THE CONDITIONAL DISTRIBUTION OF THE EIGENVALUES OF THE GINIBRE ENSEMBLE

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ABSTRACT. It is known that the limiting distribution of the normalized eigenvalues of an $n \times n$ matrix with independent complex Gaussian entries is uniform in the unit disk $D \subset \mathbb{C}$, as n tends to infinity. We are interested in the conditional distribution of the eigenvalues in the presence of obstacles restricting their location. That is, requiring the eigenvalues to lie outside an open subset G of D. In the limit, the optimal distribution consists of a singular component, supported on the boundary of the obstacle G, and a uniform component supported on $D \setminus G$. In special cases (e.g. circular or elliptical regions), it is possible to find an analytic solution, however, in general this is difficult and one has to rely on numerical methods. We employ the balayage method, in order to find the optimal distribution via a solution of a suitable Dirichlet problem. Several examples are provided to illustrate the use of the method.

CONTENTS

1. INTRODUCTION

Originally introduced by Wishart in 1920s [Wis28], random matrices can be used in different areas of science including mathematics, statistics, and physics. [BKS+06, Meh04, AGZ05]. For example, a large variety of problems can be given in terms of n-dimensional system of linear ordinary differential equations of the form

$$\frac{d\mathbf{x}}{dt} = M\mathbf{x}, \tag{1}$$

where $\mathbf{x} \in \mathbb{C}^n$, and the coefficient matrix $M \in \mathcal{M}^{\mathbb{C}}_{n\times n}$ has complex random entries. Since the properties of solutions to the system in (1) are affected by the location of the eigenvalues of M in the complex plane, it becomes necessary to study the random matrix M.

For a real symmetric random matrix with independent entries, it is known that the limiting distribution of the eigenvalues is given by the semi-circle law [Meh04, TV08]. For a non-symmetric random matrix with complex independent entries, it has been shown that the limiting distribution

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of the eigenvalues is given by the circular law [TV08]. However, it is not yet known how the limiting distribution of the eigenvalues changes if there are obstacles or restrictions on the locations of the eigenvalues. In this paper, we will focus on random matrices with standard complex normal entries and examine how the limiting distribution of the eigenvalues of this matrix, which is also known as the *Ginibre ensemble* [Gin65], changes based on the restrictions imposed on the locations of eigenvalues.

The structure of this report is as follows: in Section 2 we derive the joint probability distribution of the eigenvalues of an $n \times n$ matrix with complex normal entries. Section 3 includes a discussion of the reduction that allows for a description of the probability of certain constraints on eigenvalues in terms of continuous measures. Section 4 deals with the *balayage method* for finding the distribution of the eigenvalues in terms of a solution of a suitable *Dirichlet problem*. Lastly, in Section 5, we finish this paper by discussing some examples of the *balayage method* corresponding to different restrictions on the location of the eigenvalues of M.

2. Joint Distribution of the eigenvalues

In this section, we derive the joint probability density function (p.d.f.) of the eigenvalues of a random matrix M, whose elements M_{ij} are independent and identically distributed (i.i.d.) complex normal random variables. We call this random matrix M, the standard complex random matrix, where a standard complex normal random variable is defined as follows:

Definition 2.1. *A random variable Q is said to be standard complex normal random variable if its density is given by*

$$\frac{1}{\pi} e^{-|q|^2} dA(q),$$

where $A(q)$ is the measure on $\mathbb{C}$.

The joint distribution of n independent standard complex normal random variables is

$$\frac{1}{\pi^n} e^{-\sum_{i=1}^{n} |q_i|^2} \prod_{i=1}^{n} dA(q_i). \tag{2}$$

Now let $G \subseteq \mathbb{C}$ be open, and define the set of matrices

$$\mathcal{A}_G = \{M \in \mathcal{M}_{n\times n}^{\mathbb{C}} : \forall i,\ \lambda_i \notin G\},$$

where $\{\lambda_i\}_{i=1}^n$ is the set of (random) eigenvalues of the matrix M, and $\mathcal{M}_{n\times n}^{\mathbb{C}}$ denotes the set of all $n \times n$ random matrix with complex normal random variables as its entries. Given G, our goal is to compute the probability $\mathbb{P}(M \in \mathcal{A}_G)$ of choosing, among all random matrices of size $n \times n$, a standard complex random matrix whose eigenvalues lie outside G. From (2):

$$\mathbb{P}(M \in \mathcal{A}_G) = \frac{1}{\pi^{n^2}} \int_{\mathcal{A}_G} e^{-\sum_{i,j} |m_{ij}|^2} dA(m_{ij}).$$

Before we start to find the joint distribution of the eigenvalues, recall that a non-symmetric square matrix M can be decomposed as

$$M = V(Z + T)V^*, \tag{3}$$

where V is a unitary matrix, i.e $VV^* = I$ and V^* denotes the conjugate transpose of V, Z is a diagonal matrix and T is a strictly upper triangular matrix. This is known as the *Schur decomposition*. See chapter 2 of [AR12] for the further details.

Theorem 2.1. [HKPV09, Section 6.3] *Given an $n \times n$ complex standard random matrix M and $\mathbf{Z} = (z_1, z_2, \dots, z_n)$, the n-tuple of eigenvalues of M in uniform random order, the joint distribution of $\mathbf{Z}$ is*

$$f_{\mathbf{Z}}(\boldsymbol{z}) = \frac{1}{C_n} \cdot e^{-\sum_{j=1}^{n} |z_j|^2} \cdot \prod_{i>j} |z_i - z_j|^2 dA(\boldsymbol{z}),$$

where C_n is a normalizing constant and $A(\boldsymbol{z})$ is an area measure on $\mathbb{C}^n$.

Our approach is to use the change of variable method to re-express (2) in terms of the eigenvalues instead of the elements of the random matrix M. We proceed as follows.

Step 1: The term $e^{-\sum_{i,j} |m_{ij}|^2}$ can be rewritten as an expression involving the eigenvalues of the matrix M. Note that $\sum_{i,j} |m_{ij}|^2 = \mathbf{tr}(M^* M)$, where M^* is the conjugate transpose of M. From (3), one can write M as $V(Z+T)V^*$. Note that this decomposition is not unique. In order to make it unique, we can, for example, require that the diagonal elements of V are positive, i.e. $V_{ii} > 0$. We may assume that these diagonal values are non-zero. Therefore,

$$\begin{aligned}
\mathbf{tr}(M^*M) &= \mathbf{tr}(V(Z+T)^*V^*V(Z+T)V^*) \\
&= \mathbf{tr}(V^*V(Z+T)^*V^*V(Z+T)) && \mathbf{tr}(AB) = \mathbf{tr}(BA) \\
&= \mathbf{tr}((Z+T)^*(Z+T)) && V \textit{ is a unitary matrix} \\
&= \mathbf{tr}(Z^*Z) + \mathbf{tr}(T^*T),
\end{aligned}$$

where the last line holds due to the fact that Z is a diagonal matrix and T is a strictly upper triangular matrix. Therefore we have

$$e^{-\sum_{i,j} |m_{ij}|^2} = e^{-(\sum_{j=1}^{n} |z_j|^2 + \sum_{i<j} |t_{ij}|^2)}.$$

Step 2: From the Schur decomposition and using matrix differentiation rules, we have

$$dM = dV(Z+T)V^* + V(dZ + dT)V^* + V(Z+T)dV^*.$$

Since $VV^* = I$, it follows that V^*dV is a skew-Hermitian form, which leads to

$$dM = V(V^*dV(Z+T) - (Z+T)V^*dV + dZ + dT)V^*.$$

Therefore,

$$V^*dMV = V^*dV(Z+T) - (Z+T)V^*dV + dZ + dT.$$

Now put $V^*dMV = \Lambda = (\lambda_{ij})$, $Z + T = S = (s_{ij})$, $V^*dV = \Omega = (\omega_{ij})$, then

$$\Lambda = \Omega S - S\Omega + dS. \tag{4}$$

Since Λ is the image under unitary transformation of dM,

$$\bigwedge_{i,j} (dm_{ij} \wedge \overline{dm_{ij}}) = \bigwedge_{i,j} (\lambda_{ij} \wedge \overline{\lambda_{ij}}) = \bigwedge_{i,j} |\lambda_{ij}|^2.$$

We express $\bigwedge_{i,j} |\lambda_{ij}|^2$ in terms of the eigenvalues in the following way:

$$\begin{aligned}
\lambda_{ij} &= \sum_{k=1}^{j} \omega_{ik} s_{kj} - \sum_{k=i}^{n} s_{ik}\omega_{kj} + ds_{ij} \\
&= \begin{cases} \omega_{ij}(s_{jj} - s_{ii}) + \sum_{k=1}^{j-1} \omega_{ik} s_{kj} - \sum_{k=i+1}^{n} s_{ik}\omega_{kj} & \text{i > j,} \\ ds_{ij} + (\omega_{ii} - \omega_{jj}) s_{ij} + \sum_{k=1, k\neq i}^{j} \omega_{ik} s_{kj} - \sum_{k=i, k\neq j}^{n} s_{ik}\omega_{kj} & \text{i} \leq \text{j.} \end{cases}
\end{aligned}$$

We evaluate $\bigwedge_{i,j} |\lambda_{ij}|^2$ in the order of $\lambda_{n1}, \overline{\lambda_{n1}}, \lambda_{n2}, \overline{\lambda_{n2}}, \cdots, \lambda_{nn}, \overline{\lambda_{nn}}, \lambda_{(n-1)1}, \overline{\lambda_{(n-1)1}}, \lambda_{(n-1)2}, \overline{\lambda_{(n-1)2}}, \cdots, \lambda_{11}, \overline{\lambda_{11}}$, which allows for the cancellations in $\wedge_{i,j}\omega_{ij}$. Then using the above expression and the properties of the wedge product, we get

$$\bigwedge_{i,j} |\lambda_{ij}|^2 = \prod_{i>j} |z_i - z_j|^2 \bigwedge_i |dz_i|^2 \bigwedge_{i>j} |\omega_{ij}|^2 \bigwedge_{i<j} |dt_{ij} + t_{ij}(\omega_{ii} - \omega_{jj})|^2.$$

Recall that $\Omega = V^* dV$ is a skew-Hermitian form and that the dimension of the subspace of skew-Hermitian forms is $n^2 - n$. In addition, note that for all k, $\omega_{kk} \wedge (\bigwedge_{i>j} |\omega_{ij}|^2)$ is an $n^2 - n + 1$ form, and $\omega_{kk} \wedge (\bigwedge_{i>j} |w_{ij}|^2) = 0$ (It is an $n^2 - n + 1$ form on an $n^2 - n$ manifold). Finally, we get

$$\bigwedge_{i,j} (dm_{ij} \wedge \overline{dm_{ij}}) = \bigwedge_{i,j} |\lambda_{ij}|^2 = \prod_{i>j} |z_i - z_j|^2 \bigwedge_i |dz_i|^2 \bigwedge_{i>j} |\omega_{ij}|^2 \bigwedge_{i<j} |dt_{ij}|^2.$$

Step 3: Combining step 1 and 2, we can re-express the joint distribution of elements of the matrix as follows:

$$\frac{1}{\pi^{n^2}} \cdot e^{-(\sum_{j=1}^n |z_j|^2 + \sum_{i<j} |t_{i,j}|^2)} \cdot \Big(\prod_{i>j} |z_i - z_j|^2 \bigwedge_i |dz_i|^2 \bigwedge_{i>j} |\omega_{ij}|^2 \bigwedge_{i<j} |dt_{ij}|^2 \Big).$$

Integrating out the ω_{ij} and t_{ij} terms, we get

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{C_n'} \cdot e^{-\sum_{j=1}^n |z_j|^2} \cdot \prod_{i>j} |z_i - z_j|^2 dA(\mathbf{z}),$$

where C_n' is a normalizing constant that depends on n.

