Exploring the Geometric Model of Riffle Shuffling

Jieun Lee, Janine Lewandoski, and Kevin Refior

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Abstract

Card shuffling is an interesting topic to explore because of its complexity. Initially, card shuffling seems simple because it is ubitquitous. The majority of people know how to shuffle a deck of cards but few consider the math behind it. However, when it comes to analyzing the elements of card shuffling, it incorporates linear algebra, group theory, probability theory, and Markov Chains. When playing card games, people use various techniques to keep others from having an unfair advantage, the most prevalent technique being cutting and randomizing the deck. In this paper, we will investigate how well two models—the standard model and Geometric Model—describe how humans shuffle cards. We will find the optimal region of fast riffle shuffles for a small deck and measure the randomization of a deck using variation distance. To gain a better understanding of the Geometric Model of card shuffling we will also examine the transition matrix that is formed from the probabilities of transitioning from one deck to another. We found the eigenvalues and eigenvectors for the transition matrix for small decks.

1 Introduction

For centuries people have been playing card games. But, nobody attempted to model how people shuffle cards until the 1950's when Gilbert and Shannon created the standard model of riffle shuffling. Another mathematician, Reeds, came up with this model on his own a few years later and the standard model was known, from then on, as the Gilbert-Shannon Reeds (GSR) Model. In 1988, Diaconis and Fill tested the GSR Model against real data. They each shuffled 100 times and concluded that the model was a good fit [4]. Diaconis wrote his most famous analysis on the GSR Model in 1992, where he discovered the minimal number of times to shuffle was 7 to randomize the deck. This was reported in the New York Times later that year [7].

In 2009, Diaconis suggested a new model to describe riffle shuffling. In 2013, Drouillard researched this in a model he called the Geometric Model. This is a generalization of the standard model by including an extra parameter α, which measures the neatness of a shuffler.

Up until 2011, no data had been collected on human shuffling. Cope [3] engineered an efficient way to gather data and had 45 people shuffle approximately 50 times each. This data was used in our research to compare the Geometric Model to the standard model, using Chi-Squared Goodness of Fit tests. We determined that the Geometric Model fits approximately 3 times as many shufflers as does the standard model.

Drouillard was able to find the eigenvalues of the Geometric Model for small decks [6]. We expanded his research by exploring not just the eigenvalues with respect to α, but also the eigenvectors. While parsing through the data, we discovered the signature eigenvector, which is a constant vector recording the parity of permutations. We proved this is an eigenvector for any size deck.

The Geometric Model, surpringly, was shown by Drouillard to contain shuffles faster than the GSR Model. We used a Monte Carlo calculation to characterize the region of fast shuffles for a 3 card deck and found approximately, 20% of shuffles fall into this region.

2 Background

2.1 Riffle Shuffle

We researched the Geometric Model of riffle shuffling in this project, it is therefore important to understand the basics. *Riffle shuffling* is the process of cutting a deck of cards into two piles and dropping the cards by alternating between the left and right piles. A *shuffle* is represented as a series of 0's and 1's, where 1 stands for the cards from the bottom section of the deck and 0 stands for the cards from the top section after the deck is cut. For example, a six card shuffle could be

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

This shuffle is read from left to right. In this example, assume the top section is in the left hand and the bottom section is in the right hand. The first card dropped is from the left hand, followed by one from the right hand, two from the left hand, then the final two from the right hand.

A *transition* is when the cards switch between the left and right piles. In this example there are three transitions, which take place after the first card, second card, and fourth card are dropped.

$$\begin{pmatrix} 0 & | & 1 & | & 0 & 0 & | & 1 & 1 \end{pmatrix}$$

2.2 GSR Model

The vast majority of card shuffling research relies on the GSR Model, the standard model for riffle shuffling. In this model, the assumptions are:

1. The probability that the deck is cut after k cards is $\binom{n}{k}\frac{1}{2^n}$, with a deck of n cards

2. Cards are dropped one at a time from either the left or right hand

3. If there are L cards in the left hand and R cards in the right hand, the probability that the next card is dropped from the left hand is $\frac{L}{L+R}$ and likewise for the right hand, $\frac{R}{L+R}$

Under the GSR Model, all shuffles are equally likely. In an experiment performed in 1988, Fill and Diaconis each shuffled a deck of cards 100 times and recorded the results [4]. The subjects had different packet size distributions - subject 1 had a packet size of 1 62% of the time while subject 2 had packet size of 1 80% of the time. In other words, both subjects were "neater" than the model, which predicts about half of all packets will have size 1. Diaconis wrote his most famous paper with Bayer in 1992 [1]. They were able to find explicit formulas for how quickly a deck of distinct cards randomizes under the GSR Model.

