Thurston Norm of 2 Generator, 1 Relator Groups

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Abstract

We discuss an algorithm for drawing the unit ball of the Thurston Norm on the second homology of an oriented 3-manifold. We explore which unit balls are obtainable through this algorithm and provide geometric interpretation. We also explore ways to enumerate the obtainable unit balls.

Introduction

We are interested in 3-manifolds M with fundamental groups that admit (2,1)-presentations, i.e.

$$\pi_1(M) = \langle x, y | r \rangle \tag{1}$$

We usually assume that these manifolds have two-dimensional first homology groups ($b_1(M) = 2$). These properties allow us to apply the results of [4] in order to provide an algorithm for drawing the unit ball of the Thurston norm.

The Thurston norm generalizes the *genus* of a knot $K \subset S^3$. Recall that every knot K bounds an embedded surface in the knot complement $S^3 - K$, the *Siefert surface* $S(K)$. Of course, if a knot bounds a single surface, it in fact bounds arbitrarily many surfaces of different genus. The knot genus records the *simplest* possible Siefert surface, i.e. the Siefert surface of minimal genus. For example, the unknot is the only knot with genus 0.

The Siefert surface represents a class in the homology group $H_2(S^3 - K, \partial(S^3 - K))$. In fact this same procedure works for any 3-manifold M. Given any element ϕ in $H_2(M, \partial M)$, let

$$x_M([\phi]) := \min\{-\chi(S) | S \subset M \text{ and } [S] = \phi\}$$

Thurston showed that $x_M(\phi)$ is a seminorm on $H_2(M, \partial M, \mathbb{R})$. By Poincare duality, this also gives a norm on $H^1(M, \mathbb{R})$. The Thurston Norm measures the 'simplicity' of surfaces representing homology classes, and can tell us a lot about the geometry of a 3-manifold.

Friedl and Tillmann [4] provided a way to assign to a group G admitting 2-generator, 1-relator presentation with non-empty, cyclically reduced relator and $b_1 = 2$ a marked polytope in $H_1(G, \mathbb{Z})$ that contains information about the group G. Polytopes in the first homology induce a norm on the first cohomology, which determines the Thurston norm in the second homology with Poincare Duality. We provide an algorithm for obtaining information about the Thurston norm of a 3-manifold obtained by gluing a 2-handle along a separating simple closed curve on the surface a genus 2 handlebody. Given this construction of the manifold, the algorithm thus applies to any manifold with a fundamental group as in 1.

We aim to determine which polygons can be obtained through this algorithm, particularly whether all polygons can be obtained and how (i.e. to which manifold they correspond).

1. Background

The three-manifolds we construct are obtained by gluing a 2-handle along a separating simple closed curve on the surface of a genus 2 handlebody, i.e. identifying four consecutive faces of a unit cube along an annular neighborhood of the separating simple closed curve. The property that the curve is separating guarantees that $b_1 = 2$ (see Section 1.2). To generate every simple closed curve on $\mathbb{T}^2 \# \mathbb{T}^2$, the surface of the genus two handlebody, we use the mapping class group (defined below) of $\mathbb{T}^2 \# \mathbb{T}^2$ to obtain every separating simple closed curve from transformations of the separating simple closed curve $[x, z] = xzx^{-1}z^{-1}$ depicted on the top-left corner of Figure 5.

1.1. "Nice" way of generating Mod(S)

Definition 1.1.1. *Let S be a surface and c a curve on S, and let U be an annular neighborhood of c on S. Then U is homeomorphic to a cylinder. Let $\phi : U \to S^1 \times I$ be this homeomorphism. Define a map $f : S^1 \times I \to S^1 \times I$ with $f : (e^{i\theta}, t) \mapsto (e^{i(\theta + 2\pi t)}, t)$. The **Right Dehn Twist** about c, denoted T_c, is a self-homeomorphism of S given by $T_c = \phi^{-1} \circ f \circ \phi$ on U and $T_c = id$ everywhere else on S. The **Left Dehn Twist** about c is T_c^{-1}.*

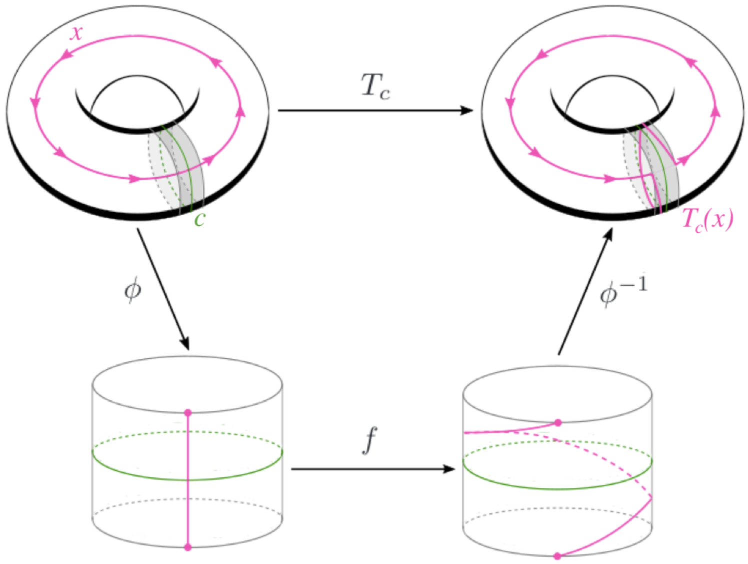


Figure 1: Right Dehn Twist about c on a Torus

For a surface $S_{g,p}$ with genus g and p punctures, let $\mathrm{Hom}^+(g,p)$ denote the group of all orientation preserving homeomorphisms of $S_{g,p}$ that fix the set of punctures, and let $\mathrm{Hom}_0(g,p)$ be the path-component of the indentity in $\mathrm{Hom}^+(g,p)$, i.e. all elements of $\mathrm{Hom}^+(g,p)$ homotopy equivalent to the identity map. We can now define the Mapping Class Group (MCG) of $S_{g,p}$.

Definition 1.1.2. *The* ***Mapping Class Group*** *of $S_{g,p}$ is the group*

$$\mathrm{Mod}(S_{g,p}) = \mathrm{Hom}^+(g,p)/\mathrm{Hom}_0(g,p)$$

of homotopy classes of orientation preserving homeomorphisms of $S_{g,p}$ that fix the set of punctures.

Fact 1.1.1. (Humphries 1979) *For a surface $S_{g,0}$ of genus g, $2g+1$ Dehn Twists generate* $\mathrm{Mod}(S_{g,0})$.

We are concerned with generating the mapping class group $\mathrm{Mod}(\mathbb{T}^2\#\mathbb{T}^2)$ of a genus 2 torus with no punctures. The $2g+1=5$ Dehn Twists that generate $\mathrm{Mod}(\mathbb{T}^2\#\mathbb{T}^2)$ are T_a, T_b, T_c, T_d, T_e for curves a, b, c, d, e as in Figure 2.