Remark 2.1. [HKPV09, p.60] *One can explicitly compute the normalizing constant C_n in the last expression by evaluating the integral*

$$C_n' = \int e^{-\sum_{j=1}^n |z_j|^2} \cdot \prod_{i>j} |z_i - z_j|^2 dA(\boldsymbol{z}).$$

One can show that

$$C_n' = \pi^n \cdot \prod_{k=1}^n k!.$$

The eigenvalues tend to be uniformly distributed on the disk with radius $\sqrt{n}$. See Figure 1 for the case when n is 1000. After rescaling the eigenvalues by $\sqrt{n}$, the eigenvalues now tend to lie inside the unit disk centered at the origin. By substituting z_j with $\sqrt{n} w_j$, the joint distribution of $\mathbf{W} = (w_1, w_2, \cdots, w_n)$ is given by

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{1}{C_n} \cdot \exp\left(-n^2 \left[\sum_{i=1}^n \frac{|w_i|^2}{n} - \frac{1}{n^2} \sum_{i \neq j} \log |w_i - w_j| \right] \right) dA(\mathbf{w})$$

where $A(\mathbf{w})$ is an area measure on $\mathbb{C}^n$.

3. A large deviation principle

In this section, using the joint distribution of the eigenvalues derived in the previous section, we introduce a large deviation principle to find the optimal limiting distribution of the eigenvalues under the presence of obstacles. The whole derivation closely follows [PH98]. We first list some notations. Let $G \subseteq D = \{|w| \leq 1\}$ be the union of finitely many simply connected domains and

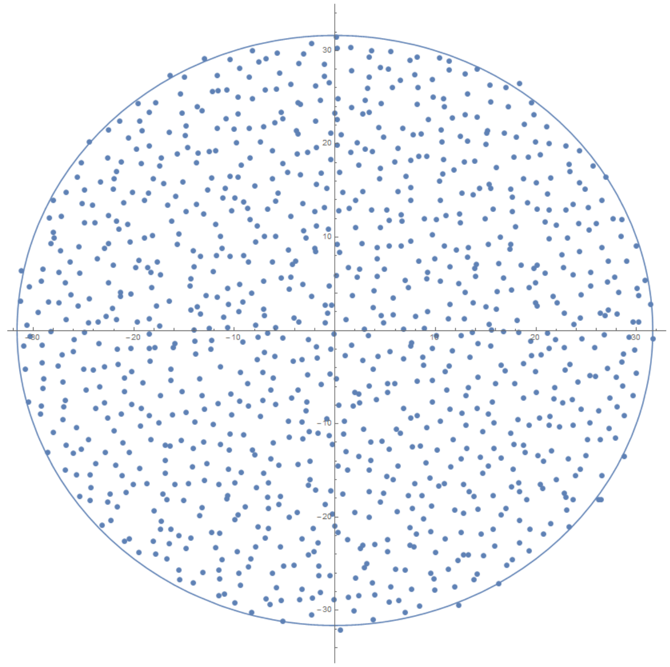


FIGURE 1. The distribution eigenvalues without normalization

$A_G = \mathbb{C} \setminus G$. A_G^n is the n-dimensional cartesian product of A_G. Then the probability that no eigenvalues lie in G is

$$\mathbb{P}_n(\forall i,\ w_i \notin G) = \int\limits_{A_G^n} f_{\mathbf{W}}(\mathbf{w}) dA(\mathbf{w}) = \frac{1}{C_n'} \int\limits_{A_G^n} \exp\left(-n^2 \left[\sum_{i=1}^n \frac{|w_i|^2}{n} - \frac{1}{n^2}\sum_{i\neq j} \log|w_i - w_j|\right]\right) dA(\mathbf{w}).$$

Note that the major contribution to the integral comes from the point $\mathbf{w} \in A_G^n$ that makes the expression inside the square brackets the smallest. Therefore, after taking logarithm on both sides, we expect the integral to be asymptotically approximated as:

$$\log \mathbb{P}_n(\forall i,\ w_i \notin G) \approx -n^2 \cdot \inf_{\mathbf{w}\in A_G^n} \left[\sum_{i=1}^n \frac{|w_i|^2}{n} - \frac{1}{n^2}\sum_{i\neq j} \log|w_i - w_j|\right] - C_n'. \tag{5}$$

The above expression can be re-written in terms of the empirical measure of the eigenvalues.

Definition 3.1. *The empirical measure μ_n is defined as*

$$\mu_n = \frac{1}{n}\sum_{j=1}^n \delta_{w_j}$$

where δ_{w_j} is a Dirac delta measure at w_j.

Recall that if f is a continuous function in $\mathbb{C}$,

$$\int f d\mu_n = \frac{1}{n}\sum_{j=1}^n f(w_j).$$

Using the definition of empirical measure, (5) can be re-written as

$$\log \mathbb{P}_n(\mu_n \in B_G) \approx -n^2 \cdot \inf_{\mu_n \in B_G} \left[\left(\int_{\mathbb{C}^n} |w|^2 d\mu_n(w) - \iint_{z\neq w} \log|z - w| d\mu_n(z) d\mu_n(w)\right)\right] - C_n',$$

where $B_G = \{\mu \mid \mu(G) = 0\}$. Now we introduce the functional,

$$J(\mu) = \int_{\mathbb{C}} |w|^2 d\mu(w) - \iint_{\mathbb{C}^2} \log|z-w| d\mu(z)d\mu(z). \tag{6}$$

Theorem 3.1. *If μ_n is a sequence of empirical measures of the eigenvalues of the Ginibre ensemble, then*

$$\lim_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(\mu_n \in B_G) = -\left(\frac{3}{4} + \inf_{\mu\in B_G} J(\mu)\right).$$

A rough strategy of the proof for the above theorem is to find an appropriate upper bound for $\limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}(\mu_n \in A_G)$ and a lower bound for $\liminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}(\mu_n \in A_G)$. After finding the upper and lower bound, we will use the squeeze theorem to get the value of the limit. In this paper, we demonstrate how the upper bound is achieved. For the lower bound, see [AKR16]. One can also show the conditional limiting measure is the minimizer of the functional $J(\mu)$ over $\mu \in B_G$. To begin the proof of the upper bound, we must first introduce some standard definitions and state relevant theorems. See [DZ09] for further explanations.

Definition 3.2. *A Polish space is a separable, completely metrizable topological space.*

Definition 3.3. *Let τ be a Polish space, then the weak topology on $M_1(\tau)$ is the topology generated by the following sets*

$$U_{\phi,x,\delta} = \left\{ v \in M_1(\tau) : \left| \int_\tau \phi \; dv - x \right| < \delta \right\}$$

where $\delta > 0, x \in \mathbb{R}, \phi \in C(\tau)$, a set of continuous functions in τ.

Definition 3.4. *A probability measure μ on τ is tight if for each $\eta > 0$, there exists a compact set $K_\eta \subset \tau$ such that $\mu(K_\eta^c) < \eta$.*

Remark 3.1. *Every probability measure on a Polish space is tight.*

Definition 3.5. *A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures on a Polish space X satisfies a large deviations principle (LDP) with speed $a_n \to \infty$ as $n \to \infty$ and rate function I if:*

1. *(1) $I : X \to [0,\infty]$ is lower semi-continuous, that is, $\liminf_{x\to x_0} I(x) \geq I(x_0)$*
2. *(2) For any open set $O \subset X$, $-\inf_{x\in O} I(x) \leq \liminf_{n\to\infty} \frac{1}{a_n} \log \mu_n(O)$*
3. *(3) For any closed set $F \subset X$, $\limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(F) \leq -\inf_{x\in F} I(x)$.*

Remark 3.2. *A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures on a Polish space X satisfies a weak large deviations principle if*

1. *(1) The rate function I is positive and lower semi-continuous.*
2. *(2) There is a lower bound for open sets, which is equivalent to the condition (2) in the definition 3.5.*
3. *(3) There is an upper bound for compact sets. Here the condition is the same as in the previous definition, except closed sets are replaced by compact sets, resulting in a weaker principle.*

Definition 3.6. *A sequence $\{\mu_n\}$ of probability measures on a Polish space X is exponentially tight if there exists a sequence $\{K_L\}_{L\in\mathbb{N}}$ of compact sets such that*

$$\limsup_{L\to\infty} \limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(K_L^c) = -\infty.$$

Definition 3.7. *A rate function I is good if the level sets $\{x \in X : I(x) \leq M\}$ are compact for all $M \in \mathbb{R}$.*

Theorem 3.2. [DZ09, Lemma 1.2.18] *(a) If $\{\mu_n\}$ satisfies the weak LDP and is exponentially tight, then it satisfies the full LDP and the rate function I is good. (b) If $\{\mu_n\}$ satisfies the upper bound of the LDP, with a good rate function I, then it is exponentially tight.*

To prove the upper bound in theorem (3.1), recall the definition of $J(\mu)$:

$$J(\mu) = \int_{\mathbb{C}} |w|^2 d\mu(w) - \iint_{\mathbb{C}^2} \log|z-w| d\mu(z)d\mu(w).$$

In our case, the rate function is $J(\mu)$ and $a_n = n^2$. One can show that the level sets $\{\mu \in M_1(\mathbb{C}) : J(\mu) \leq M\}$ are compact, thus $J(\mu)$ is a good rate function.

