Weak Choice Principles and Forcing Axioms

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1 Introduction

Forcing is a technique that was discovered by Cohen in the mid 20th century, and it is particularly useful in proving independence results. The process involves adding new, generic elements to your universe and using them to prove a result. In many applications this is encapsulated by the addition of suitable axioms, often these are Forcing Axioms. One of the first such principles is known as Martin's Axiom(MA). For explanation and background of MA see Kunen [3]. In general we consider the basic format of a forcing axiom to be as follows:

$FA_\kappa(P)$: For every partial order in the class P, for any family $\mathcal{D}$ of at most κ-many dense subsets, there is a $\mathcal{D}$-generic filter.

Due to the nature of Choice being independent of ZF, along with the general concern over what propositions use Choice, it is natural to think of certain weak choice principles as equivalent to forcing axioms. This can also make certain proofs easier and more natural for individuals who are familiar with forcing but less so with some other equivalents. This paper will prove equivalents between several choice principles and appropriate forcing axioms.

2 Preliminary Results

The following lemma can be useful in working with a broad class of partial orders that tend to come up often in applications.

Lemma 1 *For any partial order such that every element is either minimal or has two incompatibe extensions, G is a generic filter iff* $G = \{y \in \mathbb{P} \mid y \geq x\}$ *for some minimal element* x

Proof:
$\Leftarrow$ is clear from the definition of a generic filter.
$\Rightarrow$ We know that G is a generic filter. Therefore we conclude that $P \setminus G$ is not dense, for if it was then it would be a dense set that is not intersected by G. So there is an element $p \in G$ such that every extension of p is in G. This implies

that p must be minimal, for otherwise the filter would contain incompatible elements. For any $D \in \mathcal{D}$, D contains every minimal element in $\mathbb{P}$ since if not then D would not be dense, since that minimal element would be an element for which the set contained nothing less than or equal to. Now we must show that G is the upward closure of p. Take any other element $q \in G$. Then by definition there must be a common extension $r \in G$ that extends both p and q. However p is minimal, so $r = p$. Therefore $p \leq q$ for any q in P. $\square$

Some examples of formerly known equivalents between certain choice principles and forcing axioms:

It is a well established fact that the Axiom of Choice, which states that for any family of sets there exists a choice function, is equivalent to Zorn's Lemma.

$\mathbf{Zorn'sLemma}$:
Every partial order with lower bounds on every chain contains a minimal element.

It is natural that before working with weak choice principles we show how full choice can be represented as a Forcing Axiom. In the next theorem let a 'Zorn' Partial Order be any partial order that satisfies the conditions of Zorn's lemma (every chain has a lower bound) and satisfies the conditions from lemma 1 (every element is either minimal or has two incompatible extensions).

Theorem 1 *Zorn's Lemma* $\Leftrightarrow$ *For every Zorn partial order, with every family* $\mathcal{D}$ *of dense subsets, there exists a* $\mathcal{D}$*-generic filter.*

Proof:
$\Rightarrow$ Now consider any minimal element $m \in \mathbb{P}$. Let $G_m = \{q \in \mathbb{P} \mid q \geq m\}$. Then G_m is a generic filter, since m is a common extension for any two elements in G, and for any $D \in \mathcal{D}$ we know $m \in G_m \cap D$ by Lemma 1.
$\Leftarrow$ We know we can find a generic filter G on any Zorn partial order. G must be the upward closure of a minimal element by Lemma 1. Therefore the partial order must contain some minimal element. $\square$

Next we consider a formulation of the weaker choice principle, Dependent Choice.

$\mathbf{DependentChoice(DC)}$: Let R be a binary relation on X such that $\forall x \in X \quad \exists y \in X(xRy)$. Then there exists a sequence $\langle x_n \mid n < \omega \rangle$ such that $\forall n < \omega (x_n R x_{n+1})$.

