde{\Gamma_1}} P_n(s)s^j e^{-s}s^\gamma ds = 0$$

with $0 \leq j \leq n-1$*,* $Re(\gamma) > -1$ *and not an integer,* $P_n(x)$ *is monic.*
The corresponding normalizing constant is given by

$$h_n(\gamma) = \int_0^\infty P_n(s)s^n e^{-s}s^\gamma ds = \frac{1}{1-e^{2\pi i\gamma}} \int_{\widetilde{\Gamma_1}} P_n(s)s^n e^{-s}s^\gamma ds = \Gamma(n+\gamma+1)n!$$

Since the range of argument in the second multi-integral is different from the first one, we should fix Claim 3.3.1 as following:

Claim 3.3.2. *By theorem of uniqueness of analytic continuation,we can extend the contour in previous theorem to a loop* $\widetilde{\Gamma_2}$ *on the complex plane. For the generalized Laguerre polynomials with weight* $w(s) = e^{-s}s^{\gamma}$*, the orthogonality condition after the extension is*

$$\int_0^\infty P_n(s)s^j e^{-s}s^\gamma ds = \frac{1}{1-e^{-2\pi i\gamma}} \int_{\widetilde{\Gamma_2}} P_n(s)s^j e^{-s}s^\gamma ds = 0$$

with $0 \leq j \leq n-1$*,* $Re(\gamma) > -1$ *and not an integer,* $P_n(x)$ *is monic.*
The corresponding normalizing constant is given by

$$h_n(\gamma) = \int_0^\infty P_n(s)s^n e^{-s}s^\gamma ds = \frac{1}{1-e^{-2\pi i\gamma}} \int_{\widetilde{\Gamma_2}} P_n(s)s^n e^{-s}s^\gamma ds = \Gamma(n+\gamma+1)n!$$

Remark: One may notice that the formula of normalizing constant in Theorem 3.3.1 is slightly different from the one in the literature. That is because in [DLMF](18.2.7), the orthogonal polynomial $p_n(x)$ is defined as:

$$p_n(x) = k_n x^n + \widetilde{k_n}x^{n-1} + \widetilde{\widetilde{k_n}}x^{n-1} + \dots$$

where k_n is not always 1. However, $P_n(x)$ introduced in Definition 2.9.1 is monic (i.e. $k_n = 1$). Hence we need to make some change here. By [DLMF](18.2.1), we know the orthogonality condition is the following:

$$\int_a^b p_n(x)p_m(x)w(x)dx = \widetilde{h_n(\gamma)}\delta_{nm}$$

where $w(x)$ is the weight and δ_{nm} is the Kronecker Delta. It is also known that $P_n(x) = \frac{p_n(x)}{k_n}$, so the orthogonality condition for $P_n(x)$ becomes

$$\int_a^b P_n(x)P_m(x)w(x)dx = \frac{1}{k_n^2}\widetilde{h_n(\gamma)}\delta_{nm}$$

According to [DLMF] Table 18.3.1, we know

$$\widetilde{h_n(\gamma)} = \frac{\Gamma(n+\gamma+1)}{n!}$$

$$k_n = \frac{(-1)^n}{n!}$$

Thus in our case,

$$h_n(\gamma) = \frac{\widetilde{h_n(\gamma)}}{k_n^2} = \frac{\Gamma(n+\gamma+1)}{n!}(n!)^2 = \Gamma(n+\gamma+1)n!$$

With the aid of the two claims above, we need to rewrite our multi-integrals again in terms of line contour. The first one becomes

$$H_{r_c-1} = (-1)^{r_c}(e^{-i\pi(-\frac{\alpha}{2}-2r_c+n)})^{r_c}(e^{2\pi i(-\frac{\alpha}{2}-2r_c+n)}-1)^{r_c}\int_0^\infty\cdots\int_0^\infty \prod_{\substack{j<k\\ j,k\in I}}(\widetilde{s_j}-\widetilde{s_k})^2\prod_{l\in I}e^{-\widetilde{s_l}}\widetilde{s_l}^{-\frac{\alpha}{2}-2r_c+n}\prod_{q\in I}d\widetilde{s_q}$$
$$= (-1)^{r_c}(2i\sin((-\frac{\alpha}{2}-2r_c+n)\pi))^{r_c}H^L_{r_c-1}$$

where $H^L_{r_c-1}$ denotes the Hankel Determinant associated with Laguerre polynomial.
On the other hand, the second one becomes

$$H_{n-r_c-1} = (-1)^{n-r_c}\int_{\widetilde{\Gamma_2}}\cdots\int_{\widetilde{\Gamma_2}}\prod_{\substack{j<k\\ j,k\in I^c}}(\widetilde{s_j}-\widetilde{s_k})^2\prod_{l\in I^c}e^{-\widetilde{s_l}}\widetilde{s_l}^{\frac{\alpha}{2}+2r_c-n}\prod_{q\in I^c}d\widetilde{s_q}$$
$$= (-1)^{n-r_c}(1-e^{-2\pi i(\frac{\alpha}{2}+2r_c-n)})^{n-r_c}\int_0^\infty\cdots\int_0^\infty\prod_{\substack{j<k\\ j,k\in I^c}}(\widetilde{s_j}-\widetilde{s_k})^2\prod_{l\in I^c}e^{-\widetilde{s_l}}\widetilde{s_l}^{\frac{\alpha}{2}+2r_c-n}\prod_{q\in I^c}d\widetilde{s_q}$$
$$= (-1)^{n-r_c}(e^{-\pi i(\frac{\alpha}{2}+2r_c-n)})^{n-r_c}(2i\sin((\frac{\alpha}{2}+2r_c-n)\pi))^{n-r_c}H^L_{n-r_c-1}$$
$$= (-1)^{n-r_c}(e^{-\pi i\frac{\alpha}{2}})^{n-r_c}(e^{-\pi i(2r_c-n)(n-r_c)})(2i\sin((\frac{\alpha}{2}+2r_c-n)\pi))^{n-r_c}H^L_{n-r_c-1}$$
$$= (-1)^{n-r_c}(e^{-\pi i\frac{\alpha}{2}})^{n-r_c}e^{-\pi i2r_c(n-r_c)}e^{-\pi in(n-r_c)}(2i\sin((\frac{\alpha}{2}+2r_c-n)\pi))^{n-r_c}H^L_{n-r_c-1}$$
$$= (-1)^{n-r_c}(e^{-\pi i\frac{\alpha}{2}})^{n-r_c}(-1)^{-n(n-r_c)}(2i\sin((\frac{\alpha}{2}+2r_c-n)\pi))^{n-r_c}H^L_{n-r_c-1}$$
$$= (-1)^{(1-n)(n-r_c)}(e^{-\pi i\frac{\alpha}{2}})^{n-r_c}(2i\sin((\frac{\alpha}{2}+2r_c-n)\pi))^{n-r_c}H^L_{n-r_c-1}$$