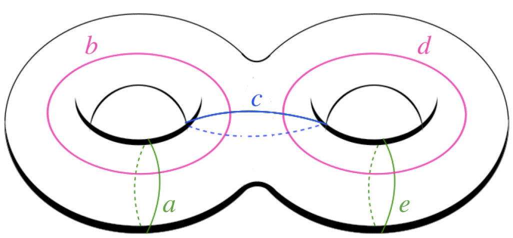


Figure 2: Dehn Twists about a, b, c, d, e generate $\mathrm{Mod}(\mathbb{T}^2\#\mathbb{T}^2)$

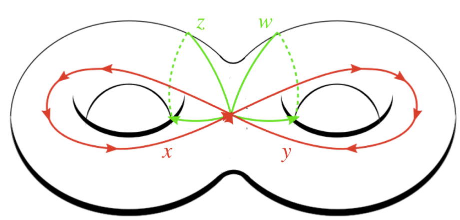


Figure 3: Fundamental group of $\mathbb{T}^2\#\mathbb{T}^2$

Fact 1.1.2. *T_a, T_b, T_c, T_d, T_e are given by the following formulae and are the identity elsewhere, where w, x, y, z are the generators of $\pi_1(\mathbb{T}^2\#\mathbb{T}^2)$ as in Figure 3.*

$$
\begin{array}{ll}
T_a : x \mapsto z^{-1}x & T_a^{-1} : x \mapsto zx \\
T_b : z \mapsto xz & T_b^{-1} : z \mapsto x^{-1}x \\
T_c : x \mapsto xwz^{-1} & T_c^{-1} : x \mapsto xzw^{-1} \\
\quad\;\; y \mapsto ywz^{-1} & \qquad\;\; y \mapsto yzw^{-1} \\
T_d : w \mapsto y^{-1}w & T_d^{-1} : w \mapsto yw \\
T_e : y \mapsto wy & T_e^{-1} : y \mapsto w^{-1}y
\end{array}
$$

1.2. The Thurston Norm

By an **n-cell** we will mean a space homeomorphic to the open n-ball B_n.

Definition 1.2.1. *A **cell complex** or **CW-complex** is a Hausdorff space X along with a partition P of X into n-cells such that for each n-cell c_n there exists a continuous map $f : B_n \to X$, called the **attaching map**, such that ${B_n}^{\circ}$ maps homeomorphically onto c_n and $f(\partial B_n) \subseteq \bigcup_1^m c$ for $c \in P$ a k-cell with $k < n$.*

Definition 1.2.2. *The n^{th}**-chain group with real coefficients**, denoted $C_n(X)$, of a CW-complex X is the real vector space with the n-cells of X as a basis. The n^{th}**-boundary map** $\partial_n : C_n(X) \to C_{n-1}(X)$ of a CW-complex X is the linear map that takes each n-cell C in the basis of c_n to the sum (up to sign) of the basis elements corresponding to the (n-1)-cells in ∂C.*

Definition 1.2.3. *The i^{th}**-homology group with real coefficients** of a CW-complex X is the vector space*

$$H_i(X) = \ker \partial_i / im\ \partial_{i+1}$$

Fact 1.2.1. *$H_i(X)$ is a topological invariant of the space X.*

Remark. *There are analagously defined homology groups with coefficients in any ring R. In particular, the integral homology groups are denoted $H_i(X, \mathbb{Z})$.*

From now on some definitions will be provided in a specific form that is as general as is needed for the scope of this project.

Definition 1.2.4. *Let S be a surface and P a partition of S as a CW-complex. Let V denote the number of 0-cells or vertices in P, E denote the number of 1-cells or edges in P, and F denote the number of 2-cells or faces in P. The **Euler characteristic** of S, denoted $\chi(S)$ is*

$$\chi(S) := V - E + F$$

The Euler characteristic is a topological invariant of S and does not depend on P. The Euler characteristic of surfaces with equal genus and boundary components is the same and in fact, $\chi(S_{g,p})$ can be defined in terms of g and p as follows

$$\chi(S_{g,p}) = 2 - 2g - p$$

Definition 1.2.5. *The n^{th}**-cochain group** of a space X is defined as*

$$C^n(X) := \mathrm{Hom}(C_n(X), R)$$

for a ring R. An element of $C^n(X, R)$ is called a n-cochain.

The boundary maps of the chains of X naturally induce *coboundary maps* between the cochains of X in the same way that linear maps between vector spaces induce dual maps between dual vector spaces. Thus, if the boundary map ∂_i is represented by a matrix $\mathbf{A}$, the corresponding co-boundary map δ^i is represented by the transpose $\mathbf{A}^{\intercal}$. Just as we defined homology, we can define cohomology in exactly the way one would expect

Definition 1.2.6. *The* ***ith cohomology*** *of a space X is*

$$H^i(X) = \ker \delta^i / im\ \delta^{i-1} = \mathrm{Hom}(H_1(X), F)$$

for some field F.

Definition 1.2.7. *Suppose $A \subset X$ is a CW-subcomplex of X. The relative chain groups $C_n(X, A)$ are generated by the n-cells in $X - A$. There are relative homology groups $H_i(X, A)$ defined analagously as for homology groups. A relative cycle can be thought of as a chain of cells in $X - A$ such that the boundary of this chain is contained in A. Relative cohomology groups are defined similarly.*

Theorem 1.2.1 (Poincarè duality). *Suppose M is a finite CW-complex which is also an orientable n-manifold. Then exists an isomorphism ψ_i between C_i and $C^{n-i}(M, \partial M)$ $\forall i \leq n$ which induces an isomorphism $H_i(M) \cong H^{n-i}(M, \partial M)$. More generally, this is true for any coefficient ring R.*

Note that any closed orientable n-manifold admits a canonical element in $H_n(M, \mathbb{Z})$. This is usually called the *fundamental class*, and can be visualized as the chain which assigns the same constant K to each n-cell. Note that any $(n-1)$-cell is contained in exactly two n-cells, so by choosing orientations correctly this is an n-cycle. A similar idea works to show that if M has boundary there is a canonical element in $H_n(M, \partial M)$. In low dimensions it turns out that all the homology groups can be thought of as fundamental classes.

Fact 1.2.2. *Let M be a 3-manifold. Then every element in $H_2(M, \partial M)$ is represented by the fundamental class of an immersed surface S.*

Now, by definition and by 1.2.1 we know that for a 3-manifold M

$$\begin{aligned} \mathrm{Hom}(H_1(M), \mathbb{R}) = &H^1(M) \cong H_2(M, \partial M) \\ &H_1(M) \cong H^2(M, \partial M) \end{aligned}$$

For each loop c_i in the basis of $H_1(M)$, the corresponding basis element ϕ_i in $H^1(M)$ is such that

$$\phi_i(c_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

and each c_i can be uniquely assigned an element of $H_2(M, \partial M)$ represented by a surface S_i.

1.3. Previous relevant results

We introduce some previous results that are relevant to our work.

Definition 1.3.1 (Friedl and Tillmann). *A (2,1)-presentation $\langle x, y \mid r \rangle$ for a group G is said to be a* ***nice presentation*** *if r is a nonempty, cyclically reduced word and if $b_1(G) = 2$.*

Lemma 1.3.1 (Friedl and Tillmann). *Let G be a group admitting a nice (2,1)-presentation and $\phi : G \to \mathbb{Z}$ an epimorphism. There exists a nice (2,1)-presentation $\langle x, y \mid r \rangle$ for G with $\phi(x) = 0$, $\phi(y) = 1$ that gives rise to the same polytope as the original presentation.*

A *fibered* 3-manifold M admits the structure of a fiber bundle over S^1, i.e. there is a bundle

$$S \to M \to S^1$$

where S is some surface. A cohomology class in $H^1(M)$ is fibered if it is the pullback of the fundamental class of S^1.

Theorem 1.3.2 (Thurston). *Let N be a 3–manifold. There exists a unique symmetric marked polytope $\mathcal{P}_N$ in $H_1(N;\mathbb{R})$ such that for any $\phi \in H^1(N;\mathbb{R}) = \mathrm{Hom}(\pi_1(N),\mathbb{R})$ we have*

$$x_N(\phi) = \max\{\phi(p) - \phi(q) \,|\, p, q \in \mathcal{P}_N\}$$

Furthermore, ϕ is fibered if and only if it pairs maximally with a marked vertex.