Theorem 2 *DC* $\Leftrightarrow$ *For all partial orders and every countable family of dense sets* $\mathcal{D}$ *there is a* $\mathcal{D}$*-generic filter.*

Proof:
$\Rightarrow$ Let $\mathbb{P}$ be a partial order. Let $\mathcal{D} = \{D_n \mid n < \omega\}$. Define a relation R as the following: $xRy \quad \leftrightarrow y \leq x$ and $(x \in \bigcap_{k=0}^{n} D_n \rightarrow y \in \bigcap_{k=0}^{n+1} D_n)$ and $(x \notin D_0 \rightarrow y \in D_0)$. Use dependent choice to obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n R x_{n+1}$. Now let $G = \{y \in \mathbb{P} \mid \exists n \in \omega(y \geq x_n)\}$.
Claim : G is a generic filter.
G is a filter, since for any two elements y_1, y_2 there must be some x_n such that $x_n \leq y_1$ and $x_n \leq y_2$ by definition of G. It is clear that G is closed upward. G is $\mathcal{D}$-generic since each x_n intersects the first n+1 dense sets D_n based on the defined relation R. If x is in the intersection of the first n dense sets then its successor in the sequence chosen must be in the first n + 1 sets. We know that D_0 must be intersected by either x_0 or x_1, and therefore every D_n will eventually be intersected. So for every D there is some member of the sequence that intersects that D (since there are only countably many). Each of these x_n is in the filter, so G is a $\mathcal{D}$-generic filter for this partial order.

$\Leftarrow$ We need to construct a partial order from the set X with the binary relation R. Let $\mathbb{P} = \{p \in X^{<\omega} \mid \forall i < dom(p)(p(i)Rp(i+1)\}$ with $p \leq q \leftrightarrow q \subseteq p$. Let $E_n = \{p \in \mathbb{P} \mid n \in dom(p)\}$. Each E_n is dense, since every finite function can be extended to some element with larger domain, specifically each element $p \in E_n$ has a witness in $E_{dom(p)+1}$. Therefore our conditions are satisfied and there is a E-generic filter on this partial order. Since G intersects every E_n G contains functions of arbitrarily large finite length. If we take $\bigcup G$ this gives us a full function from ω to X, since every n is in the domain of this function. Therefore since we defined our partial order using the relation R there is a sequence $\langle x_n \mid n \in \omega\rangle$ such that $x_n R x_{n+1}$ $\square$

The next theorem is from Shannon [4], removing an unnecessary condition and simplifying the proof in order to clarify and provide another example of the link between forcing axioms and choice principles.

Define a class of partial orders $\Gamma = \{\mathbb{P} \mid \mathbb{P}$ is the countable union of finite sets$\}$

Theorem 3 $AC^{\omega}_{fin} \Leftrightarrow FA_{\omega}(\Gamma)$

Proof:
$\Rightarrow$ This proof follows the proof of $MA(\omega)$ from [3]. We know that AC^{ω}_{fin} is equivalent to "The union of a denumerable collection of finite, pairwise disjoint non-empty sets is denumerable". Let $\mathcal{D} = \{D_n : n \in \omega\}$ be our countable family of dense subsets. Define p_n as follows: let p_0 be any element of $\mathbb{P}$. Let $p_{n+1} \in D_n$ and $p_{n+1} \leq p_n$. These can be selected using a well-ordering of $\mathbb{P}$ by choosing least elements. We then let $G = \{q \in \mathbb{P} \mid \exists n(q \geq p_n)\}$. Then G is a $\mathcal{D}$-generic filter.