where $H^L_{n-r_c-1}$ denotes the Hankel Determinant associated with Laguerre polynomial.
Up to this point, we can easily compute $H^L_{r_c-1}$ and $H^L_{n-r_c-1}$ by Proposition 2.9.2 and Theorem 3.3.1. We obtain that

$$H^L_{r_c-1} = \prod_{j=0}^{r_c-1}h_j(-\frac{\alpha}{2}-2r_c+n) = \prod_{j=0}^{r_c-1}\Gamma(j-\frac{\alpha}{2}-2r_c+n+1)j!$$
$$H^L_{n-r_c-1} = \prod_{j=0}^{n-r_c-1}h_j(\frac{\alpha}{2}+2r_c-n) = \prod_{j=0}^{n-r_c-1}\Gamma(j+\frac{\alpha}{2}+2r_c-n+1)j!$$

Moreover, we try to give a precise formula for r_c using piecewise functions.

Proposition 3.3.1. *The piecewise function for the critical point r_c is given by*

$$r_c(\alpha,n) = \begin{cases} 0 & \text{if } \alpha > 2n-2 \\ n-j & \text{if } -2n+4j-2 < \alpha < -2n+4j+2 \text{ and } j = 1,2,\ldots,n-1 \\ n & \text{if } \alpha < -2n+2 \end{cases}$$

Proof. Since $0 \le r \le n$ and $r \in \mathbb{Z}$, so $p(r)$ only takes values on that discrete set. It is clear that $p(r)$ is a upward parabola and it takes minimum value at $r_{min}(n,\alpha) = \frac{2n-\alpha}{4}$. We will discuss different cases of relative positions between r_{min} and r_c. If $r_{min} \le 0$, then $r_c = 0$. If $r_{min} \ge n$, then $r_c = n$. Let $0 \le j \le n$ and $j \le r_{min} \le j+1$. If $k \le r_{min} \le j+\frac{1}{2}$, then $r_c = j$. If $j+\frac{1}{2} \le r_{min} \le j+1$, then $r_c = j+1$. From the figure, we can formulate the discussion above mathematically as following:

- $r_c(\alpha,n) = 0$ when $\frac{2n-\alpha}{4} < \frac{1}{2}$

- $r_c(\alpha, n) = n - j$ when $n - \frac{2j+1}{2} < \frac{2n-\alpha}{4} < n - \frac{2j-1}{2}$ and $j = 1, 2, \ldots, n-1$
- $r_c(\alpha, n) = n$ when $\frac{2n-\alpha}{4} > n - \frac{1}{2}$

Hence, the piecewise function for the critical point r_c can be written as

$$r_c(\alpha, n) = \begin{cases} n & \text{if } \frac{\alpha}{2} < 1 - n \\ n - j & \text{if } 2j - 1 - n < \frac{\alpha}{2} < 2j + 1 - n \text{ and } k = 1, 2, \ldots, n-1 \\ 0 & \text{if } \frac{\alpha}{2} > n - 1 \end{cases}$$

□

Remark: Floor functions gives a more compact form for $r_c(\alpha, n)$. $\forall\ 0 \leq k \leq n$, we know $r_c = k$ if and only if $k - \frac{1}{2} \leq \frac{n}{2} - \frac{\alpha}{4} \leq k + \frac{1}{2}$ so $k \leq \frac{n}{2} - \frac{\alpha}{4} + \frac{1}{2} \leq k + 1$. This makes sure $0 < \lfloor \frac{n}{2} - \frac{\alpha}{4} + \frac{1}{2} \rfloor < n$. Let $n - j = k$, then range of the cases in between becomes $n - 2k - 1 < \frac{\alpha}{2} < n - 2k + 1$. When $k = 0$, we get the rightmost endpoint $n - 1$. When $k = n$, we get the leftmost endpoint $1 - n$. That is

$$r_c(\alpha, n) = \begin{cases} n & \text{if } \frac{\alpha}{2} < 1 - n \\ \lfloor \frac{n}{2} - \frac{\alpha}{4} + \frac{1}{2} \rfloor & \text{if } 1 - n < \frac{\alpha}{2} < n - 1 \\ 0 & \text{if } \frac{\alpha}{2} > n - 1 \end{cases}$$

Finally we reach the conclusion.