2. The Algorithm

We start with a separating simple closed curve on the surface of a genus 2 handlebody. Particularly, we start with the curve $\gamma_0 = xzx^{-1}z^{-1}$. This curve is chosen so as to guarantee $b_1 = 2$ in the resulting manifold, and details on why this holds can be found in Section 2.3. By Fact 1.1.2 we know a series of Dehn Twists applied to γ_0 can generate any separating simple closed curve, along which we can then paste a 2-handle to get a 3-manifold. The inputs to the algorithm are then n-words in T_a, T_b, T_c, T_d, T_e for any n, which encode any possible 3-manifold admitting a nice $(2,1)$-presentation for its fundamental group. Denoting by γ_k the word in w, x, y, z after the first k Dehn Twist applications to γ_0, the relator for the fundamental group of the resulting 3-manifold is obtained by simply dropping the w's and z's from γ_n (because they are non-trivial in $\pi_1(\mathbb{T}^2\#\mathbb{T}^2)$ but trivial in π_1 of the handlebody) and cyclically reducing the resulting word in x, y. We will call $r(\gamma_n)$ the cyclically reduced relator corresponding to γ_n and $W_T(x,y)$ (a word in xy) the equivalent of γ_n in the fundamental group of M_T (i.e. the result of "dropping" w, z from γ_n).

We will let T be the input word in T_a, T_b, T_c, T_d, T_e and their inverses, n be the length of T, so that $\gamma_n = T(\gamma_0)$. M_T will denote the manifold resulting from pasting a 2-handle along γ_n, $\pi_1(M_T) = \langle x, y \mid r\rangle$ with r being the xy components of γ_n. We will use $\pi_1(M_T)$ to construct a marked polytope in $H_1(M_T)$ as follows [4] and as shown in Figure 4:

1. Trace out a walk on the plane by reading $r(\gamma_n)$ from left to right, starting at $(0,0)$ and adding $(\epsilon, 0)$ for any x^ϵ and $(0,\epsilon)$ for any y^ϵ. Mark all points in the walk that were crossed only once.

2. Take the convex hull C of these points, preserving the markings of vertex points and let V_C denote its set of vertices.

3. For $n, m \in \mathbb{Z}$, place a point on $(n+\frac{1}{2}, m+\frac{1}{2})$ whenever each of the points in $S_{m,n} = \{(m,n),(m+1,n),(m,n+1),(m+1,n+1)\}$ is either inside C or in V_C. Mark $(n+\frac{1}{2}, m+\frac{1}{2})$ if $S \cap V_C$ is fully marked. Let the set of all such $(n+\frac{1}{2}, m+\frac{1}{2})$ be K.

4. Take the convex hull of K, preserving markings. The resulting polygon P_T is the output polygon.

2.1. Meaning of $\mathcal{P}_T$ and deriving the Thurston unit ball

The algorithm described in the previous subsection determines the Thurston norm by the following theorem.

Theorem 2.1.1. *[3] Let M be an irreducible 3-manifold that admits a cyclically reduced $(2,1)$-presentation $\pi = \langle x, y \,|\, r\rangle$. Let P_T be constructed as in the algorithm above. Then*

$$\mathcal{P}_N \doteq \mathcal{P}_T$$

where P_N is as in Theorem 1.3.2 and $\doteq$ means equal up to translation.

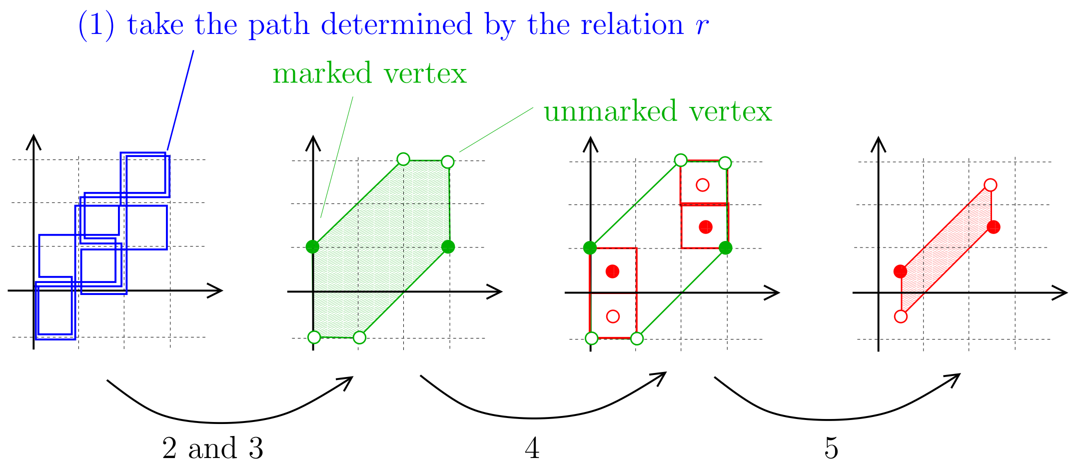


Figure 4: [4] Construction of P_T from $\pi_1(M)$

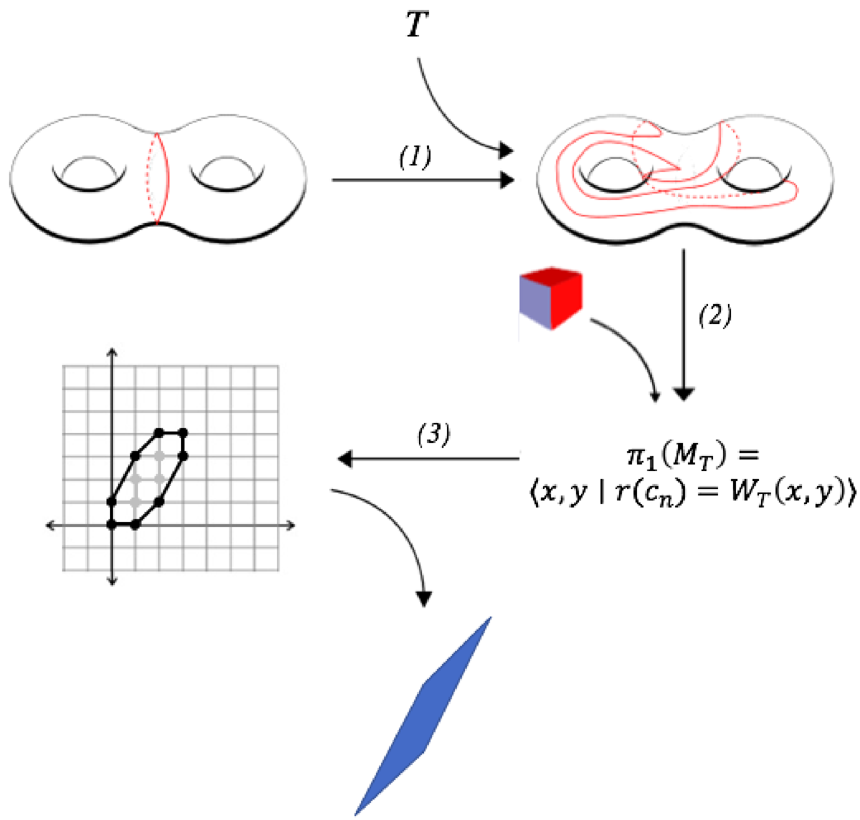


Figure 5: The Algorithm: (1) Apply T to $\gamma_0 = [x, z]$ to get γ_n (2) Paste a 2-handle along γ_n to get M_T (3) Construct the polytope associated to $r(\gamma_n)$, the relator of $\pi_1(M_T)$.

Therefore, P_T is the polynomial dual to the unit ball of the Thurston norm and it encodes exactly the same information. Let $\phi \in H^1(M_T)$ then $\ker\phi \subset H_1(M_T)$ has dimension one and is a line on our superposition of H_1 on the plane. Let $(\ker\phi)^\perp$ denote the orthogonal complement of $\ker\phi$. The unit norm of ϕ is then "the thickness of P in the direction of $(\ker\phi)^\perp$ as measured by ϕ". That is,

$$x_M(\phi) = \max\{\phi(p) - \phi(q) \mid p, q \in (\ker\phi)^\perp \cap P\}$$

as shown in Figure 6.