2.3 Geometric Model

In 2009, Diaconis suggested a generalization of the GSR Model that would account for some shufflers being neater than others [5]. In 2013, Drouillard named it the Geometric Model [6]. There are three parameters in the Geometric Model:

- n is the total number of cards
- β is the probability that the first card dropped in a shuffle is from the top section of the deck after the deck is cut
- α is the probability of switching hands, i.e., the probability of a transition at any point

So, α is a measure of "neatness" of the shuffler:

- "messy" shufflers drop fewer and larger packets, and have small α's
- "neat" shufflers drop more and smaller packets, and have large α's

The Geometic Model becomes the GSR Model if $\alpha = \frac{1}{2}$.

The probability of obtaining a particular shuffle under the Geometric Model is

$$\beta^i(1-\beta)^{1-i}\alpha^j(1-\alpha)^{n-j-1}$$

where j = number of transitions, and

$$i = \begin{cases} 1 & \text{if the first card is dropped from the bottom section of the deck} \\ 0 & \text{otherwise} \end{cases}$$

In our research, we assumed $\beta = \frac{1}{2}$. With this assumption, the probability of obtaining a particular shuffle simplifies to

$$\frac{1}{2}\alpha^j(1-\alpha)^{n-j-1}$$

Drouillard's biggest discovery was how neatness affects the speed that a deck approaches uniform distribution. When looking at the eigenvalues of the transition matrix, it is apparent that the second highest eigenvalue controls the speed a deck approaches uniform distribution. Drouillard determined that for small decks, $(3 \leq n \leq 9)$ the deck approaches uniform the fastest when α is between 0.5 and approximately 0.8 [6].

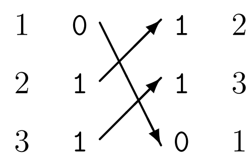
2.4 Transition Matrix

The *transition matrix* is an $n! \times n!$ matrix which is formed from the probabilities of transitioning from one deck to another. This is best explained with an example. For every deck, there are 2^n shuffles. For a deck of 3 cards, the shuffles are

Shuffles
000
001
010
011
100
101
110
111

Each shuffle has a corresponding permutation. A *permutation* is a way of rearranging the ordering of a deck of cards. There are $n!$ permutations for every deck. For shuffle $\begin{pmatrix}0 & 1 & 1\end{pmatrix}$, the corresponding permutation can be revealed by constructing a table with 4 columns:

- In column 1, write the identity permutation $\left(1, 2, 3, \ldots, n\right)$
- In column 2, write down the shuffle starting with the 0's on the top, then following with the 1's
- In column 3, write down the shuffle with the right most bit on the top and the leftmost bit on the bottom. So for shuffle $\begin{pmatrix}0 & 1 & 1\end{pmatrix}$, first write down 1, followed by 1, and end with 0
- Match the topmost 0 from column 2, to the topmost 0 from column 3. Continue this way by drawing arrows to the second topmost 0 and so on. Follow this procedure for all the 1's
- From here, one can see that the 1 in column 1 moved to row 3 in column 4. The 2 in column 1 moved to row 1 in column 4, and the 3 moved to row 2 in column 4. Column 4 is then the corresponding permutation, which is $\begin{pmatrix}2 & 3 & 1\end{pmatrix}$ in this example



This excercise can be completed for each shuffle to get the following:

Shuffle	*Permutation*
000	123
001	312
010	132
011	231
100	123
101	213
110	123
111	123

Using the Geometric Model, we can solve for the probabilities for each shuffle in terms of α.

Shuffle	*Permutation*	*Probability*
000	123	$\frac{1}{2}(1-\alpha)^2$
001	312	$\frac{1}{2}(1-\alpha)\alpha$
010	132	$\frac{1}{2}\alpha^2$
011	231	$\frac{1}{2}(1-\alpha)\alpha$
100	123	$\frac{1}{2}(1-\alpha)\alpha$
101	213	$\frac{1}{2}\alpha^2$
110	123	$\frac{1}{2}(1-\alpha)\alpha$
111	123	$\frac{1}{2}(1-\alpha)^2$

By finding the probability that each permutation occurs, we can calculate the permutation's probability with respect to α.