Theorem 3.3.2 (Pan-Prokhorov)**.** *The asymptotic as $x \to 0$ of $\tau_n(x)$ for the case of Painlevé-III equation in the case of $\varepsilon = 1$ and $\frac{\alpha}{2} \notin \mathbb{Z}$ is given by*

$$\tau_n(x) \sim \frac{2^{-\alpha r_c + 2r_c(n-r_c) + \frac{\alpha n}{2}}}{(2\pi i)^n} d_1^{r_c} d_2^{n-r_c} (-e^{\frac{i\pi\alpha}{2}})^{n-r_c} x^{n(n-1)+\alpha r_c - \frac{n\alpha}{2} - 2r_c(n-r_c)} H_{r_c-1} H_{n-r_c-1}$$

where

$$d_1 = c_1(1 + \cot(\pi\frac{\alpha}{2})i) + c_2(1 - \cot(\pi\frac{\alpha}{2})i),$$
$$d_2 = c_2 \csc(\pi\frac{\alpha}{2})i - c_1 \csc(\pi\frac{\alpha}{2})i,$$
$$c_1, c_2 \in \mathbb{C},$$
$$r_c(\alpha, n) = \begin{cases} 0 & \textit{if } \alpha > 2n - 2 \\ n - j & \textit{if } -2n + 4j - 2 < \alpha < -2n + 4j + 2 \textit{ and } j = 1, 2, \ldots, n-1 \\ n & \textit{if } \alpha < -2n + 2 \end{cases},$$
$$H_{r_c-1} = (-1)^{r_c}(2i\sin((-\frac{\alpha}{2} - 2r_c + n)\pi))^{r_c} \prod_{j=0}^{r_c-1} \Gamma(j - \frac{\alpha}{2} - 2r_c + n + 1)j!,$$
$$H_{n-r_c-1} = (-1)^{(1-n)(n-r_c)}(e^{-\pi i\frac{\alpha}{2}})^{n-r_c}(2i\sin((\frac{\alpha}{2} + 2r_c - n)\pi))^{n-r_c} \prod_{j=0}^{n-r_c-1} \Gamma(j + \frac{\alpha}{2} + 2r_c - n + 1)j!$$

3.4 Asymptotic of $q_n(x)$

To simplify our computation, we are going to rewrite the formula in Theorem 3.3.2. It is necessary to introduce the notation for Barnes G-function $G(z)$. $G(z)$ satisfies the following property:

$$G(z+1) = \Gamma(z)G(z), \text{ with normalization } G(1) = 1,\ \forall z \in \mathbb{C} \tag{37}$$

We also need to introduce the following lemmas:

Lemma 3.4.1. *For Gamma function $\Gamma(z)$ and Barnes G-function $G(z)$, the following relation holds:*

$$\prod_{j=0}^{n} \Gamma(j+1) = \frac{G(n+2)}{G(1)}$$

Proof. The proof is relatively easy, which just use (37) recursively.

$$\prod_{j=0}^{n}\Gamma(j+1)=\frac{\Gamma(n+1)\Gamma(n)\ldots\Gamma(1)G(1)}{G(1)}=\frac{\Gamma(n+1)\Gamma(n)\ldots\Gamma(2)G(2)}{G(1)}=\ldots=\frac{\Gamma(n+1)G(n+1)}{G(1)}=\frac{G(n+2)}{G(1)}$$

□

Lemma 3.4.2. *The Hankel determinants associated with Laguerre polynomial in terms of Barnes G-function are given by*

$$H^L_{r_c-1}=\frac{G(r_c+1)G(-\frac{\alpha}{2}+n-r_c+1)}{G(-\frac{\alpha}{2}+n-2r_c+1)}$$

$$H^L_{n-r_c-1}=\frac{G(n-r_c+1)G(r_c+\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}-n+2r_c+1)}$$

Proof. Applying Lemma 3.4.1, we can rewrite the Hankel determinants in section 3.4 in terms of $G(x)$ as following:

$$\begin{aligned}H^L_{r_c-1}&=\prod_{j=0}^{r_c-1}\Gamma(j-\frac{\alpha}{2}-2r_c+n+1)j!\\&=\prod_{j=0}^{r_c-1}j!\prod_{j=0}^{r_c-1}\Gamma(j-\frac{\alpha}{2}-2r_c+n+1)\\&=\prod_{j=0}^{r_c-1}\Gamma(j+1)\prod_{j=0}^{r_c-1}\Gamma(j-\frac{\alpha}{2}-2r_c+n+1)\\&=\frac{G(r_c+1)G(-\frac{\alpha}{2}+n-r_c+1)}{G(-\frac{\alpha}{2}+n-2r_c+1)}\end{aligned}$$

Similarly, we have

$$\begin{aligned}H^L_{n-r_c-1}&=\prod_{j=0}^{n-r_c-1}\Gamma(j+\frac{\alpha}{2}+2r_c-n+1)j!\\&=\prod_{j=0}^{n-r_c-1}j!\prod_{j=0}^{n-r_c-1}\Gamma(j+\frac{\alpha}{2}+2r_c-n+1)\\&=\prod_{j=0}^{n-r_c-1}\Gamma(j+1)\prod_{j=0}^{n-r_c-1}\Gamma(j+\frac{\alpha}{2}+2r_c-n+1)\\&=\frac{G(n-r_c+1)G(r_c+\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}-n+2r_c+1)}\end{aligned}$$

□

To deduce the asymptotic of $q_n(x)$, we need to use our main result (3.3.2) and (27). We have to shift the index of Tau function n and its dependent parameter α. However, all the parameters in our main result depends on them. Specifically, d_1, d_2 are purely written in terms of α. On the other hand, r_c is written in terms of both n and α. H_{r_c-1} and H_{n-r_c-1} depend on all of r_c, n and α, which is much more complicated. Because of periodicity, it is easy to see that: $d_1(\alpha-2)=d_1(\alpha)$ and $d_2(\alpha-2)=-d_2(\alpha)$. Thus it is obvious that r_c plays a central role in such dependency relations. We need to reexamine r_c carefully and try to suggest a piecewise function for it in terms of n and α.