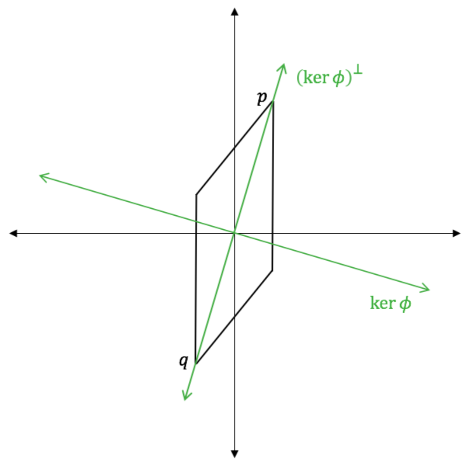
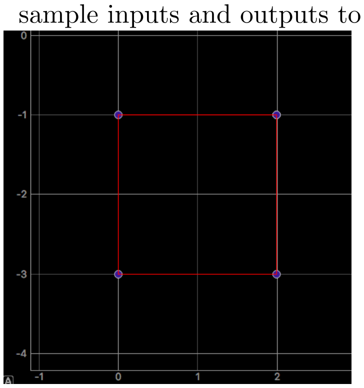


Figure 6: $x_M(\phi) = \phi(p) - \phi(q)$

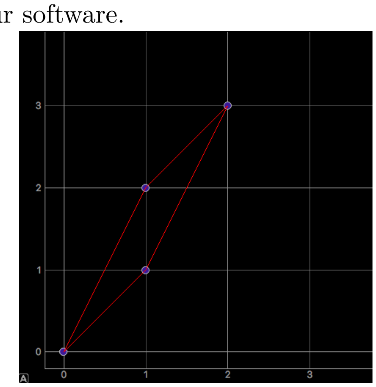
2.2. Sample Outputs

We implemented this algorithm in Python and the pyqtgraph graphics library, and below are some sample inputs and outputs to our software.



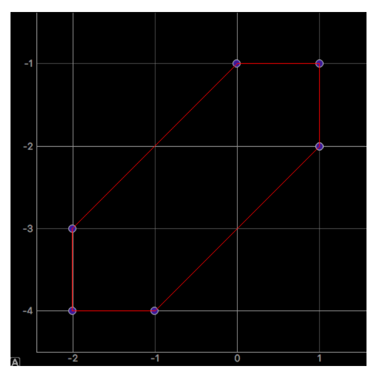
$T = T_d^3 T_b^3 T_c$

$\pi_1(M_T) = \langle x, y \mid y^{-3}x^3y^3x^{-3}\rangle$



$T = T_b^{-1}T_d^{-1}T_cT_d^{-1}T_c$

$\langle x, y \mid xyxy^2xyx^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^{-1}\rangle$



$T = T_d^2T_b^2T_c^2$

$\langle x, y \mid y^{-2}x^{-2}y^{-2}x^2y^2x^2y^2x^{-2}\rangle$

2.3. $b_1 = 1$ case

One can easily tell whether $b_1 = 1$ from the relator fo the fundamental group. Let $r = x^{\epsilon_1}y^{\delta_1}\ldots x^{\epsilon_k}y^{\delta_k}$ then if $\sum_{i=1}^k \epsilon_1 = 0$ and $\sum_{i=1}^k \delta_1 = 0$, r represents a separating closed curve and bounds two genus 1 surfaces, making it trivial in the homology. If on the other hand $\sum_{i=1}^k \epsilon_1 \neq 0$ or $\sum_{i=1}^k \delta_1 \neq 0$ then $b_1 = 1$.

Assume $\pi = \langle x, y \mid r\rangle$ is the fundamental group of M and $b_1 = 1$. We apply Lemma 1.3.1 to get a presentationn $\pi = \langle x, y \mid r\rangle$ such that $\phi(x) = 0$, $\phi(y) = 1$ and draw the corresponding polytope as in the $b_1 = 2$ case. When calculating Thurston norms as in Section 2.1, the fact that $\phi(y) = 0$ just means we can "ignore" the horizontal direction and only care about the vertical thickness of our polytope. Perhaps this will become clearer by example, refer to Appendix A.2.

3. Results

We now show the existence of some polygons that can appear as dual polygons to the unit ball of the Thurston norm of certain 3-manifolds. We say a polygon P is *obtainable* if there is a 3-manifold M with $\pi_1(M) = \langle x, y|r\rangle$ and $P_M \doteq P$.

Lemma 3.0.1. *Every fully marked rectangle is obtainable.*

Proof. We first prove inductively that

$$T_d^{-n}T_b^mT_c(xzx^{-1}z^{-1}) = xy^nwz^{-1}x^{-m}x^mzx^mzw^{-1}y^{-n}x^{-1}z^{-1}x^{-m} \tag{2}$$

with T_d, T_b, T_c as in Fact 1.1.2. Equation 2 clearly holds for $n = m = 1$. Inducting on n,

$$\begin{aligned} T_d^{-(n+1)}T_b^mT_c(xzx^{-1}z^{-1}) &= T_d(xy^nwz^{-1}x^{-m}x^mzx^mzw^{-1}y^{-n}x^{-1}z^{-1}x^{-m}) \\ &= xy^n\ T_d(w)\ z^{-1}x^{-m}x^mzx^mz\ T_d(w^{-1})\ y^{-n}x^{-1}z^{-1}x^{-m} \\ &= xy^{n+1}wz^{-1}x^{-m}x^mzx^mzw^{-1}y^{-(n+1)}x^{-1}z^{-1}x^{-m} \end{aligned}$$

as desired. Next, inducting on m,

$$\begin{aligned} T_d^{-n}T_b^{m+1}c(xzx^{-1}z^{-1}) &= T_b(T_d^{-n}T_b^mT_c(xzx^{-1}z^{-1})) \\ &= xy^nw\ T_b(z^{-1})\ x^{-m}x^m\ T_b(z)\ x^m\ T_b(z)\ w^{-1}y^{-n}x^{-1}\ T_b(z^{-1})\ x^{-m} \\ &= xy^nwz^{-1}x^{-(m+1)}x^{m+1}zx^{m+1}zw^{-1}y^{-n}x^{-1}z^{-1}x^{-(m+1)} \end{aligned}$$

as desired, where the first equality is true since b and d clearly commute.

Now, we claim that $T = T_d^{-(h+1)}T_b^{w+1}T_c$ generates a rectangle of height h and width w with edges parallel to the axes. By 2 we know that the fundamental group of M_T has relator $r = y^{h+1}x^{-(w+1)}x^{w+1}x^{w+1}y^{-(h+1)}x^{-(w+1)}$ which cyclically reduces to

$$r = y^{h+1}x^{w+1}y^{-(h+1)}y^{-(w+1)}$$

it is easy to see that the walk obtained from this relator never crosses the same point more than one, thus it is fully marked, and that the convex hull C of the walk is a rectangle of width $w + 1$ and height $h + 1$ and that the resulting polygon P is thus a fully marked rectangle of width w and height h. ☐

Corollary 3.0.1.1. *Every fully marked interval on the x or y axes is obtainable.*

Proof. Intervals are rectangles of height or width zero. Vertical intervals of length m can be obtained by $T = T_d^{-(m+1)}T_bT_c$ and horizontal intervals of length m can be obtained by $T = T_d^{-1}T_b^{m+1}T_c$. ☐

Lemma 3.0.2. *Given a Polygon P, the polygon P' obtained by reflecting P about the x or y axis is obtainable.*

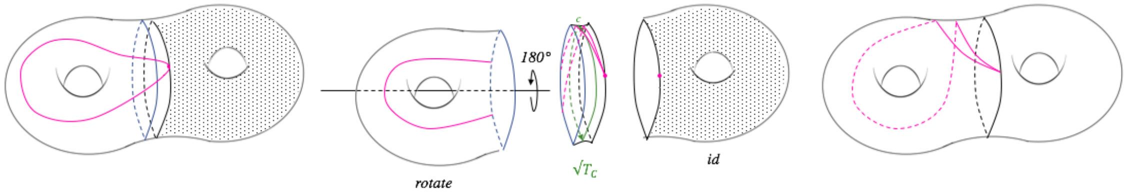


Figure 7: In order to mirror a polygon along the y axis, we want an automorphism of $\mathbb{T}^2\#\mathbb{T}^2$–that we can append to the input word T–which sends x to x^{-1} but keeps y fixed. Pictured above is one such map. It keeps the right side (the y side) of the genus 2 torus fixed, and rotates the left side (the x side) by 180°, applying a half Dehn twist in the middle annulus around c so that the boundaries of each shape will agree when "pasting" the torus back together.