We first show that the functional $J(\mu)$ is lower semi-continuous and strictly convex for measures with compact support. Consider $\mu \in M_1(\mathbb{C})$, where $M_1(\mathbb{C})$ is a set of all probability measures on $\mathbb{C}$, and let

$$\Sigma(\mu) = \iint_{\mathbb{C}^2} \log|x-y| d\mu(x)d\mu(y),$$

which is also known as the logarithmic energy of the measure μ. Notice that

$$\Sigma(\mu) = \inf_{\epsilon>0} \iint \log(\epsilon + |x-y|) d\mu(x)d\mu(y).$$

Σ is an upper semi-continuous functional in μ, as the expression inside the infimum is continuous. Finally, since

$$J(\mu) = \int_{\mathbb{C}} |z|^2 d\mu(z) - \Sigma(\mu) \tag{7}$$

one can observe that J is a lower semi-continuous functional because the first term is linear and continuous. One can show this in an alternative way by introducing

$$F(x,y) = -\log|x-y| + \frac{1}{2}(|x|^2 + |y|^2)\ ,\ x,y \in \mathbb{C},$$

$$F_\alpha(x,y) = \min\{F(x,y), \alpha\},\ \alpha > 0.$$

Note that F is bounded from below by some constant. And F_α is bounded above and below and continuous. Therefore, the integral $\int_{\mathbb{C}^2} F_\alpha(x,y)d\mu(x)d\mu(y)$ is a continuous functional. Note that,

$$\begin{aligned}\sup_{\alpha>0} \int_{\mathbb{C}^2} F_\alpha(x,y)d\mu(x)d\mu(y) &= \int_{\mathbb{C}^2} F(x,y)d\mu(x)d\mu(y) \\ &= \int_{\mathbb{C}} |x|^2 d\mu(x) - \int_{\mathbb{C}^2} \log|x-y| d\mu(x)d\mu(y) = J(\mu).\end{aligned}$$

The above expression is lower semi-continuous since it is the supremum of continuous functionals. To show that J is a strictly convex functional for measures with compact support, we first introduce the following lemma.

Lemma 3.1. [ST97, Lemma 1.8] *Let $\nu = \mu_1 - \mu_2$, where μ_1, μ_2 are probability measures with compact support and finite logarithmic energy. Then, $\Sigma(\nu) \leq 0$ and $\Sigma(\nu) = 0$ if and only if $\nu = 0$.*

Now suppose that μ_1 and μ_2 satisfy the condition of the above lemma. In order to prove the convexity of J, we show concavity for Σ and it is sufficient to prove mid-point concavity. In other words, we wish to show that

$$\Sigma\left(\frac{\mu_1+\mu_2}{2}\right) \geq \frac{1}{2}(\Sigma(\mu_1) + \Sigma(\mu_2)). \tag{8}$$

By definition,

$$\Sigma\left(\frac{\mu_1+\mu_2}{2}\right) = \int_{\mathbb{C}^2} \log|x-y| d\left(\frac{\mu_1+\mu_2}{2}\right)(x) d\left(\frac{\mu_1+\mu_2}{2}\right)(y)$$

$$= \frac{1}{4}\Sigma(\mu_1) + \frac{1}{4}\Sigma(\mu_2) + \frac{1}{2}\int_{\mathbb{C}^2} \log|x-y| d\mu_1(x) d\mu_2(y). \tag{9}$$

By Lemma 3.1, we also have

$$0 \geq \Sigma\left(\frac{\mu_1-\mu_2}{2}\right) = \frac{1}{4}\Sigma(\mu_1) + \frac{1}{4}\Sigma(\mu_2) - \frac{1}{2}\int_{\mathbb{C}^2} \log|x-y| d\mu_1(x) d\mu_2(y). \tag{10}$$

Taking the sum of (9) and (10), we get (8). In addition, Σ is strictly concave since the equality in (8) only holds when $\mu_1 = \mu_2$. Hence, from (7), $J(\mu)$ is strictly convex for measures with compact support.

Now we will introduce some preliminary lemmas to prove the upper bound of the Theorem 3.1.

Lemma 3.2. [PH98, Lemma 10] *Let C_n be the normalizing constant,*

$$C_n = \int_{\mathbb{C}^n} \exp\left(-n\sum_{j=1}^{n} |z_j|^2\right) \prod_{j<k} |z_j - z_k|^2 dA(\boldsymbol{z}).$$

then,

$$\limsup_{n\to\infty} \frac{1}{n^2} \log C_n \leq -\inf_{\mu\in M_1(\mathbb{C})} J(\mu).$$

Remark 3.3. *We will show later that*

$$\inf_{\mu\in M_1(\mathbb{C})} J(\mu) = \frac{3}{4}$$

Proof. We use the following simple fact

$$\sum_{1\leq j<k\leq n} (a_j + a_k) = (n-1)\cdot\sum_{j=1}^{n} a_j$$

Using this, we can rewrite C_n as

$$C_n = \int_{\mathbb{C}^n} \exp\left(-\sum_{j=1}^{n} |z_j|^2\right) \cdot \exp\left(2\cdot\left(\sum_{j<k} \log|z_j - z_k| - \frac{|z_j|^2}{2} - \frac{|z_k|^2}{2}\right)\right) dA(\mathbf{z}).$$

The above expression can be rewritten as

$$= \int_{\mathbb{C}^n} \exp\left(-\sum_{j=1}^{n} |z_j|^2\right) \cdot \exp\left(-2\sum_{j<k} F(z_j, z_k)\right) dA(\mathbf{z})$$

$$= \left(\int_{\mathbb{C}^n} \exp\left(-\sum_{j=1}^{n} |z_j|^2\right)\right) \cdot \exp\left(-n^2 \iint_{\{x\neq y\}} F(x,y) d\mu_{\mathbf{z}}^n(x) d\mu_{\mathbf{z}}^n(y)) dA(\mathbf{z})\right).$$

where $\mu_{\mathbf{z}}^n = \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j}$. We can bound the above integral by

$$\exp\left(-n^2 \inf_{\mu\in M_1(\mathbb{C})} \iint_{\mathbb{C}^2} F(x,y) d\mu(x) d\mu(y)\right) \cdot \int_{\mathbb{C}^n} \exp(-\sum_{j=1}^{n} |z_j|^2)\ dA(\mathbf{z}).$$

Rewriting the integral over $\mathbb{C}^n$, we have

$$C_n \leq \left(\exp\left(-n^2 \inf_{\mu\in M_1(\mathbb{C})} \iint_{\mathbb{C}^2} F(x,y) d\mu(x) d\mu(y)\right)\right) \cdot \left(\int e^{|z|^2} dA(z)\right)^n.$$

After taking the logarithm on both sides, dividing by n^2, and taking the upper limit, we get the following inequality:

$$\limsup_{n\to\infty} \frac{1}{n^2} \log C_n \leq - \inf_{\mu\in M_1(\mathbb{C})} \iint_{\mathbb{C}^2} F(x,y) d\mu(x) d\mu(y).$$

□

Definition 3.8. *For all $\mu \in M_1(\mathbb{C})$ and $G \subset M_1(\mathbb{C})$, set*

$$\widetilde{G} = \{\boldsymbol{z} \in \mathbb{C}^n : \mu_{\boldsymbol{z}}^n \in G\},$$

where $\mu_{\boldsymbol{z}}^n$ is the empirical measure of z.

Lemma 3.3. [PH98, Lemma 11] *$\forall \mu \in M_1(\mathbb{C})$ and G, an open neighborhood of μ,*

$$\inf_{G:\mu\in G} [\limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(\widetilde{G})] \leq -J(\mu) - \liminf_{n\to\infty} \frac{1}{n^2} \log C_n \tag{11}$$

Proof. Since $F_\alpha = \min\{F, \alpha\} \leq F$, it follows that $\frac{1}{n^2} \sum_{j\neq k} F_\alpha(z_j, z_k) \leq \frac{1}{n^2} \sum_{j\neq k} F(z_j, z_k) + \frac{\alpha}{n}$. Using this fact, it follows that

$$\mathbb{P}_n(\widetilde{G}) \leq \frac{1}{C_n} \int_{\widetilde{G}} \exp\left(-\sum_{j=1}^n |z_j|^2\right) \cdot \exp\left(-n^2 \iint_{\mathbb{C}^2} F_\alpha(x,y) d\mu_{\mathbf{z}}^n(x) d\mu_{\mathbf{z}}^n(y) + n\alpha\right) dA(\mathbf{z}).$$

This, in turn, is

$$\mathbb{P}_n(\widetilde{G}) \leq \frac{1}{C_n} \left(\int e^{-|z|^2} dA(z))\right)^n \cdot \exp\left(-n^2 \inf_{\mu'\in G} \iint_{\mathbb{C}^2} F_\alpha(x,y) d\mu'(x) d\mu'(y) + n\alpha\right).$$

Thus, we get

$$\limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(\widetilde{G}) \leq - \inf_{\mu'\in G} \iint_{\mathbb{C}^2} F_\alpha(x,y) d\mu'(x) d\mu'(y) - \liminf_{n\to\infty} \frac{1}{n^2} \log C_n.$$

Taking the infimum over all open neighborhoods G of μ and using the continuity and boundedness of F_α, we are left with

$$\inf_G [\limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(\widetilde{G})] \leq - \iint_{\mathbb{C}^2} F_\alpha(x,y) d\mu(x) d\mu(y) - \liminf_{n\to\infty} \frac{1}{n^2} \log C_n.$$

The above inequality holds for all α and taking the limit as $\alpha \to \infty$, we get (11) by the monotone convergence theorem. □

Next we show $\mathbb{P}_n$ is exponentially tight. To show this, for any positive α, we must find a compact set, $K_\alpha \subset M_1(\mathbb{C})$ such that

$$\limsup_{\alpha\to\infty} \limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(\widetilde{K}_\alpha^c) = -\infty$$

Lemma 3.4. [PH98, Lemma 14] *$\mathbb{P}_n$ is exponentially tight.*

Proof. For $\alpha > 0$, let $K_\alpha = \{\mu \in M_1(\mathbb{C}) : \int_{\mathbb{C}} |z|^2 d\mu(z) \leq \alpha\}$. It can be proven that K_α is closed. See [PH98] for more details. Therefore if we show that K_α is tight, K_α is compact. Consider $\mu \in K_\alpha$, we know that

$$\begin{aligned}
\alpha \geq \int_{\mathbb{C}} |z|^2 d\mu(z) &= \int_{|z|\leq r} |z|^2 d\mu(z) + \int_{|z|>r} |z|^2 d\mu(z) \\
&\geq \int_{|z|>r} r^2 d\mu(z) = r^2 \mu(\{|z| > r\}) = r^2 \mu(D_r^c),
\end{aligned}$$

where $D_r = \{|z| \leq r\}$. Note that the uniform estimate $\mu(D_r^c) \leq \frac{\alpha}{r^2}$ implies that the family K_α forms a tight family. Now we find an upper bound for $\mathbb{P}_n(\widetilde{K_\alpha^c})$. Note that

$$\mathbb{P}_n(\widetilde{K_\alpha^c}) = \frac{1}{C_n} \int_{\widetilde{K_\alpha^c}} \exp\left(-n \sum_{j=1}^{n} |z_j|^2\right) \cdot \prod_{j<k} |z_j - z_k|^2 dA(\mathbf{z}),$$