3.4.1 Piecewise function for the exponent of leading term $e(\alpha)$

From Proposition 3.3.1, we can modify the following formulae of r_c with shifted indices and parameters

$$r_c(\alpha, n+1) = \begin{cases} 0 & \text{if } \alpha > 2n \\ n-j+1 & \text{if } -2n+4j-4 < \alpha < -2n+4j \text{ and } j = 1, 2, \ldots, n \\ n+1 & \text{if } \alpha < -2n \end{cases}$$

$$r_c(\alpha-2, n) = \begin{cases} 0 & \text{if } \alpha > 2n \\ n-j & \text{if } -2n+4j < \alpha < -2n+4j+4 \text{ and } j = 1, 2, \ldots, n-1 \\ n & \text{if } \alpha < -2n+4 \end{cases}$$

$$r_c(\alpha-2, n+1) = \begin{cases} 0 & \text{if } \alpha > 2n+2 \\ n-j+1 & \text{if } -2n+4j-2 < \alpha < -2n+4j+2 \text{ and } j = 1, 2, \ldots, n-1 \\ n+1 & \text{if } \alpha < -2n+2 \end{cases}$$

Proposition 3.4.1. *The piecewise function for the exponent of leading term e is given by*

$$e(\alpha) = \begin{cases} 1 & \text{if } \alpha > 2+2n \\ \alpha+2n-4j-1 & \text{if } -2n+4j < \alpha < -2n+4j+2 \text{ and } j = 0, 1, \ldots, n \\ -\alpha-2n+4j-1 & \text{if } -2n+4j-2 < \alpha < -2n+4j \text{ and } j = 1, 2, \ldots, n \\ -1 & \text{if } \alpha < -2n \end{cases}$$

Proof. We just simply plug Proposition 3.3.1 and modified formulae of r_c with shifted indices and parameters into 27 to compute the exponent of leading terms of $q_n(x)$.
If $\alpha > 2+2n$, then we need to choose:

$$r_c(\alpha, n) = r_c(\alpha, n+1) = r_c(\alpha-2, n) = r_c(\alpha-2, n+1) = 0$$

By 27, we obtain that:

$$q_n(x) \sim (\text{some coefficients}) \ \frac{x^{(n+1)n - \frac{(n+1)(\alpha-2)}{2}} x^{n(n-1) - \frac{n\alpha}{2}}}{x^{(n+1)n - \frac{(n+1)\alpha}{2}} x^{n(n-1) - \frac{n(\alpha-2)}{2}}}$$

$$\sim (\text{some coefficients}) \ x$$

Similarly, if $-2n+4j < \alpha < -2n+4j+2$ and $j = 0, 1, \ldots, n$, then we need to choose:

$$\begin{aligned} r_c(\alpha, n) &= n-j \\ r_c(\alpha, n+1) &= n-j \\ r_c(\alpha-2, n) &= n-j \\ r_c(\alpha-2, n+1) &= n+1-j \end{aligned}$$

We obtain that:

$$q_n(x) \sim (\text{some coefficients}) \ \frac{x^{(n+1)n+(\alpha-2)(n+1-j) - \frac{(n+1)(\alpha-2)}{2} - 2(n+1-j)j} x^{n(n-1)+\alpha(n-j) - \frac{n\alpha}{2} - 2(n-j)j}}{x^{(n+1)n+\alpha(n-j) - \frac{(n+1)\alpha}{2} - 2(n-j)(j+1)} x^{n(n-1)+(\alpha-2)(n-j) - \frac{n(\alpha-2)}{2} - 2(n-j)j}}$$

$$\sim (\text{some coefficients}) \ x^{\alpha-1+2n-4j}$$

If $-2n+4j-2 < \alpha < -2n+4j$ and $j = 1, 2, \ldots, n$, then we need to choose:

$$\begin{aligned} r_c(\alpha, n) &= n-j \\ r_c(\alpha, n+1) &= n+1-j \\ r_c(\alpha-2, n) &= n+1-j \\ r_c(\alpha-2, n+1) &= n+1-j \end{aligned}$$

We obtain that:

$$q_n(x) \sim (\text{some coefficients}) \; \frac{x^{(n+1)n+(\alpha-2)(n+1-j)-\frac{(n+1)(\alpha-2)}{2}-2(n+1-j)j} x^{n(n-1)+\alpha(n-j)-\frac{n\alpha}{2}-2(n-j)j}}{x^{(n+1)n+\alpha(n+1-j)-\frac{(n+1)\alpha}{2}-2(n+1-j)j} x^{n(n-1)+(\alpha-2)(n-j+1)-\frac{n(\alpha-2)}{2}-2(n-j+1)(j-1)}}$$

$$\sim (\text{some coefficients}) \; x^{-\alpha-1-2n+4j}$$

If $\alpha < -2n$, then we need to choose:

$$\begin{aligned} r_c(\alpha, n) &= n \\ r_c(\alpha, n+1) &= n+1 \\ r_c(\alpha-2, n) &= n \\ r_c(\alpha-2, n+1) &= n+1 \end{aligned}$$

We obtain that:

$$q_n(x) \sim (\text{some coefficients}) \; \frac{x^{(n+1)n+(\alpha-2)(n+1)-\frac{(n+1)(\alpha-2)}{2}} x^{n(n-1)+\alpha n-\frac{n\alpha}{2}}}{x^{(n+1)n+\alpha(n+1)-\frac{(n+1)\alpha}{2}} x^{n(n-1)+(\alpha-2)n-\frac{n(\alpha-2)}{2}}}$$

$$\sim (\text{some coefficients}) \; x^{-1}$$

□

3.4.2 Piecewise formulae for asympototic of $q_n(x)$

In this part, we will compute the coefficients shown in the previous section case by case. For convenience, firstly we need to rewrite the formulae of r_c in terms of the same ranges as Proposition 3.4.1.