$\Leftarrow$ Let C_n be a sequence of finite nonempty sets. Let $\mathbb{P}$ be the set of choice functions on finite initial segments of the sequence. Define $\leq$ on $\mathbb{P}$ by reverse inclusion. $\mathbb{P}$ is the countable union of finite sets, since there are countably many finite initial segments, each with finitely many choice functions. Let $\mathcal{D} = \{D_n \mid n < \omega\}$ where $D_n = \{p \in \mathbb{P} \mid \forall(i \leq n)C_n \subseteq dom(p)\}$. It is clear that D_n is dense for each n. Therefore we can get G a $\mathcal{D}$-generic filter. Let $f = \bigcup G$. First we must justify that f is a function. f cannot choose two elements for the same set, since these would have to come from two incompatible finite functions, which cannot both be in a filter together. f must also have domain equal to all of the C_n since G meets D_n. Therefore f is a choice function on these countably many finite sets. $\square$

3 Main Results

The Boolean Prime Ideal Theorem (BPI) is a relatively strong version of choice, which can often be substituted in many proofs in place of full choice. As such there are many known equivalences and consequences of BPI. The purpose of this section is to introduce a new equivalent in the realm of forcing and show how this can be used in place of other versions of BPI in proofs.

Boolean Prime Ideal Theorem :
Every Boolean Algebra has a prime ideal.

Most equivalent statements to BPI involve using that a property is consistent for every finite case in order to prove the full case. The equivalent statement we are most concerned with in this paper is the Consistency Principle. For proof that the consistency principle is equivalent to the boolean prime ideal theorem see Jech [2].

Consistency Principle : Every binary mess has a consistent function.

We would like to put forward a new equivalent to BPI in the form of a Forcing Axiom, or as similar to a standard Forcing Axiom as possible. Following the example of Cowen in [1] equivalents to the BPI seem to require moving away from a strict adherence to the forcing axiom form discussed above. The requirements on antichains in the following theorem are in the same vein as the piercing sets of levels discussed in [1].

Theorem 4 *BPI* $\Leftrightarrow$ *For every partial order* $\mathbb{P}$ *with* $\mathcal{A}$ *a family of finite antichains such that* $\forall F \in [\mathcal{A}]^{<\omega} \exists p \in \mathbb{P} \forall A \in F \exists q \in A(p \leq q)$ *then there is a* $\mathcal{A}$*-generic linked family on* $\mathbb{P}$.

Proof:
$\Leftarrow$ We have a binary mess M on a set X and we would like to define an appropriate partial order and family of finite antichains. Let $\mathbb{P} = \{m \mid m \in M\}$

with $p \leq q \leftrightarrow q \subseteq p$. So extension is determined by function extension. Next define $\forall B \in [X]^{<\omega}$ the set $A_B = \{m \in M \mid dom(m) = B\}$. Each A_B is a finite antichain. They are finite since each A is finite and there can only be finitely many distinct functions from some finite domain into 2. They must also form an antichain since any two distinct elements must disagree at some point in the domain, and therefore cannot both be extended to the same function.

Next we would like to show that the family of A_B satisfies our condition. For any finite subset $\{A_{B_1}...A_{B_n}\}$ we can take the union of the corresponding B_i, which will still be finite since the finite union of finite sets is finite. Since our partial order is made of every element of the mess (which must contain an element corresponding to every finite domain), there must be some $p \in \mathbb{P}$ such that $dom(p) = \bigcup_{i=1}^{n} B_i$. Furthermore the restriction of p to any of the B_i must also be in the mess and therefore in the partial order. Thus p extends an element in each of the A_{B_i}.Thus the family of all of the A_B satisfies our condition on the finite antichains. Therefore there exists G, a linked family that intersects each of the A_B.

Now we claim that $f = \bigcup G$ is our consistent function. First we want to show that f is a function, i.e. it does not contradict itself at any point in the domain. This is however guaranteed by the fact that G is a linked family. Any two elements of a linked family are compatible and therefore they must agree on any common domain. Next we want to ensure that $dom(f) = X$. This is clear, since G is $\mathcal{A}$-generic, for any finite subset B of X, including the singletons, G intersects A_B which means $B \subset dom(f)$. From the above argument as to why the A_B are antichains, it is clear that G intersects each A_B in exactly one element. So for every A, if we take $f \upharpoonright B$, this element is in the mess/partial order. Therefore f is a consistent function.