Using the fact that $\frac{1}{n}\sum_1^n |z_j|^2 > \alpha$, we know that

$$\mathbb{P}_n(\widetilde{K_\alpha^c}) \leq \frac{1}{C_n} \exp\left(-\frac{1}{2}\alpha n^2\right) \cdot I_n,$$

where

$$I_n = \int_{\mathbb{C}^n} \exp\left(-\frac{n}{2}\sum_1^n |z_j|^2\right) \cdot \prod_{j<k} |z_j - z_k|^2 dA(\mathbf{z}).$$

Similarly to Lemma 3.2 (with $|z|^2$ replaced by $\frac{|z|^2}{2}$),

$$I_n \leq \exp\left(-n^2 \left(\inf_{\mu \in M_1(\mathbb{C})} \left[\int \frac{|z|^2}{2} d\mu - \Sigma(\mu)\right]\right) + 1\right)$$

holds for large enough n. We will later show that the infimum is attained. It remains to find a lower bound for C_n. Recall that

$$C_n = \int_{\mathbb{C}^n} \exp\left(-n \sum_{j=1}^{n} |z_j|^2\right) \cdot \prod_{j<k} |z_j - z_k|^2 dA(\mathbf{z}).$$

To find a lower bound, we integrate over the compact subset $B_n = \{\mathbf{z} \in \mathbb{C}^n, \mathbf{z} = (z_1, z_2, ..., z_n) : |z_k - w_k| \leq \frac{1}{n^2}\}$. Here, $w_k = \exp(\frac{2\pi i k}{n}), k \in \{1, 2, \cdots, n\}$, are the nth root of unity. If $\mathbf{z} \in B_n$,

$$\exp\left(-n \sum_{j=1}^{n} |z_j|^2\right) \geq \exp\left(-n \cdot n\left(1 + \frac{1}{n^2}\right)\right) = \exp(-n^2 - 1).$$

We know that

$$\prod_{j<k} |z_j - z_k|^2 = \prod_{j \neq k} |z_j - z_k|.$$

Using the triangle inequality and the fact that each eigenvalue is within $\frac{1}{n^2}$ of their respective root of unity, we get

$$\prod_{j \neq k} |z_j - z_k| = \prod_{j \neq k} |w_j - w_k + z_j - w_j - z_k + w_k| \geq \prod_{j \neq k} \left(|w_j - w_k| - \frac{2}{n^2}\right).$$

For large n, the two terms w_j and w_k are about $\frac{1}{n}$ apart, leading us to

$$\prod_{j \neq k} |z_j - z_k| \geq \prod_{j \neq k} \left(\frac{1}{2}|w_j - w_k|\right) = \left(\frac{1}{2}\right)^{n(n-1)} \cdot \prod_{j \neq k} |w_j - w_k|.$$

In order to proceed our argument, we make a following claim.

Claim 3.1. $D_n(w_1, w_2, \cdots, w_n) = (\sqrt{n})^n$

where D_n is the determinant of an $n \times n$ Vandermonde matrix.

Proof. Recall

$$D_n(w_1, w_2, \cdots, w_n) = \begin{vmatrix} 1 & w_1 & w_1^2 & \dots & w_1^n \\ 1 & w_2 & w_2^2 & \dots & w_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w_n & w_n^2 & \dots & w_n^n \end{vmatrix}.$$

Let $v_j = (1, w_j, w_j^2, \cdots, w_j^{n-1}) \in \mathbb{C}^n, \forall j \in \{1, 2, \cdots, n\}$. Note that w_j, $j \in \{1, 2, \cdots, n\}$ are the n th root of unity. Therefore, $|v_j| = \underbrace{(1 + 1 + \cdots + 1)}_{\text{n}}^{\frac{1}{2}} = \sqrt{n}$. In addition, the fact that $v_j \perp v_k, j \neq k$ can be seen from:

$$\langle v_j, v_k \rangle = \sum_{l=0}^{n-1} w_j^l (\overline{w_k})^l = \sum_{l=0}^{n-1} \exp\left(\frac{2\pi i l(j-k)}{n}\right) = \frac{\exp\left(\frac{2\pi i n(j-k)}{n}\right) - 1}{\exp\left(\frac{2\pi i(j-k)}{n}\right) - 1} = 0.$$

The determinant of the Vandermonde matrix is the volume of an n dimensional parallelepiped, which leads to the fact that $D_n(w_1, w_2, \cdots, w_n) = (\sqrt{n})^n$. ☐

In addition to the claim, notice that $D_n(w_1, w_2, \cdots, w_n) = \prod_{j<k} |w_j - w_k| = (\sqrt{n})^n$. Thus $\prod_{j \neq k} |w_j - w_k| = n^n$, and

$$C_n \geq \int_{B_n} \exp\left(-n \sum_{j=1}^{n} |z_j|^2\right) \cdot \prod_{j<k} |z_j - z_k|^2 dA(\mathbf{z}) \geq \exp(-Cn^2 + n \log n) \cdot \int_{B_n} dA(\mathbf{z}).$$

The integral on the right hand side of the last term in previous expression equals to $(\frac{\pi}{n^4})^n$, which is bounded by $\exp(-Cn \log n)$. Finally, we get

$$C_n \geq \exp(-Cn^2 - Cn \log n).$$

Recall that

$$\mathbb{P}_n(\widetilde{K_\alpha^c}) = \frac{1}{C_n} \int_{\widetilde{K_\alpha^c}} \exp(-n \sum |z_j|^2) \cdot \prod_{j<k} |z_j - z_k|^2 dA(\mathbf{z}),$$

using our previous estimates, we get

$$\limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(\widetilde{K_\alpha^c}) \leq -\frac{\alpha}{2} + C.$$

for some constant C. Taking the limit of the right hand side as $\alpha \to \infty$, we get that

$$\limsup_{n\to\infty, \alpha\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(\widetilde{K_\alpha^c}) \to -\infty,$$

and this implies exponential tightness of the sequence μ_n (or $\mathbb{P}_n$). ☐

Since $\mathbb{P}_n$ satisfies exponential tightness, we obtain the the upper bound for the left hand side term in the Theorem 3.1. One can also prove the lower bound for the one similar to Lemma (3.3) aforementioned term(see [PH98].). In addition, one can find a lower bound in Theorem 3.1 [AKR16].

3.1. **The Equilibrium Measure.** In this section, we find the unique global minimizer of the functional J, also known as the equilibrium measure. We first evaluate the functional J for the uniform probability measure on the unit disk,

$$\nu_{eq}(z) = \mathbb{1}_D \frac{dA(\mathbf{z})}{\pi},$$

where D is the unit disk centered at the origin. Then we will show that the global minimizer for the functional J is in fact this measure. To show this result, we introduce the notion of logarithmic potential.

Definition 3.9. *The logarithmic potential of a probability measure $\mu \in M_1(\mathbb{C})$ is given by*

$$U_\mu(z) = \int_{\mathbb{C}} \log|z-w| d\mu(w).$$

Now consider the functional J applied to ν_{eq},

$$J(\nu_{eq}) = \int_{\mathbb{C}} |z|^2 d\nu_{eq}(w) - \iint_{\mathbb{C}^2} \log|z-w| d\nu_{eq}(z) d\nu_{eq}(w). \tag{12}$$

To compute the leftmost integral, we change to polar coordinates and arrive at the following integral:

$$\int_0^{2\pi} \left(\int_0^1 r^2 \cdot r dr \right) \frac{d\theta}{\pi} = \frac{1}{2}.$$

To compute the rightmost integral in (12), we first find the logarithmic potential of the measure ν_{eq}. Working in polar coordinates, we get

$$U_{\nu_{eq}}(z) = \int_0^1 \left(\frac{1}{2\pi} \int_0^{2\pi} \log|z - re^{i\theta}| d\theta \right) 2r dr.$$

Next, we claim the following result.

Claim 3.2.

$$U_{\nu_{eq}}(z) = \begin{cases} \frac{|z|^2}{2} - \frac{1}{2} & z \leq 1, \\ \log|z| & z > 1. \end{cases}$$

Proof. By a well-known formula [Lan13, p. 342], for $\alpha, \beta \in \mathbb{C}$ (not both 0),

$$\frac{1}{2\pi} \int_0^{2\pi} \log|\alpha e^{i\theta} - \beta| d\theta = \log\max\{|\alpha|, |\beta|\}.$$

One can proceed by considering two different cases. When $|z| > 1$, we get

$$U_{\nu_{eq}}(z) = \int_0^1 \log\max\{|z|, r\} \; 2r dr = \log|z| \int_0^1 2r dr = \log|z|.$$

On the other hand, for $|z| \leq 1$,

$$U_{\nu_{eq}}(z) = \int_0^{|z|} \log|z| \cdot 2r dr + \int_{|z|}^1 \log r \cdot 2r dr = \log|z| \cdot |z|^2 + \left(r^2 \log r - \frac{r^2}{2} \right)\Big|_{r=|z|}^1 = \frac{|z|^2}{2} - \frac{1}{2}.$$

$\square$

By Claim 3.2 the rightmost integral in (12) becomes

$$\int\limits_{|z|\leq 1} \left(\frac{|z|^2}{2} - \frac{1}{2}\right) d\nu_{eq}(z) = \frac{1}{2} \int\limits_{|z|\leq 1} |z|^2 d\nu_{eq}(z) - \frac{1}{2} \int\limits_{|z|\leq 1} d\nu_{eq}(z).$$

Recall that we already computed the first term of the right hand side of the above equation, and for the second term, the integral is simply an integral with respect to a probability measure over its support, giving a value of 1. Thus,

$$J(\nu_{eq}) = \frac{1}{2} - \left(\frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2}\right) = \frac{3}{4}$$

We will use this result later in Section 4. Now we prove that the uniform probability measure is the global minimizer of the functional J. To prove this statement, we introduce the notion of a *pseudo inner product* of probability measures.

Definition 3.10. *Let μ_1, μ_2 be measures with finite logarithmic energy and compact support. Then the pseudo inner product of μ_1 and μ_2 is*

$$\langle \mu_1, \mu_2 \rangle = -\iint_{\mathbb{C}^2} \log|z - w| d\mu_1(z) d\mu_2(w)$$

and the pseudo norm associated with this pseudo inner product is denoted by $\|\mu\|^2 = \langle \mu, \mu \rangle$.