Permutation	*Probability*
123	$1-\alpha$
132	$\frac{1}{2}\alpha^2$
213	$\frac{1}{2}\alpha^2$
231	$\frac{\alpha-\alpha^2}{2}$
312	$\frac{\alpha-\alpha^2}{2}$
321	0

The transition matrix is then easily obtained. Along the left column are decks and across the top are decks. The transition matrix gives the probability of transition from one deck to another after one shuffle.

$$\begin{array}{cc} & \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \begin{array}{c} 123 \\ 132 \\ 213 \\ 231 \\ 312e \\ 321 \end{array} & \begin{bmatrix} 1-\alpha & \frac{1}{2}\alpha^2 & \frac{1}{2}\alpha^2 & \frac{(\alpha-\alpha^2)}{2} & \frac{(\alpha-\alpha^2)}{2} & 0 \\ \frac{1}{2}\alpha^2 & 1-\alpha & \frac{(\alpha-\alpha^2)}{2} & 0 & \frac{1}{2}\alpha^2 & \frac{(\alpha-\alpha^2)}{2} \\ \frac{1}{2}\alpha^2 & \frac{(\alpha-\alpha^2)}{2} & 1-\alpha & \frac{1}{2}\alpha^2 & 0 & \frac{(\alpha-\alpha^2)}{2} \\ \frac{(\alpha-\alpha^2)}{2} & 0 & \frac{1}{2}\alpha^2 & 1-\alpha & \frac{(\alpha-\alpha^2)}{2} & \frac{1}{2}\alpha^2 \\ \frac{(\alpha-\alpha^2)}{2} & \frac{1}{2}\alpha^2 & 0 & \frac{(\alpha-\alpha^2)}{2} & 1-\alpha & \frac{1}{2}\alpha^2 \\ 0 & \frac{(\alpha-\alpha^2)}{2} & \frac{(\alpha-\alpha^2)}{2} & \frac{1}{2}\alpha^2 & \frac{1}{2}\alpha^2 & 1-\alpha \end{bmatrix} \end{array}$$

3 Results

3.1 Accuracy of the Model

Cope collected data by having 45 test subjects shuffle approximately 50 times each [3]. Each shuffle was recorded as a string of 52 0's and 1's, where 0's represent cards dropped from the top section and 1's represent cards dropped from the bottom section of the deck. Using this data, we determined a best fit α value for each test subject, where α denotes the transition probability. The best fit α is calculated by

$$\frac{\textit{Number of Observed Transitions}}{51 \cdot m}$$

where m is the number of times the subject shuffles and there are 51 possible transitions per shuffle.

We used the best fit α to create a distribution for the predicted number of times the subject shuffled with k transitions, where k is the count of transitions per shuffle, ranging from 0 to 51. This follows a binomial distribution with parameters m and α. The predicted number of times the subject shuffles with k transitions is

$$\binom{51}{k}\alpha^k(1-\alpha)^{51-k}m.$$

We used a Chi-Squared Goodness of Fit test to assess the accuracy of the observed distribution compared to the predicted distribution for each test subject. The test statistic for this test is

$$\chi^2 = \sum_{k=0}^{51} \frac{(O_k - E_k)^2}{E_k}$$

with degrees of freedom equal to $m-1$. We found p values for each subject. The p value indicates the probability of obtaining a Chi-Squared value at least as high as the observed Chi-Squared value for the test subject. For example, test subject 31 had a Chi-Squared value of 39.5846. The corresponding p value is 0.87704, with 51 degrees of freedom, . This signifies that the probability of calculating a Chi-Squared value greater than or equal to 39.5846 is 0.87704.

Upon examining the p value for each test subject, we found that 29 of the 45 test subjects had results that were statistically significant at a 5% significance level. This means, the p values for these 29 subjects are less than 0.05. A low p value indicates that the predicted distribution for the number of transitions is not a good fit for the distribution of the observed number of transitions. These test results suggest that the Geometric Model is accurate in describing the shuffling patterns for 16 of the 45 test subjects.

Figure 1 shows the data from a subject with a high p value. Subject 43 has a p value of 0.9999. The histogram shows that the observed distribution closely follows the predicted distribution.