Lemma 3.4.3. *The piecewise function for the critical point r_c compatible with the same ranges in Proposition 3.4.1 is given by*

$$r_c(\alpha, n) = \begin{cases} 0 & \text{if } \alpha > 2+2n \\ n-j & \text{if } -2n+4j < \alpha < -2n+4j+2 \text{ and } j = 0,1,\dots,n \\ n-j & \text{if } -2n+4j-2 < \alpha < -2n+4j \text{ and } j = 1,2,\dots,n \\ n & \text{if } \alpha < -2n \end{cases}$$

$$r_c(\alpha, n+1) = \begin{cases} 0 & \text{if } \alpha > 2+2n \\ n-j & \text{if } -2n+4j < \alpha < -2n+4j+2 \text{ and } j = 0,1,\dots,n \\ n-j+1 & \text{if } -2n+4j-2 < \alpha < -2n+4j \text{ and } j = 1,2,\dots,n \\ n & \text{if } \alpha < -2n \end{cases}$$

$$r_c(\alpha-2, n) = \begin{cases} 0 & \text{if } \alpha > 2+2n \\ n-j+1 & \text{if } -2n+4j < \alpha < -2n+4j+2 \text{ and } j = 0,1,\dots,n \\ n-j+1 & \text{if } -2n+4j-2 < \alpha < -2n+4j \text{ and } j = 1,2,\dots,n \\ n & \text{if } \alpha < -2n \end{cases}$$

$$r_c(\alpha-2, n+1) = \begin{cases} 0 & \text{if } \alpha > 2+2n \\ n-j+1 & \text{if } -2n+4j < \alpha < -2n+4j+2 \text{ and } j = 0,1,\dots,n \\ n-j+1 & \text{if } -2n+4j-2 < \alpha < -2n+4j \text{ and } j = 1,2,\dots,n \\ n+1 & \text{if } \alpha < -2n \end{cases}$$

Now we can use Theorem 3.3.2, Lemma 3.4.2, 3.4.3 and 27 to derive the piecewise formulae of asymptotic of $q_n(x)$.

Theorem 3.4.1. *The piecewise formulae of asymptotic as $x \to 0$ of $q_n(x)$ for Painlevé-III equation in the case of $\varepsilon = 1$ and $\frac{\alpha}{2} \notin \mathbb{Z}$, is given by*

$$q_n(x) \sim \begin{cases} \frac{1}{2n+2-\alpha} x & \text{if } \alpha > 2+2n \\ (-1)^n \left(\frac{d_1}{d_2}\right) \left[\left(\frac{\Gamma(-\frac{\alpha}{2}-n+2j+1)}{\Gamma(\frac{\alpha}{2}+n-2j)} \right)^2 \frac{\Gamma(n-j+\frac{\alpha}{2})\Gamma(n-j+1)}{\Gamma(-\frac{\alpha}{2}+j+1)\Gamma(j+1)} \right] (\frac{x}{2})^{\alpha+2n-4j-1} & \text{if } -2n+4j < \alpha < -2n+4j+2 \\ & \text{and } j = 0,1,\ldots,n \\ \left(\frac{d_2}{d_1}\right) \left[\left(\frac{\Gamma(\frac{\alpha}{2}+n-2j+1)}{\Gamma(-\frac{\alpha}{2}-n+2j)} \right)^2 \frac{\Gamma(-\frac{\alpha}{2}+j+1)\Gamma(j)}{\Gamma(n+\frac{\alpha}{2}-j+1)\Gamma(n-j+1)} \right] (\frac{2}{x})^{2n-4j+\alpha+1} & \text{if } -2n+4j-2 < \alpha < -2n+4j \\ & \text{and } j = 1,2,\ldots,n \\ -\frac{\alpha+2n}{x} & \text{if } \alpha < -2n \end{cases}$$

Proof. Let's denote the coefficients of the asymptotic of $\tau_n(x,\alpha)$ as $c(n,\alpha)$. If $\alpha > 2+2n$, then we have:

$$c(n,\alpha) = \frac{2^{\frac{n\alpha}{2}}}{(2\pi i)^n} d_2^n (-e^{\frac{i\pi\alpha}{2}})^n (1)(-1)^{(1-n)n} (e^{-\frac{i\pi\alpha}{2}})^n (2i\sin((\frac{\alpha}{2}-n)\pi))^n \frac{G(n+1)G(\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}-n+1)}$$

$$c(n,\alpha-2) = \frac{2^{\frac{n(\alpha-2)}{2}}}{(2\pi i)^n} d_2^n (-1)^n (-e^{\frac{i\pi(\alpha-2)}{2}})^n (1)(-1)^{(1-n)n} (e^{-\frac{i\pi(\alpha-2)}{2}})^n (2i\sin(\frac{\alpha-2}{2}-n)\pi)^n \frac{G(n+1)G(\frac{\alpha}{2})}{G(\frac{\alpha}{2}-n)}$$

$$c(n+1,\alpha) = \frac{2^{\frac{(n+1)\alpha}{2}}}{(2\pi i)^{n+1}} d_2^{n+1} (-e^{\frac{i\pi\alpha}{2}})^{n+1} (1)(-1)^{(1+n)(-n)} (e^{-\frac{i\pi\alpha}{2}})^{n+1} (2i\sin(\frac{\alpha}{2}-n-1)\pi)^{n+1} \frac{G(n+2)G(\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}-n)}$$

$$c(n+1,\alpha-2) = \frac{2^{\frac{(n+1)(\alpha-2)}{2}}}{(2\pi i)^{n+1}} d_2^{n+1} (-1)^{n+1} (-e^{\frac{i\pi(\alpha-2)}{2}})^{n+1} (1)(-1)^{(1+n)(-n)} (e^{-\frac{i\pi(\alpha-2)}{2}})^{n+1} (2i\sin(\frac{\alpha-2}{2}-n-1)\pi)^{n+1} \frac{G(n+2)G(\frac{\alpha}{2})}{G(\frac{\alpha}{2}-n-1)}$$