Proof. Let T denote a word in T_a, T_b, T_c, T_d, T_e such that the polygon corresponding to M_T is P and let c_n denote the word in w, x, y, z such that $T(xzx^{-1}z^{-1}) = c_n$. We define a "mirroring" map (Figure 7) as follows

$$\begin{aligned}(T_b \circ T_a)^3 : & x \mapsto z^{-1}x^{-1} \\ & z \mapsto z^{-1}[x,^{-1}]z^{-1}\end{aligned}$$

Then, by taking $(T_b \circ T_a)^3(c_n)$, each z gets replaced by $z^{-1}[x^{-1}, z^{-1}]$ which is also trivial in $\pi_1(M_T)$ and does not affect W_T. Each x gets replaced by $z^{-1}x^{-1}z$ which is x^{-1} in $\pi_1(M_T)$ and thus $W_{(T_b\circ T_a)^3\circ T}(x, y) = W_T(x^{-1}, y)$. Then, the polytope P' arising from $(T_b \circ T_a)^3 \circ T$ is the mirror image along the y-axis of P. The proof for mirroring along the x axis is analogous but uses T_d, T_e instead. □

LEMMA 3.0.3. *Let n denote the maximum horizontal distance between two points at any step of the walk and m denote the maximum vertical distance. If $n + m \geq 2k + 4$ then the marked polygon corresponding to this fundamental group cannot be contained within a k by k box in H_1*

Proof. Notice that all the angles in the convex hull C of a walk on the integer lattice will be $\geq 90°$, since the cyclically reduced condition on r implies every x^{ϵ_1} is followed by either an x^{ϵ_1} or a y^{ϵ_2} for $\epsilon_1, \epsilon_2 \in \{0, 1\}$. Then, $\forall (x, y) \in V_C, \exists (x', y') \in P_T$ such that $\mid x - x' \mid= 0.5$ and $\mid y - y' \mid= 0.5$.

Then, if there are two points $(x_0, y_0), (x_1, y_1) \in V_C$ with $x_1 - x_0 \geq k+2$ then there are two points $((x'_0, y'_0), (x'_1, y'_1) \in P_T$ such that $x'_1 - x'_0 \geq k + 1$. Similarly if $y_1 - y_0 \geq k + 2$ then $y'_1 - y'_0 \geq k + 1$.

Let $(x_0, y_0), (x_0 + n, y'_0) \in V_C$ maximize horizontal distance and $(x_1, y_1), (x'_1, y_1 + m) \in V_C$ maximize vertical distance, then if $m + n \geq 2k + 4$, either $n \geq k + 2$ or $m \geq k + 2$ or both. □

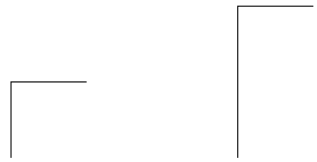
3.1. Complicated Thurston Norms

LEMMA 3.1.1. *For all $i \in \mathbb{Z}^+$, $T = T_b^{-1}(T_d^{-1}T_c)^{i+1}$ gives rise to a polytope P_T with $4i$ vertices.*

Proof. Denote $T_i = T_b^{-1}(T_d^{-1}T_c)^{i+1}$ and notice that $T_{i+1} = T_sT_i$ where $T_s = T_b^{-1}T_d^{-1}T_cT_b$. We will also define two related sequences of words on the fundamental group of $\mathbb{T}^2\#\mathbb{T}^2$

$$A_1 = ywz^{-1}x \qquad A_{n+1} = B_nA_n \tag{3}$$

$$B_1 = y^2wz^{-1}x \qquad B_{n+1} = B_nB_nA_n \tag{4}$$



A_1 B_1

Figure 8: Each A_n, B_n block determines a walk in the plane rooted at the endpoint of the block that precedes it in the word

By Fact 1.1.2, it is easy to see that

$$T_s(A_1) = y^2wz^{-1}x\ ywz^{-1}x\ x^{-1}zw^{-1}y^{-1}x^{-1}x\ ywz^{-1}x = y^2wz^{-1}x\ ywz^{-1}x = B_1A_1 = A_2$$
$$T_s(B_1) = y^2wz^{-1}x\ y^2wz^{-1}x\ ywz^{-1}x\ x^{-1}zw^{-1}y^{-1}x^{-1}x\ ywz^{-1}x = B_1B_1A_1 = B_2$$

Additionally, $T_s(A_k) = A_{k+1}$ and $T_s(B_k) = B_{k+1} \implies T_s(A_{k+1}) = T_s(B_kA_k) = T_s(B_k)T_s(A_k) = B_{k+1}A_{k+1} = A_{k+2}$ and $T_s(B_{k+1}) = T_s(B_k)T_s(B_k)T_s(A_k) = B_{k+1}B_{k+1}A_{k+1} = B_{k+2}$ thus by induction

$$A_{n+1} = T_s(A_n),\ B_{n+1} = T_s(B_n) \tag{5}$$

Furthermore, each A_n, B_n is a word in w, x, y, z which in turn represents a walk on the plane. We can obtain the height and width of said walk by noting $w(A_1) = 1 = f_0, h(A_1) = 1 = f_1, w(B_1) = 1 = f_1, h(B_1) = 2 = f_2$ and $w(A_{i+1}) = w(B_i) + w(A_i)$, $w(B_{i+1}) = w(B_i) + w(B_i) + w(A_i)$ and thus

$$\begin{aligned} w(A_n) &= f_{2n-2} \\ h(A_n) &= f_{2n-1} \\ w(B_n) &= f_{2n-1} \\ h(B_n) &= f_{2n} \end{aligned}$$

where f_k denotes the kth number in the sequence $f_0 = 1, f_1 = 1, f_k = f_{k-1} + f_{k-2}$, $w(W)$ denotes the width of the walk on the plane corresponding to the word W and $h(W)$ denotes the height of the walk on the plane corresponding to a the word W.

We claim that

$$T_i([x, z]) = xA_1A_2 \dots A_iB_iB_{i-1} \dots B_1yw\sigma_i \tag{6}$$

where $\sigma_i = T_i(x^{-1}z^{-1})$ is the walk symmetric to $xA_1A_2 \dots A_iB_iB_{i-1} \dots B_1yw = T_i(xz)$. This is clearly true for $i = 1$ and follows inductively from 5 and the fact that $T_s(x) = xA_1, T_s(yw) = B_1yw$.

Now let $A : (1, 0), (f_0, f_1), (f_2, f_3), \dots (f_{2i-2}, f_{2i-1})$ be a sequence of $i + 1$ elements of $\mathbb{Z}^2$ and $B : (f_{2i-1}, f_{2i}), (f_{2i-3}, f_{2i-2}), \dots (f_1, f_2), (0, 1)$ be another such sequence, and let (AB), the sequence of length $2i + 2$ in $\mathbb{Z}^2$ obtained from appending B to A, define a sequence of points as follows:

$$p_0 = (AB)_0 = (1, 0) \qquad p_n = p_{n-1} + (AB)_n \tag{7}$$

and define a sequence of segments $\{l_n\}_n$ by the sequence of line segments between consecutive points in $\{p_n\}_n$

$$l_0 = \overline{(0, 0), p_0} \qquad l_n = \overline{p_{n-1}p_n} \tag{8}$$

These lines have slopes $0, \frac{f_1}{f_0}, \frac{f_3}{f_2}, \dots, \frac{f_{2i-1}}{f_{2i-2}}, \frac{f_{2i}}{f_{2i-1}}, \frac{f_{2i-2}}{f_{2i-3}}, \dots, \frac{f_2}{f_1}, vertical$.