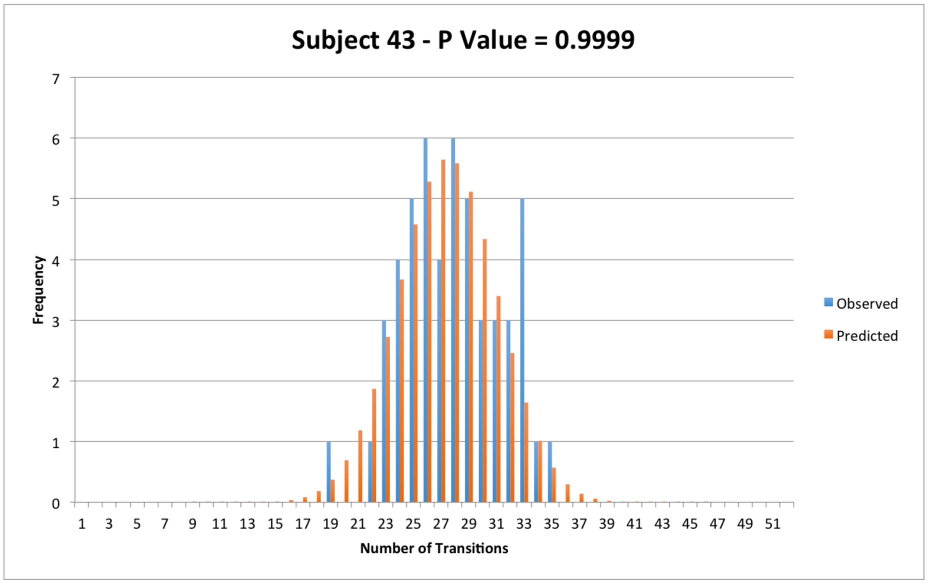


Figure 1: Subject 43

In contrast, Subject 48 has a p value that is 0.0000. Figure 2 shows that the observed distribution differs drastically from the predicted distribution. This subject shuffled once with 11 transitions and once with 43 transitions. These high and low counts heavily increase the Chi-Squared values, which brings the p value closer to 0.

The GSR Model assumes that there is an equal probability of dropping cards from both the left and right hands. This is equivalent to using an α equal to 0.5. We conducted the same Chi-Squared Goodness of Fit test for each subject using the assumed α of 0.5. In this case, the predicted number of times the subject shuffled with k transitions is

$$\binom{51}{k}\frac{m}{2^{51}}.$$

The p values from this test of the GSR Model indicate that 40 of the 45 test subjects have results that were statistically significant at the 5% significance level. Hence, the GSR Model predicts a distribution that fits the observed distribution for 5 of the 45 test subjects. These 5 subjects' shuffling patterns are also good fits for the Geometric Model, and they all have best fit α values close to 0.5.

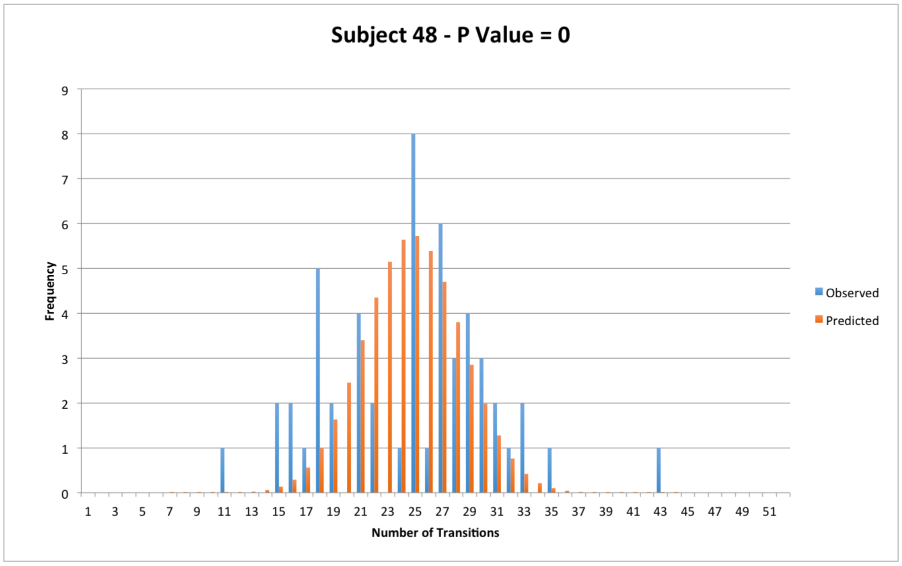


Figure 2: Subject 48

3.2 Exploring a Subspace

The GSR Model is called the “maximum entropy model” because it postulates that all shuffles are equally likely. Therefore, it seemed likely that among all ways to riffle shuffle, GSR randomizes the deck the fastest. However, the results from Drouillard’s model showed that was not the case. In fact, the deck randomizes faster if the shuffler is a little neater than the GSR Model, which assumes a constant $\alpha = 0.5$ [6].