It follows that:

$$\begin{aligned} q_n(x) &\sim \frac{c(n+1,\alpha-2)c(n,\alpha)}{c(n+1,\alpha)c(n,\alpha-2)} x \\ &\sim \frac{(-1)G(\frac{\alpha}{2}-n)G(\frac{\alpha}{2}-n)}{2G(\frac{\alpha}{2}-n-1)G(\frac{\alpha}{2}-n+1)} x \\ &\sim \frac{(-1)\Gamma(\frac{\alpha}{2}-n-1)}{2\Gamma(\frac{\alpha}{2}-n)} x \\ &\sim \frac{1}{2n+2-\alpha} x \end{aligned}$$

If $-2n+4j < \alpha < -2n+4j+2$ and $j = 0,1,\ldots,n$, then we have:

$$\begin{aligned} c(n,\alpha) = & \frac{2^{-\alpha(n-j)+2(n-j)j+\frac{n\alpha}{2}}}{(2\pi i)^n} d_1^{n-j} d_2^n (-e^{\frac{i\pi\alpha}{2}})^j (-1)^{n-j} (2i\sin((-\frac{\alpha}{2}-n+2j)\pi))^{n-j} \frac{G(n-j+1)G(-\frac{\alpha}{2}+j+1)}{G(-\frac{\alpha}{2}-n+2j+1)} \\ & (-1)^{(1-n)j} (e^{-\frac{i\pi\alpha}{2}})^j (2i\sin((\frac{\alpha}{2}-2j+n)\pi))^j \frac{G(j+1)G(n-j+\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}+n-2j+1)} \end{aligned}$$

$$c(n,\alpha-2)=\frac{2^{-(\alpha-2)(n-j)+2(n-j)j+\frac{n(\alpha-2)}{2}}}{(2\pi i)^n}d_1^{n-j}(-d_2)^j(-e^{\frac{i\pi(\alpha-2)}{2}})^j(-1)^{n-j}(2i\sin((-\frac{\alpha}{2}-n+1)\pi))^{n-j}\frac{G(n-j+1)G(-\frac{\alpha}{2}+j+2)}{G(-\frac{\alpha}{2}-n+2j+2)}$$
$$(-1)^{(1-n)j}(e^{-\frac{i\pi(\alpha-2)}{2}})^j(2i\sin((\frac{\alpha}{2}-2j+n-1)\pi))^j\frac{G(j+1)G(n-j+\frac{\alpha}{2})}{G(\frac{\alpha}{2}+n-2j)}$$

$$c(n+1,\alpha)=\frac{2^{\alpha(n-j)+2(n-j)(j+1)+\frac{(n+1)\alpha}{2}}}{(2\pi i)^{n+1}}d_1^{n-j}d_2^{j+1}(-e^{\frac{i\pi\alpha}{2}})^{j+1}(-1)^{n-j}(2i\sin((-\frac{\alpha}{2}-n+2j+1)\pi))^{n-j}\frac{G(n-j+1)G(-\frac{\alpha}{2}+j+2)}{G(-\frac{\alpha}{2}-n+2j+2)}$$
$$(-1)^{-(j+1)n}(e^{-\frac{i\pi\alpha}{2}})^{j+1}(2i\sin((\frac{\alpha}{2}-2j+n-1)\pi))^{j+1}\frac{G(j+2)G(n-j+\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}+n-2j)}$$

$$c(n+1,\alpha-2)=\frac{2^{-(\alpha-2)(n-j+1)+2(n-j+1)j+\frac{(n+1)(\alpha-2)}{2}}}{(2\pi i)^{n+1}}d_1^{n-j+1}(-d_2)^j(-e^{\frac{i\pi(\alpha-2)}{2}})^j(-1)^{n-j+1}(2i\sin((-\frac{\alpha}{2}-n+2j)\pi))^{n-j+1}$$
$$\frac{G(n-j+2)G(-\frac{\alpha}{2}+j+2)}{G(-\frac{\alpha}{2}-n+2j+1)}(-1)^{-nj}(e^{-\frac{i\pi(\alpha-2)}{2}})^j(2i\sin((\frac{\alpha}{2}-2j+n)\pi))^j\frac{G(j+1)G(n-j+\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}+n-2j+1)}$$

It follows that:

$$q_n(x)\sim\frac{c(n+1,\alpha-2)c(n,\alpha)}{c(n+1,\alpha)c(n,\alpha-2)}x^{\alpha+2n-4j-1}$$

$$\sim 2^{4j-2n-\alpha+1}(-1)^n\left(\frac{d_1}{d_2}\right)\frac{(G(\frac{\alpha}{2}+n-2j))^2G(-\frac{\alpha}{2}+j+1)(G(-\frac{\alpha}{2}-n+2j+2))^2G(\frac{\alpha}{2}+n-j+1)G(n-j+2)G(j+1)}{(G(\frac{\alpha}{2}+n-2j+1))^2G(-\frac{\alpha}{2}+j+2)(G(-\frac{\alpha}{2}-n+2j+1))^2G(\frac{\alpha}{2}+n-j)G(n-j+1)G(j+2)}x^{\alpha+2n-4j-1}$$

$$\sim(-1)^n\left(\frac{d_1}{d_2}\right)\left[\left(\frac{\Gamma(-\frac{\alpha}{2}-n+2j+1)}{\Gamma(\frac{\alpha}{2}+n-2j)}\right)^2\frac{\Gamma(n-j+\frac{\alpha}{2})\Gamma(n-j+1)}{\Gamma(-\frac{\alpha}{2}+j+1)\Gamma(j+1)}\right]\left(\frac{x}{2}\right)^{\alpha+2n-4j-1}$$