We want to prove that $\{p_n\}_n$ lie on $T_i(xz) = xA_1A_2 \dots A_iB_iB_{i-1} \dots B_1yw$ and that they bound $T_i(xz)$ on the right. Then the point sequence $\{\overline{p}_n\}_n$ where $\overline{p}_i$ is symmetric to p_i lies on σ_i and bounds it to the left.

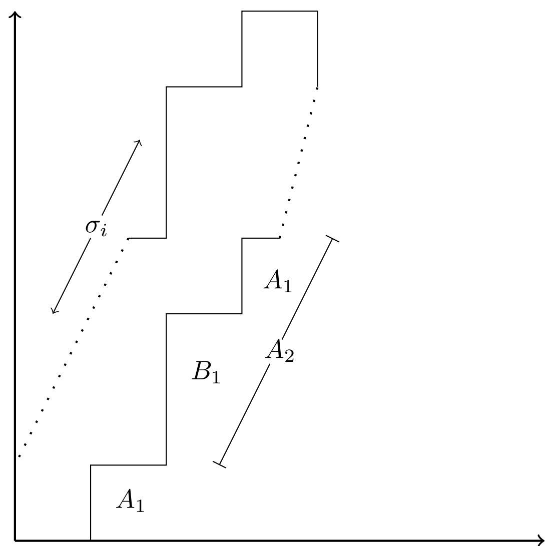


Figure 9: $T_i([x,z]) = xA_1A_2\dots A_iB_iB_{i-1}\dots B_1yw\sigma_i$

It is clear that p_0 lies on $T_i(xz) = x\dots$, and $p_1 = p_0 + (f_0, f_1)$ as well because A_1 has width f_0 and height f_1. Similarly, $\forall k \le i$, if p_{k-1} is the last point in $xA_1\dots A_{k-1}$ then p_k is clearly the last point in $xA_1\dots A_{k-1}A_k$ because $(w(A_k), h(A_k)) = (f_{2n-2}, f_{2n-1}) = (AB)_k$. Thus $p_0, p_1, \dots, p_i$ lie on $T_i(xz)$.

Now, p_{i+1} lies at the end of the path $xA_1A_2\dots A_iB_i$ because p_i lies at the end of the subpath $xA_1A_2\dots A_i$ and $p_{i+1} = p_i + (AB)_{i+1} = p_i + (w(B_i), h(B_i))$. $\forall k \mid i+2 \le k \le 2i$ if p_{k-1} lies at the end of path $xA_1A_2\dots A_iB_i\dots B_{2i+1-(k-1)}$ then p_k lies at the end of path $xA_1A_2\dots A_iB_i\dots B_{2i+1-(k-1)}B_{2i+1-k}$ since $p_k = p_{k-1}+(AB)_k = p_{k-1}+(w(B_{2i+1-k}), h(B_{2i+1-k}))$. Lastly $p_{2i+1} = p_{2i} + (0,1)$ clearly lies in $xA_1A_2\dots A_iB_iB_{i-1}\dots B_1yw$ since p_{2i} is the endpoint of $xA_1A_2\dots A_iB_iB_{i-1}\dots B_1$.

On a block A_k, it is easy to see that line connecting the endpoints q_0, q_1 of A_k bounds all points in A_k to the right. Taking a point q in $A_k = B_{k-1}A_{k-1}$ but not at the endpoints, q is either the junction point of $A_k = B_{k-1}, A_{k-1}$ or it is not. If it is then $q_0 = (x,y) \implies q = (x+w(B_{k-1}), y+h(B_{k-1})) = (x+f_{2k-3}, y+f_{2k-2})$ and $q_1 = (x+w(A_k), y+h(A_k)) = (x+f_{2k-2}, y+f_{2k-1})$ and you can check that q_0, q, q_1 are oriented clockwise. If q is not the junction point of $A_k = B_{k-1}, A_{k-1}$, let q_2 be this junction point and apply a simiplar argument to q_0, q, q_2 or q_2, q, q_1 as appropiate. A similar argument shows that a line on the endpoints of a B_k block bounds all points in B_k to the right. Thus $\{l_n\}_n$ bounds $T_i(xz)$ to the right. Furthermore, the sequence of lines $\{l_n\}_n$ is of increasing slope (this follows from the identities $f_{k+1}f_{k+2}f_kf_{k+3} = (-1)^{k+1}$ and $f_k^2 - f_{k-1}f_{k+1} = (-1)^k$ which can easily be proved inductively).

Then $\{\overline{p}_n\}_n$ lies on σ_i and bounds σ_i to the left as desired, thus the set of vertices of the convex hull of $T_i([x,])$ is $V_C = \{p_n\} \cup \{\overline{p}_n\}$ (because they bound $T_i([x,])$ and are convex by our slope argument) which has $4i+4$ elements. Then the resulting polygon $\mathcal{P}_{T_i}$ has $(4i+4)-4$ vertices, as shown in Figure 10. □

Lemma 3.1.2. *For all $i \in \mathbb{Z}^+$, $T = T_b^{-2}(T_d^{-1}T_c)^{i+1}$ gives rise to a polytope P_T with $4i+2$ vertices.*

Proof. With T_i, A_k, B_k as in the proof of Lemma 3.1.1, by Fact 1.1.2 composition of T_b^{-1} with T_i simply increases the width of A_1, B_1 by 1, thus it doubles the width of A_k, B_k. Exactly the same argument as above applies to obtain the convex hull of $4i+4$ vertices shown in Figure 12 for the walk shown in Figure 11 and as seen in Figure 12, this yields a polygon of $4i+2$ vertices.