$\Rightarrow$ In this direction we start with an arbitrary partial order with a family of finite antichains that satisfy our condition, and we wish to define an appropriate binary mess. Let X be the set of all elements of the partial order. $M = \{m : P \to 2 \mid P \in [X]^{<\omega} \wedge (\forall x, y \in P(m(x) = m(y) = 1 \to x \not\perp y) \wedge (A \in \mathcal{A} \wedge A \subseteq P \to \exists x \in A(m(x) = 1))\}$.

First we claim that M is a binary mess. It is clear that M is closed under restrictions. If an antichain is still entirely contained in the restricted domain then the same element gets mapped to one, and whatever elements that still remain that get mapped to one are still compatible. To show that every finite subset of X is the domain for some element of M we will construct such a function. Let $P \in [X]^{<\omega}$. Consider all finite antichains entirely contained in P. There can be only finitely many of these antichains, since P is finite. By our condition on these antichains, there is some element p that extends an element from each of these antichains. Take all q such that $p \leq q$ and let m send these q to one, send the rest of P to zero. This m will be in the mess. So every finite subset of X is the domain for some element of M. Therefore M is a binary mess.

By the consistency principle, we can get a function f that is consistent with the mess. We let $G = \{p \mid f(x) = 1\}$. We claim that G is a $\mathcal{A}$-generic linked family. Let $x, y \in G$. Clearly, based on our definition of M, these two elements are compatible. Its also clear, since f is a consistent function, that $f \upharpoonright P$ is in the mess for every $P \in [X]^{<\omega}$, so G must intersect each of our finite antichains, since by definition an element in this antichain must get mapped to one by f. Thus G is an $\mathcal{A}$-generic linked family. $\square$

We will show some ways in which this new equivalence can be used to prove other equivalent statements or consequences of the boolean prime ideal theorem. For the sake of convenient notation, let FA represent the equivalent formulation introduced in Theorem 4.

Example
The following theorem is actually equivalent to BPI but we will show specifically how it can be proved from this equivalent of BPI.

Theorem 5 *FA $\Rightarrow$ Let $\{A_i\}$ be a collection of finite sets and S a symmetric binary relation on $\bigcup_{i \in I} A_i$ Suppose that for every finite $W \subset I$ there is an S-consistent choice function for $\{A_i\}_{i \in W}$. Then there is an S-consistent choice function for $\{A_i\}_{i \in I}$.*

Proof:
First we will define our partial order and collection of finite antichains. So let $\mathbb{P} = \{p \mid p$ is an s-consistent choice function on a finite subcollection of $\{A_i \mid i \in I\}\}$ with $p \leq q \leftrightarrow q \subseteq p$. Then for every finite $W \in I$ let D_W be the antichain consisting of all S-consistent choice functions on $\{A_i\}_{i \in W}$. This antichain must be finite since W and each of the A_i are finite. Let the collection of all the D_W be our family of antichains $\mathcal{A}$. However for any finite collection of antichains, there must be an element that extends an element of each of the antichains. This is because the finite union of the corresponding W is also a finite subset and so the S-consistent choice function on that union can be restricted to any of the component subsets and be in the corresponding antichains. Therefore we have an $\mathcal{A}$-generic linked family, call it G.

Let $f = \bigcup G$. We claim that f is a full S-consistent choice function. Since any two elements of G must be compatible, there can be no point where the function disagrees with itself. Furthermore, its clear that since G intersects every finite antichain in our family, it must choose one element from each A_i, since $W = \{i\}$ is a finite subset. Thus we know f is a choice function now we need to show that it is S-consistent. For any two elements A_x, A_y $f \upharpoonright \{A_x, A_y\}$ is S-consistent by assumption, and so this relationship must hold for the unrestricted function as well. Therefore f is S-consistent. $\square$

Example
The Ordering Principle is a choice principle that is known to be strictly weaker than, and follow from, the boolean prime ideal theorem. The ordering principle states that any set can be linearly ordered.