When shuffling cards, the goal is to end with a completely random deck, in other words, every possible ordering is equally likely. In order to understand this, we will need a measure of how fast the deck randomizes. We use the *total variation distance*, or TVD, to measure how close a deck is to being randomized. The observed long run pattern of the TVD shows how quickly a deck randomizes. Let

$$TVD = \frac{1}{2} \sum_{\pi \in S_n} \mid \mathbb{P}(\pi) - U(\pi) \mid$$

where $U(\pi) = \frac{1}{n!}$ is the probability of obtaining π assuming perfect mixing and $\mathbb{P}(\pi)$ is the current probability of permutation π. From Drouillard’s results, as the number of shuffles gets sufficiently large, the TVD approaches an exponential decay function, in which the base of the exponential determines how quickly the TVD decays. In other words, it determines how efficient our shuffle is.

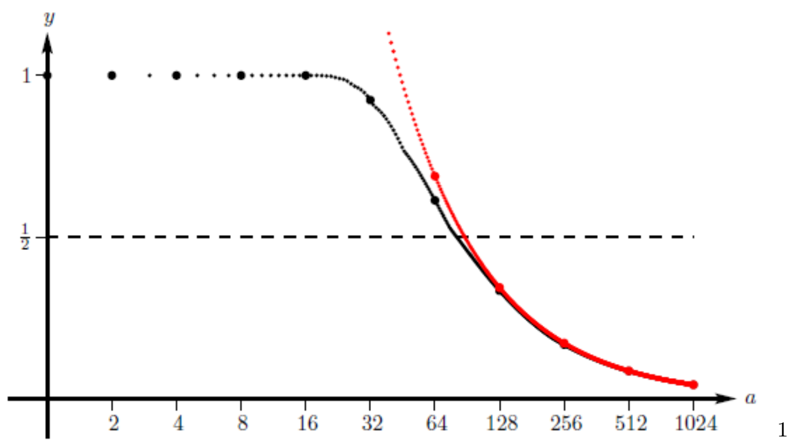


Figure 3: Variation distance from uniform distribution of a 52 card deck after $\log_2(a)$ GSR shuffles.

Figure 3 is the TVD of a 52 card deck after $\log(a)$ shuffles [2]. We see that the first few shuffles do not have a big effect on the variation distance to uniform, but eventually, the variation distance decays exponentially under the GSR Model, and the base of the exponential is $\frac{1}{2}$. This value of $\frac{1}{2}$ also happens to be the second highest eigenvalue for the GSR transition matrix. The transition matrices always have an eigenvalue of 1, and all other eigenvalues have size ≤ 1.

When a transition matrix has a full set of eigenvectors, any initial distribution can be written in terms of that basis. In the long run, only the part corresponding to eigenvalue 1 will survive. Let us look at this pattern.

Suppose we have a full set of eigenvectors

$$v_1, v_2, v_3, \dots, v_n$$

with respective eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$ satisfying

$$1 = \lambda_1 > |\lambda_2| > |\lambda_3| \geq \dots \geq |\lambda_N|.$$

We can write an distribution v in terms of the eigenbasis

$$v = \alpha_1 v_1 + \dots + \alpha_N v_N.$$

and then

$$\begin{aligned} vA^k &= \sum_{i=1}^{N} \alpha_i v_i A^k \\ &= \sum_{i=1}^{N} \lambda_i^k \alpha_i v_i \\ &= \alpha_1 v_1 + \lambda_2^k \alpha_2 v_2 + \sum_{i=3}^{N} \lambda_i^k \alpha_i v_i. \end{aligned}$$

Eventually, everything will approach 0 except $\alpha_1 v_1$. So, that is the equilibrium distribution, which for any reasonable card shuffling method is the uniform distribution[8]. Furthermore, $\lambda_2^k \alpha_2 v_2$ dominates all subsequent terms as k gets large, because of the size of λ_2. It follows that TVD should be approximately proportional to λ_2^k for large k.

So, we expect to see exponential decay in variation distance, and the base of the exponential should be the second highest eigenvalue. It is unclear how this result generalizes to non-diagonalizable transition matrices. However, it does seem to be true in the case of shuffling small decks, as shown in the graphs from Drouillard (Figure 4).

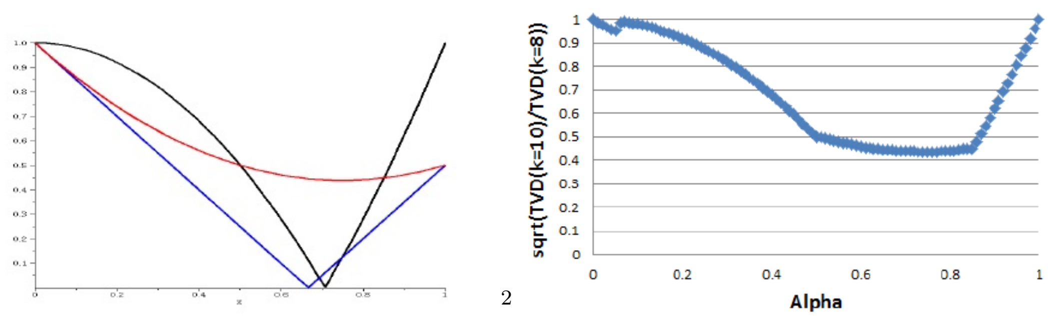


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Figure 4: Absolute value of eigenvalues and TVD decay rate for $n = 3$ under the Geometric Model.