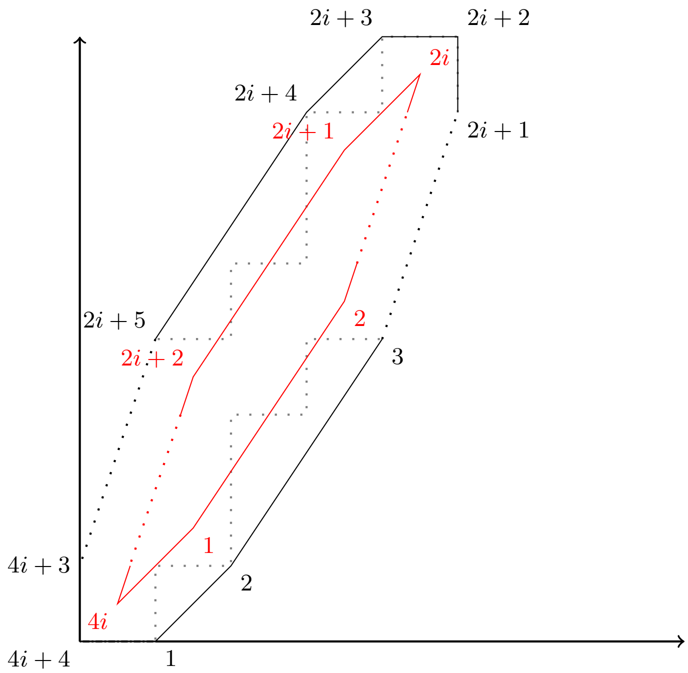


Figure 10

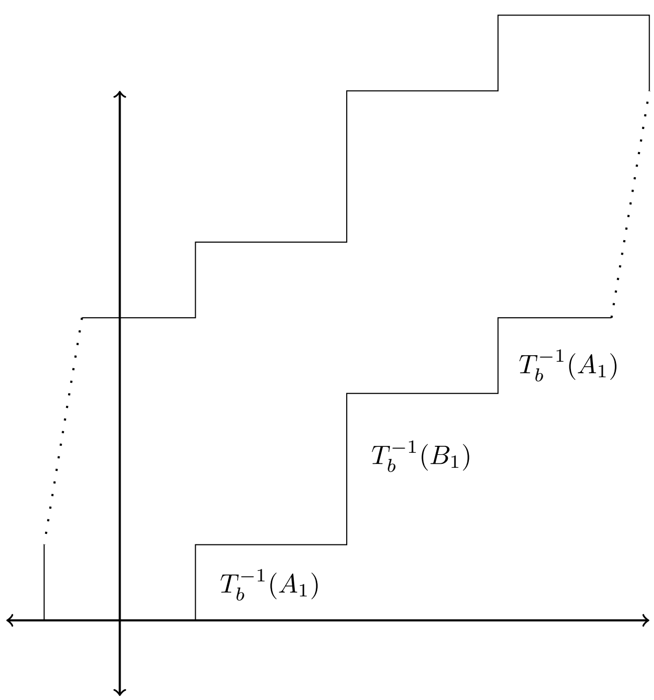


Figure 11

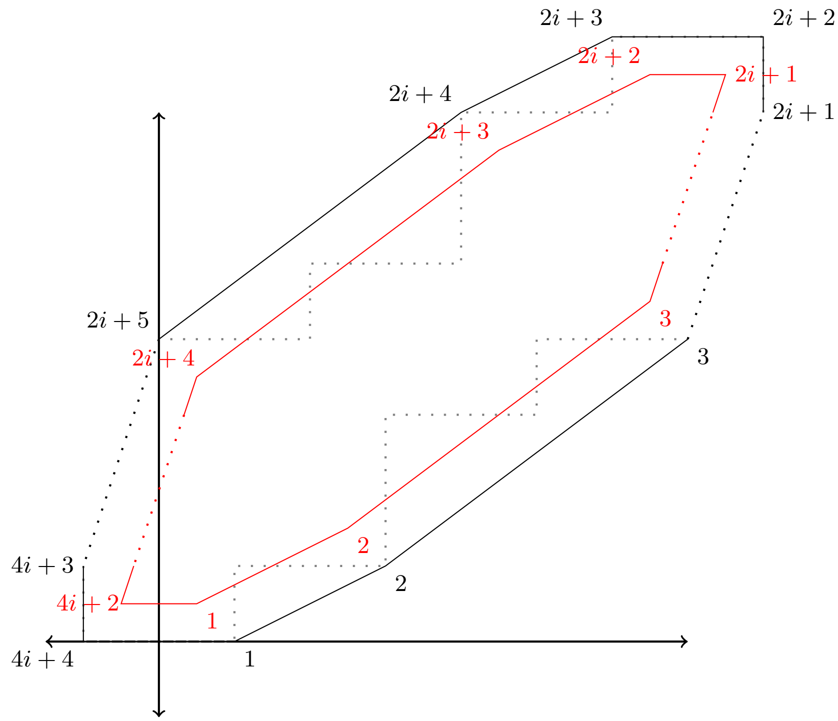


Figure 12

∎

Corollary 3.1.2.1. *Given $k \in \mathbb{Z}^+$, a polygon with $2k$ vertices is obtainable.*

Proof. This follows directly from Lemma 3.1.1 and Lemma 3.1.2. For odd k, one such polygon P_T is given by $T = T_b^{-2}(T_d^{-1}T_c)^{\frac{k+1}{2}}$. For even k, one such polygon is given by $T = T_b^{-1}(T_d^{-1}T_c)^{\frac{k+2}{2}}$. ∎

4. Future Directions

4.1. Enumerating and Detecting Simple Closed Curves

There are several reasons why predicting or generalizing the relator (and thus the polygon) that will arise from a certain sequence of Dehn Twists applied to $[x, z]$ is quite difficult.

Fact 4.1.1. *Two Dehn twists $T_{\gamma_1}, T_{\gamma_2}$ commute if and only if the two curves γ_1, γ_2 admit non-intersecting representatives.*

Because any consecutive pair of Dehn Twists in our list T_a, T_b, T_c, T_d, T_e of generators of $\mathrm{Mod}(\mathbb{T}^2\#\mathbb{T}^2)$ don't commute, there does not seem to be a canonical transformation induced on P_T by composing $T_\gamma \circ T$ for $\gamma = a, b, c, d, e$, since whatever transformation on r and thus on P_T depends on the previous Dehn twists present in T.

One reasonable approach for trying to prove the obtainability of all polygons following Lemma 3.0.3 is confining the search range to polygons within an $m \times n$ box and trying to bound the length of T as a word in T_a, T_b, T_c, T_d, T_e. Reducing the word domain for an $n \times m$ box to a finite one could lead to good "inductive" proofs of the obtainability of polygons (by inducting on its total "dimension" $m + n$).

Fact 4.1.2. *Let H denote the genus 2 handlebody.* $\mathrm{Mod}(H) < \mathrm{Mod}(\partial H) = \mathrm{Mod}(\mathbb{T}^2 \# \mathbb{T}^2)$

Some elements of $\mathrm{Mod}(\mathbb{T}^2 \# \mathbb{T}^2)$ are also in the Mapping Class Group of the two-handlebody. In particular, $T_a, T_c, T_e \in \mathrm{Mod}(H)$ and thus $\mathcal{P}_{T_\gamma \circ T} \doteq \mathcal{P}_T$ for $\gamma = a, b, c$. That is, within a word $T = T_{\gamma_n} \ldots T_{\gamma_2} T_{\gamma_1}$ there could be any length of "trivial" subwords. This makes bounding word size a lot more difficult.

This motivated us to look at other ways to enumerate separating simple closed curves without using the Mapping Class Group, and using expressions on the Fundamental Group directly. For instance, [2] provides a way of listing all simple closed curves on a once puncutred torus and [1] provides an algorithm for detecting whether a curve on a surface admits a simple representative.

4.2. The Code

We intend to make the code for drawing the Thurston unit balls available online. We could also use this computational tool to add information to web archives such as The Knot Atlas, or maybe incorporate the code to existing topology software.

A. Interpretation

A.1. The Interval Case

Let's look at what information we can gather from the following output case (Figure 13). The walk

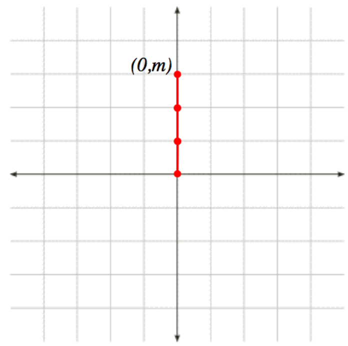


Figure 13

on $\mathbb{Z}^2$ that yields this interval is encoded by the relator $xy^mx^{-1}y^{-m}$. One way to make a manifold with the fundamental group $\langle x, y \mid xy^mx^{-1}y^{-m}\rangle$ is by identifying the orange curves and the blue curves in Figure 14.

Notice that the orange and blue curves on the surface of the solid torus trace out an annulus, which is homeomorphic to a cyllinder. Thus, restricting our attention to the interaction between this annulus and the cyllinder we are identifying onto it, we can see that x and y^m form loops on a torus and thus commute (Figure 15).

Now that we have a 3-manifold M with the appropiate fundamental group $\pi_1(M) = \langle x, y \mid xy^mx^{-1}y^{-m}\rangle$, we can derive some information about it from the polygon in Figure 13. Namely, that if $\phi \in H^1(M, \mathbb{Z})$ is the element of the first cohomology that sends $x \mapsto 1$ and $y \mapsto 0$ (first basis element of the cohomology) and $\psi \in H^1(M, \mathbb{Z})$ is the element of the first cohomology that sends $x \mapsto 0$ and $y \mapsto 1$ (second basis element of the cohomology), then their thurston norms are given by

$$x_M(\phi) = 0$$
$$x_M(\psi) = m - 1$$

because the biggest "thickness" of the polygon in the direction of x is 0 and in the direction of y is $m - 1$. Then, the simplest dual surface to the loop x has Euler characteristic 0, so we can find a surface of Euler characteristic 0 (could be a torus, an annulus, a Möbius strip, ...) embedded in M. The simplest dual surface to the loop y has Euler characteristic $m - 1$, so we can find a surface S with $\chi(S) = 1 - m$ embedded in M.