If $-2n+4j-2<\alpha<-2n+4j$ and $j=1,2,\dots,n$, then we have:

$$c(n,\alpha)=\frac{2^{-\alpha(n-j)+2(n-j)j+\frac{n\alpha}{2}}}{(2\pi i)^n}d_1^{n-j}d_2^{n}(-e^{\frac{i\pi\alpha}{2}})^j(-1)^{n-j}(2i\sin((-\frac{\alpha}{2}-n+2j)\pi))^{n-j}\frac{G(n-j+1)G(-\frac{\alpha}{2}+j+1)}{G(-\frac{\alpha}{2}-n+2j+1)}$$
$$(-1)^{(1-n)j}(e^{-\frac{i\pi\alpha}{2}})^j(2i\sin((\frac{\alpha}{2}-2j+n)\pi))^j\frac{G(j+1)G(n-j+\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}+n-2j+1)}$$

$$c(n,\alpha-2)=\frac{2^{-(\alpha-2)(n-j+1)+2(n-j+1)(j-1)+\frac{n(\alpha-2)}{2}}}{(2\pi i)^n}d_1^{n-j+1}(-d_2)^{j-1}(-e^{\frac{i\pi(\alpha-2)}{2}})^{j-1}(-1)^{n-j+1}(2i\sin((-\frac{\alpha}{2}-n+2j-1)\pi))^{n-j+1}$$
$$\frac{G(n-j+2)G(-\frac{\alpha}{2}+j+1)}{G(-\frac{\alpha}{2}-n+2j)}(-1)^{(1-n)(j-1)}(e^{-\frac{i\pi(\alpha-2)}{2}})^{j-1}(2i\sin((\frac{\alpha}{2}-2j+n+1)\pi))^{j-1}\frac{G(j)G(n-j+\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}+n-2j+2)}$$

$$c(n+1,\alpha)=\frac{2^{\alpha(n-j+1)+2(n-j+1)j+\frac{(n+1)\alpha}{2}}}{(2\pi i)^{n+1}}d_1^{n-j+1}d_2^{j}(-e^{\frac{i\pi\alpha}{2}})^j(-1)^{n-j+1}(2i\sin((-\frac{\alpha}{2}-n+2j-2)\pi))^{n-j+1}\frac{G(n-j+2)G(-\frac{\alpha}{2}+j+1)}{G(-\frac{\alpha}{2}-n+2j)}$$
$$(-1)^{-jn}(e^{-\frac{i\pi\alpha}{2}})^j(2i\sin((\frac{\alpha}{2}-2j+n+2)\pi))^j\frac{G(j+1)G(n-j+\frac{\alpha}{2}+2)}{G(\frac{\alpha}{2}+n-2j+2)}$$

$$c(n+1,\alpha-2)=\frac{2^{-(\alpha-2)(n-j+1)+2(n-j+1)j+\frac{(n+1)(\alpha-2)}{2}}}{(2\pi i)^{n+1}}d_1^{n-j+1}(-d_2)^j(-e^{\frac{i\pi(\alpha-2)}{2}})^j(-1)^{n-j+1}(2i\sin((-\frac{\alpha}{2}-n+2j)\pi))^{n-j+1}$$
$$\frac{G(n-j+2)G(-\frac{\alpha}{2}+j+2)}{G(-\frac{\alpha}{2}-n+2j+1)}(-1)^{-nj}(e^{-\frac{i\pi(\alpha-2)}{2}})^j(2i\sin((\frac{\alpha}{2}-2j+n)\pi))^j\frac{G(j+1)G(n-j+\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}+n-2j+1)}$$

It follows that:

$$q_n(x) \sim \frac{c(n+1,\alpha-2)c(n,\alpha)}{c(n+1,\alpha)c(n,\alpha-2)} x^{-2n+4j-\alpha-1}$$

$$\sim 2^{2n-4j+\alpha+1}\left(\frac{d_2}{d_1}\right) \frac{(G(-\frac{\alpha}{2}-n+2j))^2 G(-\frac{\alpha}{2}+j+2)(G(-\frac{\alpha}{2}-n+2j+2))^2 G(\frac{\alpha}{2}+n-j+1)G(n-j+1)G(j+1)}{(G(\frac{\alpha}{2}+n-2j+1))^2 G(-\frac{\alpha}{2}+j+1)(G(-\frac{\alpha}{2}-n+2j+1))^2 G(\frac{\alpha}{2}+n-j+2)G(n-j+2)G(j)} x^{-2n+4j-\alpha-1}$$

$$\sim \left(\frac{d_2}{d_1}\right)\left[\left(\frac{\Gamma(\frac{\alpha}{2}+n-2j+1)}{\Gamma(-\frac{\alpha}{2}-n+2j)}\right)^2 \frac{\Gamma(-\frac{\alpha}{2}+j+1)\Gamma(j)}{\Gamma(n+\frac{\alpha}{2}-j+1)\Gamma(n-j+1)}\right]\left(\frac{2}{x}\right)^{2n-4j+\alpha+1}$$