The question arises, what distribution on riffle shuffles will mix the deck the fastest? In order to explore the space of fast shuffling methods for a deck of size $n = 3$, we performed a Monte Carlo calculation. We used the random number generator in Mathematica to pick random points on the simplex $p_1 + \ldots + p_8 = 1$ since there are 2^n shuffles of n cards. Figure 5 is a sample of the data.

```
{0.2915748951619505, {0.005225543744248773, 0.08027200005340163,
  0.1532548682825660 7, 0.2977125792029911, 0.02568252457300302,
  0.2009576841364586 5, 0.1691873968748938 1, 0.0677074031324369 5}}
{0.4045259440827470 6, {0.0219561960517757 9, 0.2293704421060061,
  0.1992513156138715 5, 0.0266143721087412 9, 0.1044512865519862 1,
  0.098485712344754 9, 0.1254801660180904 2, 0.1943905092047737 4}}
```

Figure 5: Sample data for $n = 3$ cards

The first number represents the second highest eigenvalue of the transition matrix corresponding to the distribution. The set of eight numbers following the second higest eigenvalue is the distribution on shuffles.

Among $10,000$ models, the minimum rate of the shuffle is about 0.11399. A lower eigenvalue correlates with a faster speed of a deck approaching uniform distribution. Below is a table of the fastest model of $10,000$ shuffles.

Shuffle	000	001	010	011	100	101	110	111
Model	0.0068	0.0254	0.1764	0.3056	0.1252	0.2666	0.0315	0.0625

Below is a table of the averages of all fast models (that is, models whose second highest eigenvalue is $\leq \frac{1}{2}$. These averages estimate the center of the region of fast models.

Shuffle	000	001	010	011	100	101	110	111
Model	0.0816	0.1631	0.1759	0.1615	0.0784	0.1815	0.0799,	0.0781

An interesting pattern that we noticed was that out of all the shuffling methods that were trialed, about 20% of them resulted in the region where the second higest eigenvalue was less than 0.5, meaning about 20% of all models are faster than the GSR Model, for $n = 3$.

3.3 Eigenvectors and Eigenvalues

Another aspect of the Geometric Model was investigating the eigenvectors and eigenvalues of the transition matrices for multiple size decks with respect to α. This was possible for $n = 3$, after which Mathematica was unable to calculate the eigenvectors. So for $n = 4$ and 5, only the eigenvalues were calculated. Mathematica was able to calculate the eigenvectors for $n =$

4 and 5bBy setting α to a value between 0 and 1. After looking at the eigenvectors for $n = 3, 4,$ and 5, there was an apparent pattern with one of the constant eigenvectors.

n size	vector
3	$v_3 = (1\ \text{-1}\ \text{-1}\ 1\ 1\ \text{-1})^T$
4	$v_4 = (1\ \text{-1}\ \text{-1}\ 1\ 1\ \text{-1}\ \text{-1}\ 1\ 1\ \text{-1}\ \text{-1}\ 1\ 1\ \text{-1}\ \text{-1}\ 1\ 1\ \text{-1}\ \text{-1}\ 1\ 1\ \text{-1}\ \text{-1}\ 1)^T$

As one can see, these are entirely made up of 1's and -1's. We discovered that the 1's and -1's correspond to the parity of the permutation is odd or even. For example when $n = 3$

Permutation	v_3	Sign
123	1	even
132	-1	odd
213	-1	odd
231	1	even
312	1	even
321	-1	odd

In this example, 132 is considered odd because it is the composition of an odd number of transpositions. We will refer to v_n as the *signature vector*. Mathematica calculated the eignvalues for the signature eigenvectors for $n = 3, \ldots, 11$, which are listed in the following table.