By taking $\nu = \mu_1 - \mu_2$, in Lemma 3.1, we get the following corollary.

Corollary 3.1. *If $\mu_1, \mu_2 \in M_1(\mathbb{C})$ and μ_1 and μ_2 have finite logarithmic energy, then*

$$\|\mu_1 - \mu_2\|^2 = -\Sigma(\mu_1 - \mu_2) \geq 0$$

where the equality holds if and only if $\mu_1 = \mu_2$.

Proposition 3.1. *Let μ be a measure supported inside D with finite logarithmic energy. Then,*

$$\|\mu - \nu_{eq}\|^2 = J(\mu) - J(\nu_{eq}).$$

Proof. Note that

$$\|\mu - \nu_{eq}\|^2 = \langle \mu, \mu \rangle - \langle \mu, \nu_{eq} \rangle - \langle \nu_{eq}, \mu \rangle + \langle \nu_{eq}, \nu_{eq} \rangle$$

which is equivalent to

$$-\iint_{\mathbb{C}^2} \log|z - w| d\mu(z) d\mu(w) + 2\iint_{\mathbb{C}^2} \log|z - w| d\nu_{eq}(w) d\mu(z) - \iint_{\mathbb{C}^2} \log|z - w| d\nu_{eq}(z) d\nu_{eq}(w).$$

by Claim 3.2,

$$\|\mu - \nu_{eq}\|^2 = -\iint_{\mathbb{C}^2} \log|z - w| d\mu(z) d\mu(w) + \int (|z|^2 - 1) d\mu(z) + \frac{1}{4}$$
$$= \int |z|^2 d\mu(z) - \iint_{\mathbb{C}^2} \log|z - w| d\mu(z) d\mu(w) - 1 + \frac{1}{4} = J(\mu) - J(\nu_{eq})$$

□

Remark 3.4. *If the support of μ is not contained in D, then*

$$\|\mu - \nu_{eq}\|^2 \leq J(\mu) - J(\nu_{eq}).$$

Corollary 3.2. *The global minimizer of the functional $J(\mu)$, ν_{eq}, is the uniform probability measure on the unit disk.*

Proof. From Corollary 3.1 and Proposition 3.1, we know that $J(\mu) - J(\nu_{eq}) \geq 0$, for measures with support in D. Additionally one can see from the Corollary 3.1 that this global minimizer is unique since $J(\mu) = J(\nu_{eq})$ holds only when $\mu = \nu_{eq}$. □

We see that minimizing $J(\mu)$ is equivalent to minimizing $\|\mu - \nu_{eq}\|^2$. In addition, the measure that minimizes $\|\mu - \nu_{eq}\|^2$ out of all measures satisfying $\mu(G) = 0$, is known as the balayage of μ to $D \setminus G$. In the following section, we will discuss the balayage method in more detail.

4. The Balayage Method and the invariance property of J

Let $G \subset D$ be an open set with smooth boundary. In the previous section, we mentioned the notion of balayage of μ to $D \setminus G$ as a measure which minimizes $\|\mu - \nu_{eq}\|^2$ over all measures such that $\mu(G) = 0$, where $G \subset D = \{|w| \leq 1\}$. In this section, we list some properties of balayage and introduce the balayage method which can be used to find the optimal limiting distribution for a given set G. Lastly, we end this section by using properties of balayage to prove the translation invariance property of J.

Let the balayage of μ to $D \setminus G$ be denoted by μ_G. It can be expressed as a sum of two parts: 1) a singular component, μ_G^s, which is supported on the boundary of the obstacle G, and 2) a uniform component, μ_G^u, which is supported on the complement $D \setminus \overline{G}$. In addition, the solution satisfies

$$U_{\mu_G}(z) = \begin{cases} \frac{|z|^2}{2} - \frac{1}{2} & z \in D \setminus G, \\ \log|z| & z \in \mathbb{C} \setminus D \end{cases}$$

where μ_G is the balayage measure such that $\mu_G = \mu_G^s + \mu_G^u$. [ST97, Section II.4]

Now we introduce the balayage method, which allows us to find the singular part of the optimal limiting distribution for a nice domain $G \subset D$. The balayage method gives the singular part in terms of a solution of a *Dirichlet problem* defined on the domain G. Specifically, we will find a function w where: 1) w is harmonic in G, 2) w agrees with the logarithmic potential $U_{\nu_{eq}}$ on $D \setminus G$, where ν_{eq} is the equilibrium measure, by solving a Dirichlet problem. After solving for w, we can recover the measure from the function w using the following theorem.

Theorem 4.1. [ST97, Section II.1] *The singular part of the optimal limiting measure μ_G of the distribution of eigenvalues along ∂G is given by*

$$\mu_G^s = \frac{1}{2\pi} \cdot \left(\frac{\partial w}{\partial n_+} + \frac{\partial w}{\partial n_-} \right) ds$$

where the function w is the solution of the Dirichlet problem on G that agrees with logarithmic potential $U_{\nu_{eq}}$ on $D \setminus G$, $\frac{\partial w}{\partial n_+}$ and $\frac{\partial w}{\partial n_-}$ are, respectively, the outward and inward normal derivatives; and ds is the arc length measure defined on ∂G .

Therefore, as long as we can solve the Dirichlet problem on G, we can find the optimal limiting measure for the eigenvalues. Additionally, note that from Theorem 4.1, μ_G^s is invariant under translations of G in D. This is because the equilibrium measure is translation invariant in $\mathbb{C}$. In this paper, we provide some examples for nice domains where the Dirichlet problem can be easily solved. However, in general, solving a Dirichlet problem analytically turns out to be a very difficult problem. In Section 4, we provide examples of solutions of the Dirichlet problem for some nice domains.

4.1. **Invariance property of the functional** J.

In addition to minimizing J, using Theorem 4.1, one can investigate the dependence of J on G. Now we show the translation invariance property of the functional J. Before we show the property, recall that $J(\mu_G)$ is given by,

$$J(\mu_G) = \int_{\mathbb{C}} |z|^2 d\mu_G(z) - \iint_{\mathbb{C}^2} \log|z - w| d\mu_G(z) d\mu_G(w) = \int_{\mathbb{C}} (|z|^2 - U_{\mu_G}(z)) d\mu_G(z).$$

by the definition of the logarithmic potential. Since μ_G is supported only on $D\backslash G$, the logarithmic potential $U_{\mu_G}(z)$ takes on the value $\frac{|z|^2}{2} - \frac{1}{2}$ on the support of μ_G. Using this fact, we find

$$J(\mu_G) = \int \left(|z|^2 - \left(\frac{|z|^2}{2} - \frac{1}{2} \right) \right) d\mu_G(z) = \int \left(\frac{|z|^2}{2} - \frac{1}{2} \right) d\mu_G(z) + 1.$$

Notice that the latter integrand is equivalent to $U_{\nu_{eq}}(z)$, and we can rewrite the last expression as

$$\iint \log|z-w| d\nu_{eq}(w) d\mu_G(z) + 1.$$

Changing the order of integration, we get

$$J(\mu_G) = \int U_{\mu_G}(z) d\nu_{eq}(z) + 1.$$

Thus,

$$J(\mu_G) - J(\nu_{eq}) = \int U_{\mu_G}(z) d\nu_{eq}(z) - \int U_{\nu_{eq}}(z) d\nu_{eq}(z) = \int_G \left(U_{\mu_G}(z) - U_{\nu_{eq}}(z) \right) d\nu_{eq}(z).$$

To show that this value is translation invariant with respect to shifts of the domain G, we wish to show $J(\mu_G) = J(\mu_{G_a})$, where $G_a = \{z' \in D | z' = z + a, z \in G\}$ and $G_a \subset D$. Note that this is equivalent to showing,

$$U_{\mu_G}(z) - U_{\nu_{eq}}(z) = U_{\mu_{G_a}}(z+a) - U_{\nu_{eq}}(z+a).$$

By the definition of the logarithmic potential,

$$\begin{aligned} U_{\mu_G}(z) - U_{\nu_{eq}}(z) &= \int_{G^c} \log|z-w| \frac{dA(w)}{\pi} + \int_{\partial G} \log|z-w| d\mu_G^s(w) - \int_{\mathbb{C}} \log|z-w| \frac{dA(w)}{\pi} \\ &= \int_{\partial G} \log|z-w| d\mu_G^s(w) - \int_G \log|z-w| \frac{dA(w)}{\pi}. \end{aligned}$$

Now we must show that

$$\int_{\partial G} \log|z-w| d\mu_G^s(w) - \int_G \log|z-w| \frac{dA(w)}{\pi} = \int_{\partial G_a} \log|z'-w| d\mu_{G_a}^s(w) - \int_{G_a} \log|z'-w| \frac{dA(w)}{\pi}.$$

where $z' = z + a$. Since $z' \in D$,

$$\int_G \log|z-w| \frac{dA(w)}{\pi} = \int_{G_a} \log|z'-w| \frac{dA(w)}{\pi}.$$

Thus all we need to show is

$$\int_{\partial G} \log|z-w| \; d\mu_G^s(w) = \int_{\partial G_a} \log|z+a-w| \; d\mu_{G_a}^s(w).$$

Recall that the singular part of μ_G is the balayage of the uniform measure, and therefore, it is invariant under translations. Hence the functional J is translation invariant.

5. Examples of the Balayage Method

For certain domains, solving the Dirichlet problem in an analytical way is possible. Here, we provide two examples where we use the balayage method to find the optimal boundary distribution.

5.1. **Disk of radius r.** Suppose the domain G is a disk with radius r and center $(x_0, 0)$. Without loss of generality, we assume x_0 to be positive. The first step of the balayage method is to solve a following Dirichlet problem.