Theorem 6 *FA* $\Rightarrow$ *Every set can be linearly ordered.*

Proof:
Let X be an arbitrary set. Define a partial order $\mathbb{P} = \{\leq | \leq$ is a linear order on some $A \in [X]^{<\omega}\}$. We will use reverse inclusion, so $\leq\ R\ \preceq \leftrightarrow \preceq \subseteq \leq$. For each $P \in [X]^{<\omega}$ let A_P be all of the linear orders of P. These exist since finite sets can be linearly ordered. These are finite antichains, since there are only finitely many possible orderings of a finite set, and if two linear orders differ then they can have no common extension. For any finite collection of these, take the union of the corresponding finite subsets, this can be linearly ordered and the restriction to any of the component subsets extends an element from each of the finite antichains.

Therefore we have an $\mathcal{A}$-generic linked family, G. If we take $\bigcup G$ we claim this is a linear order on X. To show trichotomy holds, for any two distinct elements of X, one must be greater than the other, since an order on just those two elements must be in the linked family (since the family is generic and therefore must intersect that antichain). This also shows that G is antisymmetric, since if distinct elements were both less than each other, they would be incompatible and so could not be in the linked family. Reflexivity is clear, since for any x, $x \leq x$ is the only partial order that can exist on the singleton sets, and so must be included in G. In order to show transitivity consider x,y,z in X. Then, since G is $\mathcal{A}$-generic, it must contain a linear ordering on x,y, and z which is not contradicted by any other element of G. So if $x \leq y$ and $y \leq z$ then $x \leq z$ must be in the finite linear order, and thus must be contained in the full order. $\square$

Next we consider another weak choice principle which follows from the Ordering Principle. We will then use some of the ideas from [4] to show an equivalence with a forcing axiom.

$\mathbf{AC^{LO}_{fin}}$: Every linearly-orderable collection of finite sets has a choice function.

In order to prove the next theorem first we define a class of partial orders. For any $n < \omega$ let Δ_n be all partial orders that are linearly orderable unions of finite sets with antichains of size at most n. In the following theorem there is no restriction on the cardinality of the family of dense sets, and so the filter obtained is fully generic.

Theorem 7 *$AC^{LO}_{fin} \leftrightarrow$ for any n $FA(\Delta_n)$ and the filter obtained will be linearly-orderable.*

Proof:
$\Rightarrow$ Since there is an upper bound on the size of the antichains we know that there are no infinite sequences of increasing antichains. Therefore we can find antichains of maximal length in $\mathbb{P}$. Pick one such antichain and call it A. Define for some x that is an element of such an antichain, $G_x = \{y : y \not\perp x\}$. We claim that G_x is a linearly-orderable generic filter. G_x inherits the linear order from the full partial order. Clearly G_x must intersect every dense set, since for any dense D, there must be some $y \in D$ such that $y \leq x$. And that element y must be in G_x. To show that G_x is a filter we use a technique from [4]. Take $y, z \in G_x$. By definition there are elements $u, v \in G_x$ such that $u \leq x, u \leq y$ and $v \leq x, v \leq z$. There must be a $w \in G_x$ such that $w \leq u, w \leq v$. (If there was no such w then the antichain I that x belongs to could be extended by taking $A \setminus x \cup \{u, v\}$.) Therefore G_x is a generic filter.

$\Leftarrow$ Let I be a linearly orderable index on a collection $\{A_i\}_{i \in I}$ of finite sets. Take $\mathbb{P} = \bigcup_{i \in I} A_i$ and $\forall x, \forall y(x \leq y)$. It should be noted that this defines a quasi-order, however in general quasi-orders can be collapsed into partial orders. Note that each A_i forms a dense set, so our filter G must intersect each A_i. Since each A_i is finite we can take the minimal element of each of these intersections with respect to the linear order on G. Define $f = \bigcup G$ this is a choice function on our linearly orderable collection of finite sets. Since we chose minimal elements, f chooses only one element from each A_i. As mentioned above, f must choose some element for each set, since non of the intersections $G \cap A_i$ are empty. $\square$

References

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