n	Eigenvalue
3	$1 - 2\alpha^2$
4	$1 - 2\alpha + 2\alpha^2 - 2\alpha^3$
5	$1 - 6\alpha^2 + 8\alpha^3 - 4\alpha^4$
6	$1 - 3\alpha + 6\alpha^2 - 12\alpha^3 + 12\alpha^4 - 4\alpha^5$
7	$1 - 12\alpha^2 + 32\alpha^3 - 44\alpha^4 + 32\alpha^5 - 8\alpha^6$
8	$1 - 4\alpha + 12\alpha^2 - 36\alpha^3 + 68\alpha^4 - 72\alpha^5 + 40\alpha^6 - 8\alpha^7$
9	$1 - 20\alpha^2 + 80\alpha^3 - 180\alpha^4 + 256\alpha^5 - 216\alpha^6 + 96\alpha^7 - 16\alpha^8$
10	$1 - 5\alpha + 20\alpha^2 - 80\alpha^3 + 220\alpha^4 - 388\alpha^5 + 440\alpha^6 - 304\alpha^7 + 112\alpha^8 - 16\alpha^9$
11	$1 - 30\alpha^2 + 160\alpha^3 - 500\alpha^4 + 1056\alpha^5 - 1504\alpha^6 + 1408\alpha^7 - 816\alpha^8 + 256\alpha^9 - 32\alpha^{10}$

The following will prove that the signature vector is an eigenvector for all size decks.

Let G be a finite group, with $P : G \to [0,1]$ a probability distribution on G. Let $X_0, X_1, X_2, \ldots$ be random variables whose values are elements of G, with

$$\Pr\left(X_{i+1} = b \mid X_i = a\right) = P(ba^{-1})$$

Under these conditions we say that the X_i are a *random walk* on G. They represent a Markov Chain with transition probabilities

$$\Pr(a \to b) = P(ba^{-1})$$

Suppose H is a normal subgroup of G, and $G/H = \{aH : a \in G\}$ is the group of cosets of H. Let $P^* : G/H \to [0,1]$ be given by

$$P^*(aH) = \sum_{h \in H} P(ah)$$

Theorem 1. *P^* represents a random walk on G/H. That is, if $Y_i = X_i H \in G/H$, then $\Pr\left(Y_{i+1} = bH \mid Y_i = aH\right) = P^*(ba^{-1}H)$.*

Proof.

$$\begin{aligned}
&\Pr\left(Y_{i+1} = bH \mid Y_i = aH\right)\\
&= \Pr\left(X_{i+1} \in bH \mid X_i \in aH\right)\\
&= \frac{\Pr\left(X_{i+1} \in bH \cap X_i \in aH\right)}{\Pr(X_i \in aH)}\\
&= \frac{\sum_{h_0 \in H} \sum_{h_1 \in H} \Pr\left(X_{i+1} = bh_1 \cap X_i = ah_0\right)}{\Pr(X_i \in aH)}\\
&= \frac{\sum_{h_0 \in H} \sum_{h_1 \in H} \Pr\left(X_{i+1} = bh_1 \mid X_i = ah_0\right) \cdot \Pr(X_i = ah_0)}{\Pr(X_i \in aH)}\\
&= \frac{\sum_{h_0 \in H} \Pr(X_i = ah_0) \sum_{h_1 \in H} \Pr\left(X_{i+1} = bh_1 \mid X_i = ah_0\right)}{\Pr(X_i \in aH)}.
\end{aligned}$$

But

$$\sum_{h_1 \in H} \Pr\left(X_{i+1} = bh_1 \mid X_i = ah_0\right) = \sum_{h_1 \in H} P(bh_1(ah_0)^{-1}) = \sum_{h_1 \in H} P(bh_1 h_0^{-1} a^{-1}).$$

As h_1 runs through all values of H, $h_1 h_0^{-1}$ does so as well. So let $h = h_1 h_0^{-1}$ and we have

$$\sum_{h \in H} P(bha^{-1}).$$

Now as h runs through all values of H, ha^{-1} runs through the left coset Ha^{-1}. But because H is a normal subgroup of G, $Ha^{-1} = a^{-1}H$. So

$$\sum_{h \in H} P(bha^{-1}) = \sum_{h \in H} P(ba^{-1}h) = P^*(ba^{-1}H).$$

So we have

$$Pr\left(Y_{i+1} = bH \mid Y_i = aH\right) = \frac{\sum_{h_0 \in H} \Pr(X_i = ah_0) \cdot P^*(ba^{-1}H)}{\Pr(X_i \in aH)}$$
$$= \frac{P^*(ba^{-1}H) \sum_{h_0 \in H} \Pr(X_i = ah_0)}{\Pr(X_i \in aH} = P^*(ba^{-1}H).$$

□

We shall say that the random walk on G described by P *induces* the random walk on G/H described by P^*. Let A be the transition matrix for the Markov Chain given by the random walk of P on G. That is,

$$A_{a,b} = \Pr\left(X_{i+1} = b \mid X_i = a\right) = P(ba^{-1}).$$

Let A^* be the transition matrix for the induced Markov Chain on G/H.