Conditions for Dirichlet problem

(1) $\Delta W(z) = 0, z \in G$
(2) $W(z) = U_{\nu_{eq}}(z)$, $z \in \partial G$

Note that $U_{\nu_{eq}}(z) = \frac{|z|^2}{2} - \frac{1}{2}$, where ν_{eq} is an equilibrium measure. Consider a function $W(z) = \frac{r^2 - {x_0}^2 - 1}{2} + x_0 \cdot x$, where $z = x + yi \in \mathbb{C}$. The function W clearly satisfies both conditions of the Dirichlet problem above. This function can be obtained either by directly deriving from the equation for G or using the Poisson integral formula. We demonstrate the balayage method for the aforementioned function and use Poisson integral formula to check the answer. The function W is

$$W(z) = \begin{cases} \frac{r^2 - {x_0}^2 - 1}{2} + x_0 \cdot x & z \in G, \\ \frac{|z|^2}{2} - \frac{1}{2} & z \in D \setminus G, \\ \log |z| & z \in \mathbb{C} \setminus D. \end{cases}$$

The normal derivatives $\frac{\partial w}{\partial n_+}$ and $\frac{\partial w}{\partial n_-}$ are given below:

$$\frac{\partial w}{\partial n_+} = \nabla W \cdot \vec{n}_+ = (x, y) \cdot \left(\frac{x - x_0}{r}, \frac{y}{r} \right) = \frac{x^2 - x_0 \cdot x}{r} + \frac{y^2}{r},$$
$$\frac{\partial w}{\partial n_-} = \nabla W \cdot \vec{n}_- = (x_0, 0) \cdot \left(\frac{x_0 - x}{r}, \frac{-y}{r} \right) = \frac{{x_0}^2 - x_0 \cdot x}{r}.$$

Therefore, the singular part μ_G^s along the boundary is given by

$$\mu_G^s = \frac{1}{2\pi} \cdot \left(\frac{\partial w}{\partial n_+} + \frac{\partial w}{\partial n_-} \right) = \frac{1}{4\pi} ds,$$

where ds is an arc length measure. Hence, for a disk, as expected, the optimal distribution along the boundary is just the constant density depending on r, which is roughly 0.08. One can see this in Figure 2.

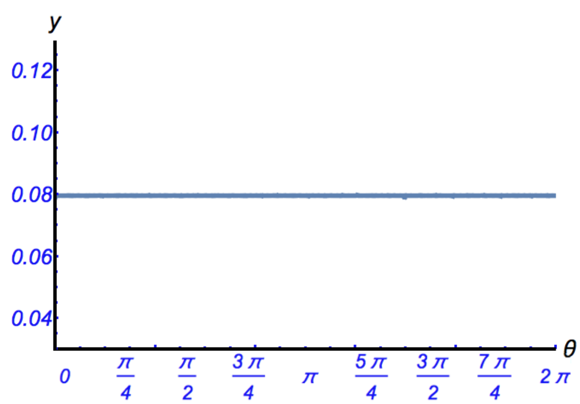
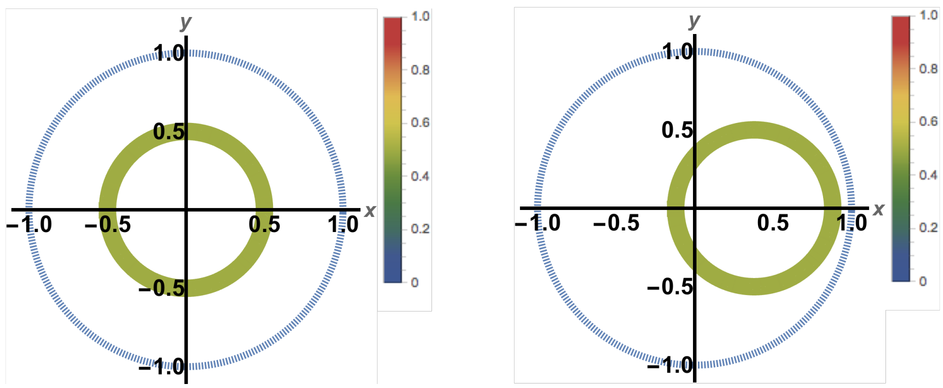


FIGURE 2. The optimal distribution plot for circle

The reason for the small oscillation in the graph is because of Mathematica's parameterization of the circle and because Mathematica approximates the solution to the Dirichlet problem. Mathematica parameterizes the circle by ordering the points of the circle by angle from the center. This leads to a parameterization that is not quite the circle, but instead a close polygonal approximation.

Here, we used 534 points for the circle. The solution $W(z)$ of the Dirichlet problem is a surface, and because Mathematica makes a mesh of this surface using many triangles, treats the circle as a polygon (discretization error), and uses Lagrange Interpolation to smooth out the surface from a function defined only on a set of nodes, the partial of $W(z)$ with respect to the outer and inner normal vectors varies along the boundary. Additionally, Mathematica treats all terms numerically in solving the Dirichlet (roundup error) and $W(z)$ is not differentiable on the boundary, giving rise to yet another source of error. Thus the reason for the slight oscillation. If Mathematica used more and more, smaller, triangles to approximate the solution, the oscillation would decrease to the value μ which we obtained above analytically.



(A) (B)

FIGURE 3. The optimal distribution plotted for circle using a heat map

From Figure 3, heuristically, we can see the distribution of eigenvalues is roughly constant (with a margin of error of ± 0.0004). Recall Figure 2, which shows the distribution plot for the circle; regardless of the θ, the dependent variable y was roughly constant with minor oscillation. In the case of Figure 3, the y-value is the numerical representation of the color (the heat map value). Since this value is constant, the resulting color is also constant.

5.2. **Ellipse.** Now suppose our domain G is an ellipse with center $(x_0, 0)$. More formally one can write this as

$$G = \left\{ (x,y) \;\middle|\; \frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$$

It is known that the solution of the corresponding Dirichlet problem is a polynomial of degree 2 in x, y. The solution that satisfies is

$$W(z) = \begin{cases} \frac{a^2-b^2}{2(a^2+b^2)}x^2 - \frac{a^2-b^2}{2(a^2+b^2)}y^2 + \frac{2{b_0}^2x_0}{2(a^2+b^2)}x + \frac{2a^2b^2-2b^2{x_0}^2-a^2-b^2}{2(a^2+b^2)} & z \in G, \\ \frac{|z|^2}{2} - \frac{1}{2} & z \in D \setminus G, \\ \log|z| & z \in \mathbb{C} \setminus D \end{cases}$$

Note that $\vec{n}_+ = \frac{1}{C} \cdot (\frac{b(x-x_0)}{a}, \frac{ay}{b})$ and $\vec{n}_- = \frac{1}{C} \cdot (\frac{b(x_0-x)}{a}, \frac{-ay}{b})$ where $C = \sqrt{\frac{b^2(x-x_0)^2}{a} + \frac{a^2y^2}{b}}$. Therefore we can recover an optimal measure μ along the boundary,

$$\mu_s = \frac{1}{2\pi C} \cdot \left((x,y) \cdot \left(\frac{b(x-x_0)}{a}, \frac{ay}{b} \right) + \left(\frac{a^2-b^2}{a^2+b^2}x + \frac{2{b_0}^2x_0}{a^2+b^2}, \frac{b^2-a^2}{a^2+b^2} \right) \cdot \left(\frac{b(x_0-x)}{a}, \frac{-ay}{b} \right) \right) ds,$$

which can be further simplified to

$$\frac{1}{2\pi C} \cdot \left(\frac{bx^2 - bxx_0}{a} + \frac{ay^2}{b} + \frac{(a^2b - 3b^3)x_0x + 2b^3x_0^2 + (b^3 - a^2b)x^2}{a^3 + ab^2} + \frac{a^3y - ab^2y}{a^2b + b^3} \right).$$

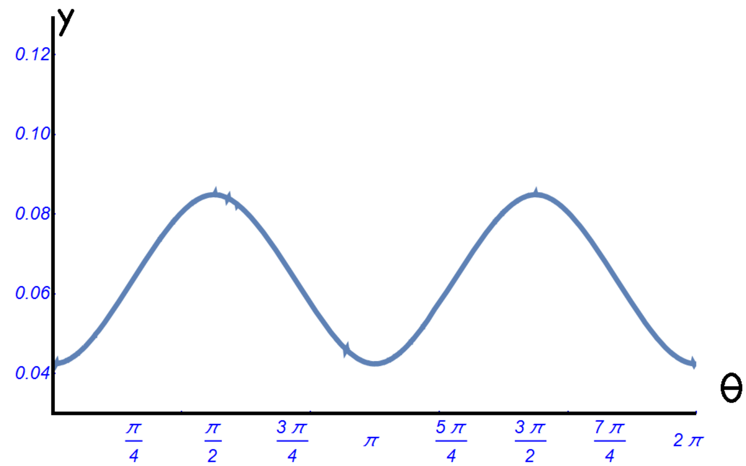
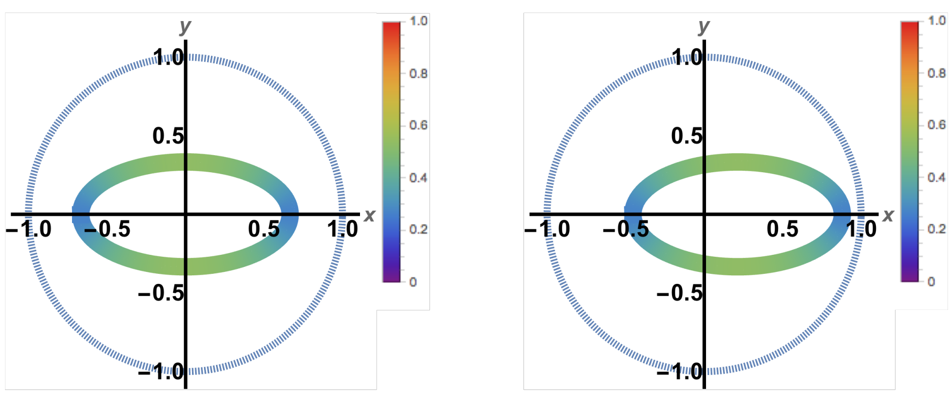


FIGURE 4. The optimal distribution plot for ellipse

One can see that, unlike the case of disk, the optimal limiting distribution density along the boundary varies with the location. Figure 4 shows how the uniform density changes for the ellipse $\frac{9x^2}{4}+9y^2 = 1$. Figure 4 is the solution of the Dirichlet problem without solving for μ_s analytically, as was done above. Like the case for the circle, Mathematica parameterizes the ellipse by ordering the points by angle from the center, again constructing a discrete approximation of the ellipse with 500 boundary points. Mathematica solves the Dirichlet problem by approximating the solution surface $W(z)$ by using a mesh of many triangles, and using Lagrange Interpolation to achieve the best smooth surface. Because of these two computational simplifications, the inner and outer normals of $W(z)$ are approximated by consequence. The scale in Figure 4 is larger than that of Figure 2, so the oscillation does not appear to be as large, but some oscillatory features are still noticeable at 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.



(A)

(B)

FIGURE 5. The optimal distribution plotted for ellipse in heat map

In Figure 5, the limiting distribution oscillates along the boundary of the ellipse, with the most eigenvalues falling on the top or the bottom of the ellipse. The density of these eigenvalues tapers off to a minimum on the right and left sides of the ellipse. Recall in Figure 4, that the distribution is at a maximum when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ and at a minimum when $\theta = 0, \pi$. This is exactly the case in Figure 5, since this figure is merely the visual representation of this distribution plotted along the boundary ellipse. From these figures, we can also observe that the points close to the center of the ellipse have higher density values.