If $\alpha < -2n$, then we have:

$$c(n,\alpha) = \frac{2^{-\frac{n\alpha}{2}}}{(2\pi i)^n} d_1^n (-e^{\frac{i\pi\alpha}{2}})^n (1)(-1)^n (2i\sin((-\frac{\alpha}{2}-n)\pi))^n \frac{G(n+1)G(-\frac{\alpha}{2}+1)}{G(-\frac{\alpha}{2}-n+1)}$$

$$c(n,\alpha-2) = \frac{2^{-\frac{n(\alpha-2)}{2}}}{(2\pi i)^n} d_1^n (-1)^n (1)(-1)^n (2i\sin((-\frac{\alpha}{2}+1-n)\pi))^n \frac{G(n+1)G(-\frac{\alpha}{2}+2)}{G(-\frac{\alpha}{2}-n+2)}$$

$$c(n+1,\alpha) = \frac{2^{-\frac{(n+1)\alpha}{2}}}{(2\pi i)^{n+1}} d_1^{n+1}(1)(-1)^{n+1}(2i\sin((-\frac{\alpha}{2}-n-2)\pi))^{n+1} \frac{G(n+2)G(-\frac{\alpha}{2}+1)}{G(-\frac{\alpha}{2}-n)}$$

$$c(n+1,\alpha-2) = \frac{2^{-\frac{(n+1)(\alpha-2)}{2}}}{(2\pi i)^{n+1}} d_1^{n+1}(1)(-1)^{n+1}(2i\sin((-\frac{\alpha}{2}-n-1)\pi))^{n+1} \frac{G(n+2)G(-\frac{\alpha}{2}+2)}{G(-\frac{\alpha}{2}-n+1)}$$

It follows that:

$$q_n(x) \sim \frac{c(n+1,\alpha-2)c(n,\alpha)}{c(n+1,\alpha)c(n,\alpha-2)} x^{-1}$$

$$\sim \frac{(-2)G(\frac{\alpha}{2}+2-n)G(-\frac{\alpha}{2}-n)}{G(-\frac{\alpha}{2}-n+1)G(\frac{-\alpha}{2}-n+1)} x^{-1}$$

$$\sim \frac{(-2)\Gamma(-\frac{\alpha}{2}-n+1)}{\Gamma(-\frac{\alpha}{2}-n)} x^{-1}$$

$$\sim -\frac{\alpha+2n}{x}$$

□

Remark: One can check the result by examine the base case when $n = 0$. By Theorem 3.4.1, we obtain the formulae of $q_0(x)$:

$$q_0(x) \sim \begin{cases} \frac{x}{2-\alpha} & \text{if } \alpha > 2 \\ \frac{d_1}{d_2} x^{\alpha-1} 2^{1-\alpha} \frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})} & \text{if } 0 < \alpha < 2 \\ -\frac{\alpha}{x} & \text{if } \alpha < 0 \end{cases}$$

On the other hand, it is known that the asymptotic behavior of J_ν as $x \to 0$ follows:

$$J_\nu(x) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu$$

By 36 and 27, we can directly compute $q_0(x)$ in the following way:

$$\begin{aligned}
q_0(x) &= \frac{\tau_1(\alpha-2,x)}{\tau_1(\alpha,x)} \\
&= \frac{d_1 J_{\frac{\alpha}{2}-1}(x) + d_2 J_{-\frac{\alpha}{2}+1}(x)}{d_1 J_{\frac{\alpha}{2}}(x) + d_2 J_{-\frac{\alpha}{2}}(x)} \\
&\sim \frac{d_1 \frac{1}{\Gamma(\frac{\alpha}{2})} \left(\frac{x}{2}\right)^{\frac{\alpha}{2}-1} + d_2 \frac{1}{\Gamma(2-\frac{\alpha}{2})} \left(\frac{x}{2}\right)^{-\frac{\alpha}{2}+1}}{d_1 \frac{1}{\Gamma(1+\frac{\alpha}{2})} \left(\frac{x}{2}\right)^{\frac{\alpha}{2}} + d_2 \frac{1}{\Gamma(1-\frac{\alpha}{2})} \left(\frac{x}{2}\right)^{-\frac{\alpha}{2}}} \\
&\sim \begin{cases}
\frac{d_2 \frac{1}{\Gamma(2-\frac{\alpha}{2})} \left(\frac{x}{2}\right)^{-\frac{\alpha}{2}+1}}{d_2 \frac{1}{\Gamma(1-\frac{\alpha}{2})} \left(\frac{x}{2}\right)^{-\frac{\alpha}{2}}} & \text{if } \alpha > 2 \\
\frac{d_1 \frac{1}{\Gamma(\frac{\alpha}{2})} \left(\frac{x}{2}\right)^{\frac{\alpha}{2}-1}}{d_2 \frac{1}{\Gamma(1-\frac{\alpha}{2})} \left(\frac{x}{2}\right)^{-\frac{\alpha}{2}}} & \text{if } 0 < \alpha < 2 \\
\frac{d_1 \frac{1}{\Gamma(\frac{\alpha}{2})} \left(\frac{x}{2}\right)^{\frac{\alpha}{2}-1}}{d_1 \frac{1}{\Gamma(1+\frac{\alpha}{2})} \left(\frac{x}{2}\right)^{\frac{\alpha}{2}}} & \text{if } \alpha < 0
\end{cases} \\
&\sim \begin{cases}
\frac{x}{2-\alpha} & \text{if } \alpha > 2 \\
\frac{d_1}{d_2} x^{\alpha-1} 2^{1-\alpha} \frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})} & \text{if } 0 < \alpha < 2 \\
-\frac{\alpha}{x} & \text{if } \alpha < 0
\end{cases}
\end{aligned}$$

Hence our result can recover the formulae of asymptotic of $q_0(x)$ derived directly from 36

3.5 Inspiration for future study

For the time limit, in this paper we can't touch all the related problems. There are still several interesting directions for the readers who are interested in such topic to consider. We list them below:

- consider the case $\varepsilon = -1$
- consider the large x asymptotics

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