Theorem 2. *If $A^* v^* = \lambda v^*$, then $Av = \lambda v$, where $v_a = v^*_{aH}$.*

Proof.

$$(Av)_b = \sum_{a \in G} A_{a,b} v_a = \sum_{a \in G} P(ba^{-1}) v^*_{aH}$$
$$= \sum_{aH \in G/H} \sum_{h \in H} P(b(ah)^{-1}) v^*_{aH} = \sum_{aH \in G/H} v^*_{aH} \sum_{h \in H} P(bh^{-1}a^{-1}).$$

But again, because H is normal, $Ha^{-1} = a^{-1}H$, so

$$
\begin{aligned}
(Av)_b &= \sum_{aH \in G/H} v^*_{aH} \sum_{h \in H} P(ba^{-1}h) = \sum_{aH \in G/H} v^*_{aH} P^*(ba^{-1}H) \\
&= \sum_{aH \in G/H} v^*_{aH} Pr\left(Y_{i+1} = bH \mid Y_i = aH\right) = \sum_{aH \in G/H} v^*_{aH} A^*_{aH,bH} \\
&= (A^* v^*)_{bH} = \lambda v^*_{bH} = \lambda v_b.
\end{aligned}
$$

□

Using these two theorems, we are able to prove the following corrollary.

Corrollary 3. *If P is a random walk on S_n, the the signature vector v given by*

$$
v_\pi = \begin{cases} \textit{1} & \textit{if permutation } \pi \textit{ is even} \\ \textit{-1} & \textit{if permutation } \pi \textit{ is odd} \end{cases}
$$

is an eigenvector of A with eigenvalue $\lambda = P^*(\textit{even } \pi) - P^*(\textit{odd } \pi)$.

Proof. The alternating group A_n is a normal subgroup of S_n with index 2. So by Theorem 1, P induces a random walk P^* on S_n/A_n. The two cosets of A_n are the even permutations (A_n itself) and the odd permutations. So if $p = P^*(A_n)$, the transition matrix for the induced chain is

$$
A^* = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}
$$

Since

$$
A^* \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} p-(1-p) \\ (1-p)-p \end{bmatrix} = (2p-1) \begin{bmatrix} 1 \\ -1 \end{bmatrix},
$$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of A^* with eigenvalue $2p-1$. Therefore by Theorem 2 the signature vector is an eigenvector of A with the same eigenvalue. □

4 Conclusion

In this research, we explored three areas—comparing the standard model to the Geometric Model, finding a region of fast shuffling methods, and examining the eigenvalues and eigenvectors of the transition matrices.

GSR and the Geometric Model are models of riffle shuffling. The Geometric Model is a generalization of the standard model by holding an extra parameter, α, that describes the neatness of a shuffler. Out of 45 test subjects who shuffled approximately 50 times each, the Geometric Model predicted a good fit distribution for 16 people. This was an improvement over the standard model, which predicted a good fit distribution for only 5 people.

Next, we considered the region where all fast shuffling methods lie. For $n = 3$ we found one that was much faster than GSR (decay rate 0.11399). About 20% of all shuffling models are faster than GSR for $n = 3$.

Lastly, we analyzed the transition matrix to find the significance of the eigenvalues and eigenvectors. Through numerous computer trials and observations of the eigenvalues of the transition matrices for small decks, we found and proved a pattern: the signature vector is an eigenvector for any shuffling method.

We conclude that the Geometric Model should be the prominently used model because it is a better fit for our samples' shuffling patterns. The Geometric Model also holds faster shuffles than the standard model. Although the relationship between the eigenvalues and eigenvectors and randomizing the deck is still unclear, these results can help us conclude that there is an evident correlation between the eigenvectors and eigenvalues with the randomization. Further research should be done to find a stronger correlation between large size decks and fast shuffles.

5 Definitions

1. A **deck** is an ordering of cards.
2. The **neatness** of a shuffler is defined as $\frac{\textit{Number of transitions}}{n-1}$
3. A **permutation** is a way of rearranging the ordering of cards.
4. A **randomization of a deck** is one that approaches uniform distribution.
5. A **random walk** is a repeated Markov Chain
6. A **riffle shuffle** is cutting the deck into two piles and dropping cards by alternating from the left and right piles.

7. The **signature vector** is a vector that is constructed based on whether the permutation is odd or even.

8. A **transition** is when a hand from which a card is dropped switches from the left and right piles.

9. A **transition matrix** is a $n! \times n!$ matrix which is made up of the probability of transitioning from one deck to another.

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