6. Acknowledgements

We first thank our two mentors, Doctor Diego Ayala and Doctor Alon Nishry for their persistent guiding. We also express our gratitude to the University of Michigan department of mathematics and the research experience for undergraduates committee for this valuable opportunity.

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Figure 1 code

```
(1)   m = RandomReal[NormalDistribution[0, 1/Sqrt[2]], {1000, 1000}] +
   I*RandomReal[NormalDistribution[0, 1/Sqrt[2]], {1000, 1000}];
   // m is a random normal distribution with 1000 values
(2)   e = Eigenvalues[m];
   // e is the set of eigenvalues of m
(3)   e1 = e/Sqrt[1000];
   // e1 normalizes e so that the majority of eigenvalues are less than 1
(4)   Show[{ListPlot[{{Re[#], Im[#]} & /@ e1}, AspectRatio -> 1],
  ContourPlot[x^2 + y^2 == 1, {x, -1, 1}, {y, -1, 1}]}]
   // this plots e1, the set of eigenvalues, within the unit circle
```

Figure 2 code

```
(1)   reg = Disk[{0, 0}, 1/2];
   // initializes a circle with radius 1/2
(2)   discReg = BoundaryDiscretizeRegion[reg, AccuracyGoal -> 5]; sol =
 NDSolveValue[{\!\(
\*SubsuperscriptBox[\(\[Del]\), \({x, y}\), \(2\)]\(u[x, y]\)\) ==
    0,
     DirichletCondition[u[x, y] == (x^2 + y^2)/2 - 1/2, True]},
  u, {x, y} \[Element] discReg, AccuracyGoal -> 20,
  PrecisionGoal -> 35];
  // this function discretizes the region "reg" such that it satisfies the Dirichlet
      condition
(3)   centerX = 0;
centerY = 0; bdryPts =
 MeshCoordinates[BoundaryDiscretizeRegion[reg, AccuracyGoal -> 5]];
bdryPts =
 Sort[bdryPts,
  ArcTan[#1[[1]] - centerX, #1[[2]] - centerY] <
    ArcTan[#2[[1]] - centerX, #2[[2]] - centerY] &]; bdryPtsX =
 Array[{(# - 1)/(Length[bdryPts] - 1), bdryPts[[#, 1]]} &,
  Length[bdryPts]];
bdryPtsY =
 Array[{(# - 1)/(Length[bdryPts] - 1), bdryPts[[#, 2]]} &,
  Length[bdryPts]]; paramX =
 Interpolation[bdryPtsX, InterpolationOrder -> 1];
paramY = Interpolation[bdryPtsY, InterpolationOrder -> 1];
pp[x_] = {paramX[x], paramY[x]};
ppt[x_] = {D[paramX[x], x], D[paramY[x], x]};
ppn[x_] = {D[paramY[x], x], -D[paramX[x], x]};
ppnn[x_] = Normalize[ppn[x]];
  // this body of code gives us the 500 or so boundary points as well as the inner and
      outer normals of "reg" after discretization
(4)   solGrad[x_, y_] = {D[sol[x, y], x], D[sol[x, y], y]};
outGrad[x_, y_] = {x, y};
(5)   dens[x_] = -solGrad[pp[x][[1]], pp[x][[2]]].ppnn[x] +
   outGrad[pp[x][[1]], pp[x][[2]]].ppnn[x];
   // this gives us the density function that is plotted with a constant in Figures 2 &
       4.
(6)   ListPlot[Sort[bdryPts,
  ArcTan[#1[[1]], #1[[2]]] < ArcTan[#2[[1]], #2[[2]]] &]]
```

```
(7)   Plot[(1/(2*Pi)) dens[u], {u, 0, 1}, AxesStyle -> Thickness[0.01],
 AxesLabel -> {Style[x, Bold, FontSize -> 20],
   Style[y, Bold, FontSize -> 20]},
 TicksStyle -> Directive[Black, Thick, Bold, 20],
 PlotRange -> {{0, 1}, {0.07965 - 1/150, 0.07965 + 1/150}},
 PlotStyle -> Thick]
  // This plots the distribution used in Figures 2 & 4.
```

Figure 3 code
Part A

```
// much of the code is the same, except for the numbers that are changed to adjust for
    the boundary shape.
(1)   reg = Disk[{0, 0}, 1/2];
   // initializes a circle with radius 1/2
(2)   discReg = BoundaryDiscretizeRegion[reg, AccuracyGoal -> 5]; sol =
 NDSolveValue[{\!\(
\*SubsuperscriptBox[\(\[Del]\), \({x, y}\), \(2\)]\(u[x, y]\)\) ==
    0,
     DirichletCondition[u[x, y] == (x^2 + y^2)/2 - 1/2, True]},
  u, {x, y} \[Element] discReg, AccuracyGoal -> 20,
  PrecisionGoal -> 35];
   // this function discretizes the region "reg" such that it satisfies the Dirichlet
       condition
(3)   centerX = 0;
centerY = 0; bdryPts =
 MeshCoordinates[BoundaryDiscretizeRegion[reg, AccuracyGoal -> 5]];
bdryPts =
 Sort[bdryPts,
  ArcTan[#1[[1]] - centerX, #1[[2]] - centerY] <
    ArcTan[#2[[1]] - centerX, #2[[2]] - centerY] &]; bdryPtsX =
 Array[{(# - 1)/(Length[bdryPts] - 1), bdryPts[[#, 1]]} &,
  Length[bdryPts]];
bdryPtsY =
 Array[{(# - 1)/(Length[bdryPts] - 1), bdryPts[[#, 2]]} &,
  Length[bdryPts]]; paramX =
 Interpolation[bdryPtsX, InterpolationOrder -> 1];
paramY = Interpolation[bdryPtsY, InterpolationOrder -> 1];
pp[x_] = {paramX[x], paramY[x]};
ppt[x_] = {D[paramX[x], x], D[paramY[x], x]};
ppn[x_] = {D[paramY[x], x], -D[paramX[x], x]};
ppnn[x_] = Normalize[ppn[x]];
   // this body of code gives us the 500 or so boundary points as well as the inner and
       outer normals of "reg" after discretization
(4)   solGrad[x_, y_] = {D[sol[x, y], x], D[sol[x, y], y]};
outGrad[x_, y_] = {x, y};
(5)   dens[x_] = -solGrad[pp[x][[1]], pp[x][[2]]].ppnn[x] +
   outGrad[pp[x][[1]], pp[x][[2]]].ppnn[x];
   // this gives us the density function of the eigenvalues.
(6)   ListPlot[Sort[bdryPts,
  ArcTan[#1[[1]], #1[[2]]] < ArcTan[#2[[1]], #2[[2]]] &]]

(7)   plt = ParametricPlot[pp[u], {u, 0, 1}, PlotStyle -> Thickness[0.05],
```

```
   ColorFunction ->
    Function[{x, y, u}, ColorData["DarkRainbow"][dens[u]]]];
Show[ParametricPlot[{Cos[2*Pi*x], Sin[2*Pi*x]}, {x, 0, 1},
  PlotStyle -> Thickness[0.02], AxesStyle -> Thickness[0.01],
  AxesLabel -> {Style[x, Bold, FontSize -> 54],
    Style[y, Bold, FontSize -> 54]},
  TicksStyle -> Directive[Black, Thick, Bold, 60]],
 ParametricPlot[pp[u], {u, 0, 1}, PlotStyle -> Thickness[0.05],
  ColorFunction ->
   Function[{x, y, u}, ColorData["DarkRainbow"][dens[u]]]]]
   // This plots the distribution and the circle, coloring the circle to describe the
       distribution of eigenvalues.
```

Part B

```
// much of the code is the same, except for the numbers that are changed to adjust for
    the boundary shape.
(1)    reg = Disk[{1/3, 0}, 1/2];
   // plots a circle with radius 1/2 centered at (1/3, 0)
(2)    discReg = BoundaryDiscretizeRegion[reg, AccuracyGoal -> 5]; sol =
 NDSolveValue[{\!\(
\*SubsuperscriptBox[\(\[Del]\), \({x, y}\), \(2\)]\(u[x, y]\)\) ==
    0,
     DirichletCondition[u[x, y] == (x^2 + y^2)/2 - 1/2, True]},
  u, {x, y} \[Element] discReg, AccuracyGoal -> 20,
  PrecisionGoal -> 35];
  // this function discretizes the region "reg" such that it satisfies the Dirichlet
      condition
(3)    centerX = 0;
centerY = 0; bdryPts =
 MeshCoordinates[BoundaryDiscretizeRegion[reg, AccuracyGoal -> 5]];
bdryPts =
 Sort[bdryPts,
  ArcTan[#1[[1]] - centerX, #1[[2]] - centerY] <
    ArcTan[#2[[1]] - centerX, #2[[2]] - centerY] &]; bdryPtsX =
 Array[{(# - 1)/(Length[bdryPts] - 1), bdryPts[[#, 1]]} &,
  Length[bdryPts]];
bdryPtsY =
 Array[{(# - 1)/(Length[bdryPts] - 1), bdryPts[[#, 2]]} &,
  Length[bdryPts]]; paramX =
 Interpolation[bdryPtsX, InterpolationOrder -> 1];
paramY = Interpolation[bdryPtsY, InterpolationOrder -> 1];
pp[x_] = {paramX[x], paramY[x]};
ppt[x_] = {D[paramX[x], x], D[paramY[x], x]};
ppn[x_] = {D[paramY[x], x], -D[paramX[x], x]};
ppnn[x_] = Normalize[ppn[x]];
  // this body of code gives us the 500 or so boundary points as well as the inner and
      outer normals of "reg" after discretization
(4)    solGrad[x_, y_] = {D[sol[x, y], x], D[sol[x, y], y]};
outGrad[x_, y_] = {x, y};

(5)    dens[x_] = -solGrad[pp[x][[1]], pp[x][[2]]].ppnn[x] +
   outGrad[pp[x][[1]], pp[x][[2]]].ppnn[x];
   // this gives us the density function of the eigenvalues.
```

```
(6)   ListPlot[Sort[bdryPts,
  ArcTan[#1[[1]], #1[[2]]] < ArcTan[#2[[1]], #2[[2]]] &]]

(7)   plt = ParametricPlot[pp[u], {u, 0, 1}, PlotStyle -> Thickness[0.05],
   ColorFunction ->
    Function[{x, y, u}, ColorData["DarkRainbow"][dens[u]]]];
Show[ParametricPlot[{Cos[2*Pi*x], Sin[2*Pi*x]}, {x, 0, 1},
  PlotStyle -> Thickness[0.02], AxesStyle -> Thickness[0.01],
  AxesLabel -> {Style[x, Bold, FontSize -> 54],
    Style[y, Bold, FontSize -> 54]},
  TicksStyle -> Directive[Black, Thick, Bold, 60]],
 ParametricPlot[pp[u], {u, 0, 1}, PlotStyle -> Thickness[0.05],
  ColorFunction ->
   Function[{x, y, u}, ColorData["DarkRainbow"][dens[u]]]]]
   // This plots the distribution and the circle, coloring the circle to describe the
       distribution of eigenvalues.
```