In the case where $m = 2$, these embedded surfaces are not too hard to visualize. Our manifold M is formed by pasting the pieces in Figure 14 along like-colored curve, with the orange and blue curves winding around the surface of the solid torus only twice. Before pasting, a dual surface to y looks like a disc (Figure 16), and we want to visualize how this disc extends when we attach the cylinder to the torus. The dual surface of y has its boundary on the boundary of M. When attaching the cylinder, the annulus between the orange and blue curves is removed from the boundary (thus we can remove two pieces from the circle) and the traced parts on the cylinder get added to the boundary (thus we can add two bumps to the circle) as in Figure 17. But the cylinder is not solid and has a hole in the middle, which is also part of the boundary, so we can add two punctures to our bumps.

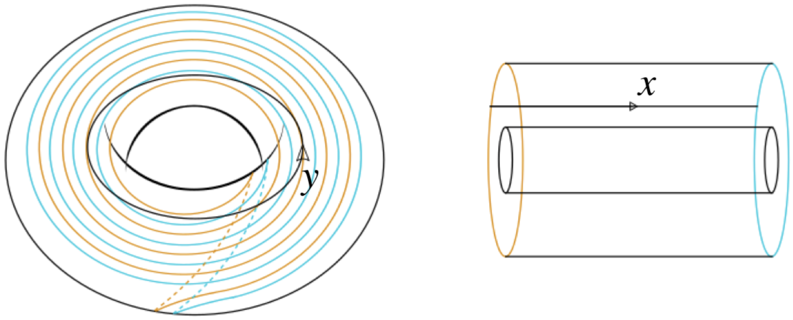


Figure 14: The orange and blue curves, separated by a distance of ϵ, wind around y m times on the surface of the solid torus. x is the loop that traverses $C \times I$ (the thickened cylinder) and closes up in the ϵ-region bounded by the blue and orange curves on the surface of the solid torus.

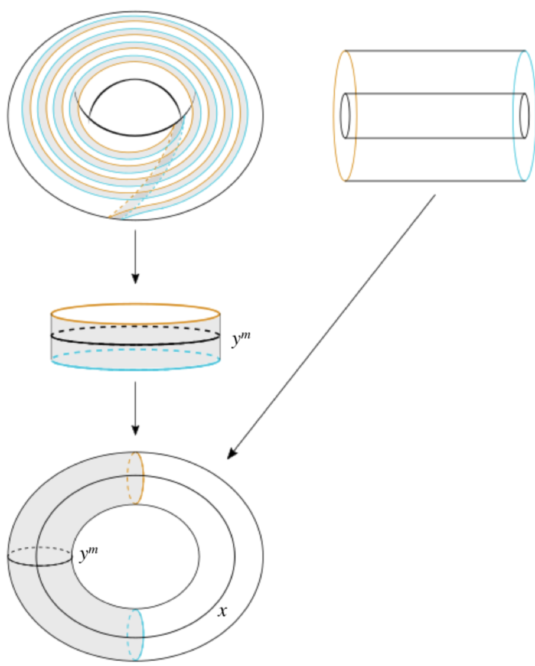


Figure 15: y^m and x lie on a torus and thus commute

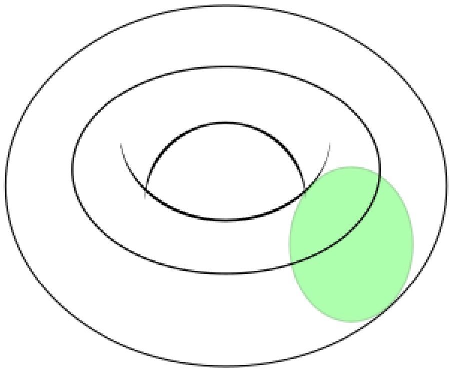


Figure 16: Dual surface to y in the solid torus

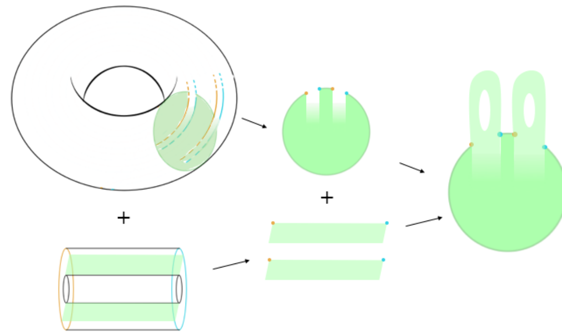


Figure 17: Extending to the dual surface of y in $\langle x, y \mid xy^2x^{-1}y^{-2}\rangle$

This surface should be dual to y in M. The surface is homotopy equivalent to two wedged circles, so the Euler characteristic is -1. The fact that the Thurston norm is 1 (which we gathered from the output of the algorithm) assures that we cannot do any better: this is the simplest surface dual to y.

A.2. $b_1 = 1$ EXAMPLE

Suppose we have a group G with a presentation $\pi' = \langle x, y \mid xy^3x^2y^{-3}x^{-1}yx^{-1}\rangle$. This group has $b_1 = 1$. To apply Lemma 1.3.1 we construct the following epimorphism $\phi : G \to \mathbb{Z}$

$$\begin{aligned} \phi : x &\mapsto -1 \\ y &\mapsto 1 \end{aligned}$$

In general, if the relator of a (2,1) group presentation with $b_1 = 1$ is of the form $x^{\epsilon_1}y^{\delta_1}\dots x^{\epsilon_k}y^{\delta_k}$ and we let $p = \sum_{i=1}^k \epsilon_i$ and $q = \sum_{i=1}^k \delta_i$ we can define an epimorphism from the group to $\mathbb{Z}$ by

$$\begin{aligned} \psi : x &\mapsto \frac{q}{\gcd(p,q)} \\ y &\mapsto \frac{-p}{\gcd(p,q)} \end{aligned}$$

In their proof of Lemma 1.3.1, Friedl and Tillmann describe an algorithm for converting π' to the right presentation, which we apply here directly. Letting $r = xy^3x^2y^{-3}x^{-1}yx^{-1}$ be the relator of π'

1. Pick $\min \phi(x), \phi(y) = \phi(x)$ and let $c = yx^\epsilon$ where $\epsilon = \begin{cases} 1 & \text{if } \phi(x), \phi(y) \text{ differ in sign} \\ -1 & \text{otherwise} \end{cases}$

2. Replace y with $cx^{-\epsilon} = cx^{-1}$ in r and let the result be s

$$\begin{aligned} s &= x(cx^{-1})^3x^2(xc^{-1})^3x^{-1}cx^{-1}x^{-1} && \text{substitute} \\ &= cx^{-1}cx^{-1}cx^2c^{-1}xc^{-1}xc^{-1}x^{-1}cx^{-1} && \text{cyclically reduce} \end{aligned}$$

3. The desired presentation for G is $\pi = \langle a, c \mid ca^{-1}ca^{-1}ca^2c^{-1}ac^{-1}ac^{-1}a^{-1}ca^{-1}\rangle$

Now we draw the polygon associated to this presentation π and derive information about the Thurston norm exactly the same way as in Section 2.1, but since $\phi(c) = 0$, $x_M(\phi_c) = \max\{\phi(p) - \phi(q) \mid p, q \in (\ker \phi_c)^\perp \cap P\} = 0$ because $(\ker \phi_c)^\perp = 0$, and any thickness of P in the c direction can thus be "ignored".

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