Figure 4 code

```
// much of the code is the same, except for the numbers that are changed to adjust for
    the boundary shape.
(1)   reg = Disk[{0, 0}, {2/3, 1/3}];
   // plots an ellipse, width 2/3, height 1/3
(2)   discReg = BoundaryDiscretizeRegion[reg, AccuracyGoal -> 5]; sol =
 NDSolveValue[{\!\(
\*SubsuperscriptBox[\(\[Del]\), \({x, y}\), \(2\)]\(u[x, y]\)\) ==
    0,
     DirichletCondition[u[x, y] == (x^2 + y^2)/2 - 1/2, True]},
  u, {x, y} \[Element] discReg, AccuracyGoal -> 20,
  PrecisionGoal -> 35];
  // this function discretizes the region "reg" such that it satisfies the Dirichlet
      condition
(3)   centerX = 0;
centerY = 0; bdryPts =
 MeshCoordinates[BoundaryDiscretizeRegion[reg, AccuracyGoal -> 5]];
bdryPts =
 Sort[bdryPts,
  ArcTan[#1[[1]] - centerX, #1[[2]] - centerY] <
    ArcTan[#2[[1]] - centerX, #2[[2]] - centerY] &]; bdryPtsX =
 Array[{(# - 1)/(Length[bdryPts] - 1), bdryPts[[#, 1]]} &,
  Length[bdryPts]];
bdryPtsY =
 Array[{(# - 1)/(Length[bdryPts] - 1), bdryPts[[#, 2]]} &,
  Length[bdryPts]]; paramX =
 Interpolation[bdryPtsX, InterpolationOrder -> 1];
paramY = Interpolation[bdryPtsY, InterpolationOrder -> 1];
pp[x_] = {paramX[x], paramY[x]};
ppt[x_] = {D[paramX[x], x], D[paramY[x], x]};
ppn[x_] = {D[paramY[x], x], -D[paramX[x], x]};
ppnn[x_] = Normalize[ppn[x]];
   // this body of code gives us the 500 or so boundary points as well as the inner and
       outer normals of "reg" after discretization
(4)   solGrad[x_, y_] = {D[sol[x, y], x], D[sol[x, y], y]};
outGrad[x_, y_] = {x, y};
```

```
(5)   dens[x_] = -solGrad[pp[x][[1]], pp[x][[2]]].ppnn[x] +
   outGrad[pp[x][[1]], pp[x][[2]]].ppnn[x];
   // this gives us the density function of the eigenvalues.
(6)   ListPlot[Sort[bdryPts,
  ArcTan[#1[[1]], #1[[2]]] < ArcTan[#2[[1]], #2[[2]]] &]]

(7)   Plot[(1/(2*Pi)) dens[u], {u, 0, 1}, AxesStyle -> Thickness[0.01],
 AxesLabel -> {Style[x, Bold, FontSize -> 20],
   Style[y, Bold, FontSize -> 20]},
 TicksStyle -> Directive[Black, Thick, Bold, 20],
 PlotRange -> {{0, 1}, {0.065 - 1/15, 0.065 + 1/15}},
 PlotStyle -> Thick]
 // This plots the distribution of eigenvalues of the ellipse.
```

Figure 5 code
Part A

```
// much of the code is the same, except for the numbers that are changed to adjust for
    the boundary shape.
(1)   reg = Disk[{0,0},{2/3,1/3}];
   // plots an ellipse, width 2/3, height 1/3
(2)   discReg = BoundaryDiscretizeRegion[reg, AccuracyGoal -> 5]; sol =
 NDSolveValue[{\!\(
\*SubsuperscriptBox[\(\[Del]\), \({x, y}\), \(2\)]\(u[x, y]\)\) ==
    0,
     DirichletCondition[u[x, y] == (x^2 + y^2)/2 - 1/2, True]},
  u, {x, y} \[Element] discReg, AccuracyGoal -> 20,
  PrecisionGoal -> 35];
  // this function discretizes the region "reg" such that it satisfies the Dirichlet
      condition
(3)   centerX = 0;
centerY = 0; bdryPts =
 MeshCoordinates[BoundaryDiscretizeRegion[reg, AccuracyGoal -> 5]];
bdryPts =
 Sort[bdryPts,
  ArcTan[#1[[1]] - centerX, #1[[2]] - centerY] <
    ArcTan[#2[[1]] - centerX, #2[[2]] - centerY] &]; bdryPtsX =
 Array[{(# - 1)/(Length[bdryPts] - 1), bdryPts[[#, 1]]} &,
  Length[bdryPts]];
bdryPtsY =
 Array[{(# - 1)/(Length[bdryPts] - 1), bdryPts[[#, 2]]} &,
  Length[bdryPts]]; paramX =
 Interpolation[bdryPtsX, InterpolationOrder -> 1];
paramY = Interpolation[bdryPtsY, InterpolationOrder -> 1];
pp[x_] = {paramX[x], paramY[x]};
ppt[x_] = {D[paramX[x], x], D[paramY[x], x]};
ppn[x_] = {D[paramY[x], x], -D[paramX[x], x]};
ppnn[x_] = Normalize[ppn[x]];
   // this body of code gives us the 500 or so boundary points as well as the inner and
       outer normals of "reg" after discretization
(4)   solGrad[x_, y_] = {D[sol[x, y], x], D[sol[x, y], y]};
outGrad[x_, y_] = {x, y};
```

```
(5)    dens[x_] = -solGrad[pp[x][[1]], pp[x][[2]]].ppnn[x] +
   outGrad[pp[x][[1]], pp[x][[2]]].ppnn[x];
   // this gives us the density function of the eigenvalues.
(6)    ListPlot[Sort[bdryPts,
  ArcTan[#1[[1]], #1[[2]]] < ArcTan[#2[[1]], #2[[2]]] &]]

(7)    plt = ParametricPlot[pp[u], {u, 0, 1}, PlotStyle -> Thickness[0.05],
   ColorFunction ->
    Function[{x, y, u}, ColorData["Rainbow"][dens[u]]]];
Show[ParametricPlot[{Cos[2*Pi*x], Sin[2*Pi*x]}, {x, 0, 1},
  PlotStyle -> Thickness[0.02], AxesStyle -> Thickness[0.01],
  AxesLabel -> {Style[x, Bold, FontSize -> 54],
    Style[y, Bold, FontSize -> 54]},
  TicksStyle -> Directive[Black, Thick, Bold, 60]],
 ParametricPlot[pp[u], {u, 0, 1}, PlotStyle -> Thickness[0.05],
  ColorFunction ->
   Function[{x, y, u}, ColorData["Rainbow"][dens[u]]]]]
   // This plots the distribution and the ellipse, coloring the ellipse to describe the
       distribution of eigenvalues.
```

Part B

```
// much of the code is the same, except for the numbers that are changed to adjust for
    the boundary shape.
(1)    reg = Disk[{1/3 - 0.06,0},{2/3,1/3}];
  // plots an ellipse, width 2/3, height 1/3. The ellipse is shifted to the right by
      approximately 1/3
(2)    discReg = BoundaryDiscretizeRegion[reg, AccuracyGoal -> 5]; sol =
 NDSolveValue[{\!\(
\*SubsuperscriptBox[\(\[Del]\), \({x, y}\), \(2\)]\(u[x, y]\)\) ==
    0,
     DirichletCondition[u[x, y] == (x^2 + y^2)/2 - 1/2, True]},
  u, {x, y} \[Element] discReg, AccuracyGoal -> 20,
  PrecisionGoal -> 35];
  // this function discretizes the region "reg" such that it satisfies the Dirichlet
      condition
(3)    centerX = 0;
centerY = 0; bdryPts =
 MeshCoordinates[BoundaryDiscretizeRegion[reg, AccuracyGoal -> 5]];
bdryPts =
 Sort[bdryPts,
  ArcTan[#1[[1]] - centerX, #1[[2]] - centerY] <
    ArcTan[#2[[1]] - centerX, #2[[2]] - centerY] &]; bdryPtsX =
 Array[{(# - 1)/(Length[bdryPts] - 1), bdryPts[[#, 1]]} &,
  Length[bdryPts]];
bdryPtsY =
 Array[{(# - 1)/(Length[bdryPts] - 1), bdryPts[[#, 2]]} &,
  Length[bdryPts]]; paramX =
 Interpolation[bdryPtsX, InterpolationOrder -> 1];
paramY = Interpolation[bdryPtsY, InterpolationOrder -> 1];
pp[x_] = {paramX[x], paramY[x]};
ppt[x_] = {D[paramX[x], x], D[paramY[x], x]};
ppn[x_] = {D[paramY[x], x], -D[paramX[x], x]};
ppnn[x_] = Normalize[ppn[x]];
```

```
   // this body of code gives us the 500 or so boundary points as well as the inner and
       outer normals of "reg" after discretization
(4)    solGrad[x_, y_] = {D[sol[x, y], x], D[sol[x, y], y]};
outGrad[x_, y_] = {x, y};

(5)    dens[x_] = -solGrad[pp[x][[1]], pp[x][[2]]].ppnn[x] +
   outGrad[pp[x][[1]], pp[x][[2]]].ppnn[x];
   // this gives us the density function of the eigenvalues.
(6)    ListPlot[Sort[bdryPts,
  ArcTan[#1[[1]], #1[[2]]] < ArcTan[#2[[1]], #2[[2]]] &]]

(7)    plt = ParametricPlot[pp[u], {u, 0, 1}, PlotStyle -> Thickness[0.05],
   ColorFunction ->
    Function[{x, y, u}, ColorData["Rainbow"][dens[u]]]];
Show[ParametricPlot[{Cos[2*Pi*x], Sin[2*Pi*x]}, {x, 0, 1},
  PlotStyle -> Thickness[0.02], AxesStyle -> Thickness[0.01],
  AxesLabel -> {Style[x, Bold, FontSize -> 54],
    Style[y, Bold, FontSize -> 54]},
  TicksStyle -> Directive[Black, Thick, Bold, 60]],
 ParametricPlot[pp[u], {u, 0, 1}, PlotStyle -> Thickness[0.05],
  ColorFunction ->
   Function[{x, y, u}, ColorData["Rainbow"][dens[u]]]]]
   // This plots the distribution and the ellipse, coloring the ellipse to describe the
       distribution of eigenvalues.
```