# Coherent Configuration on Infinite Set

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Concept of coherent configuration arises from research of permutation groups which act on a finite set. By adapting the idea to infinite set of real line and modifying its definition with Euler characteristics, we look for the analogues and differences between this algebra and the finite-set case.

## 1 Background

### 1.1 Definition

Let $\Omega$ be a **finite** set. A **coherent configuration** on $\Omega$ is a set $P = \{R_1,...,R_s\}$ of binary relations on $\Omega$ satisfying the following conditions:

(a) $P$ is a partition of $\Omega^2$;

(b) there is a subset $P_0$ of $P$, a partition of the diagonal $\Delta$=$\{(\alpha,\alpha)$:$\alpha \in \Omega\}$;

(c) for every relation $R_i \in P$, its converse $R_i^T = \{(\beta, \alpha) : (\alpha, \beta) \in R_i\}$ is in $P$, saying $R_i^T = R_{i^\star}$;

(d) there exists integer $p_{ij}^k$ for $1 \leq i, j, k \leq s$ and any $(\alpha, \beta) \in R_k$, the number of points $\gamma \in \Omega$ such that $(\alpha, \gamma) \in R_i$ and$(\gamma, \beta) \in R_j$ is equal to $p_{ij}^k$, namely intersection number.

## 1.2 Algebra

We can represent a binary relation $R$ on $\Omega$ by its basis matrix $A(R)$ where

$$A(R)_{(i,j)} = \begin{cases} 0, & \text{if } (i,j) \in R \\ 1, & \text{otherwise} \end{cases} \tag{1}$$

***Properties:***

(a)$\sum_{i=1}^{s} A(R_i)$ is the all-1 matrix.

(b)$\sum_{i=1}^{t} A(R_t)$ is the identity matrix, where $\{R_1, ..., R_t\}$ is the subset of $P$ corresponding to diagonal $\Delta$.

(c)for each $i$, there exists $i^*$ such that $A(R_i)^T = A(R_{i^*})$.

(d)for each pair $(i,j)$, we have $A(R_i)A(R_j) = \sum_{k=1}^{s} p_{ij}^{k} A(R_k)$.

*Remark:*

By (b) and (d) above, we have an algebra (a set closed under addition, scalar multiplication and bilinear product map *i.e.*matrix multiplication) generated by $A(R_1), ..., A(R_i)$ on $\mathbb{C}$, namely basis algebra of this Coherent Configuration $P$, denoted by $\mathfrak{BA}(P)$.

By Wedderburn's theorem, $\mathfrak{BA}(P) \tilde{\rightarrow} \bigoplus_i M_i$ where $\sum_i i^2 = s$, and $M_i$'s are complete matrix algebras over $\mathbb{C}$.

$$\begin{pmatrix} M_{i_1} & & & \\ & M_{i_2} & & \\ & & & \\ & & & M_{i_n} \end{pmatrix}$$

## 1.3 Permutation Groups and Coherent Configurations

*Relationship:*

·Permutation $\Rightarrow$ coherent configuration

Given a permutation $\Phi$ of a set $\Omega$, the partition of $\Omega^2$ into orbits of $\Phi$ (acting on two components respectively) is a coherent configuration.

·Coherent configuration $\not\Rightarrow$ permutation

If a coherent configuration can be connected to some permutation on a set in the way above, then it's called **Schurian**.

*Note*: The smallest non-Schurian coherent configuration has 14 points by Mikhai Klim's paper in August 2016, which is proved by computer enumeration. There is no general solution to identify whether a coherent configuration is Schurian or not so far.

## 1.4 Automorphism and Weak-Automorphism

*Definition:*

**automorphism:**

a permutation of $\Omega$ which fixes every set in $P$;

**weak-automorphism:**

a permutation of $\Omega$ which maps each member of $P$ to a member of $P$.

*Remark:*

·Both automorphisms and weak-automophisms of a partition form a group;

·If $f$ is a weak-automophism, then $f$ induces a bijection $g$ from $P$ to $P$.

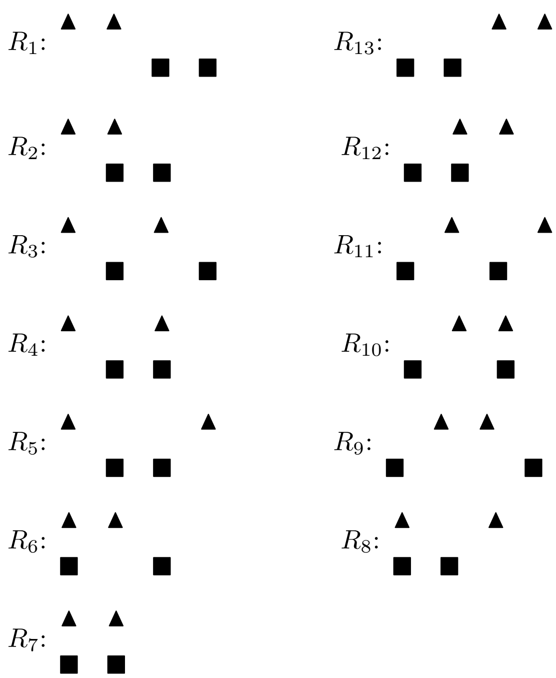
# 2 Extend Coherent Configuration to Infinite Set

## 2.1 $\Omega = \mathbb{R}$

Partition $\mathbb{R}^2$ into diagonal, upper semi-plane and lower semi-plane, and extend the concept of intersection number into the category of range of $\gamma \in \mathbb{R}$ (we will discuss it later), an automorphism of this *coherent configuration* is the order isomorphism of the real line (considering how $\mathbb{R}^2$ points are related).

## 2.2 $\Omega = \{(\alpha, \beta) | \alpha < \beta \in \mathbb{R}\}$

We make group act on pairs and consider how two-point sets may relate and there are thirteen cases ($\alpha$ is denoted by ▲ and $\beta$ by ■):



*Remark:*

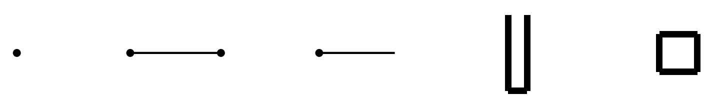
$\cdot$This is a partition of $\Omega^2$.

$\cdot R_7 = \Delta$, thus this is homogeneous.

$\cdot R_i$ has converse $R_{14-i}$.

$\cdot P_{ij}^k$ acts as intersection number where $R_iR_j = \sum_{k=1}^{13} P_{ij}^k R_k$ (details later).

Introduced in 2.1 case $\Omega = \mathbb{R}$, we try to extend the concept of intersection number $P_{ij}^k$ to polyhedron geometrically which reflects the range of $\gamma \in \Omega$, all possible ranges of $\gamma$ on $\mathbb{R}^2$ are as following:



(last two pictures by thick line contains its closed edges and internal area)

By introducing the idea of S.Schanuel[1], we can transfer $\gamma$'s from geometry to algebra, which furthermore helps to extend the concept of coherent configuration to infinite set.

In a category $\mathbb{P}$ whose objects are bounded polyhedrons (boolean algebra generated by affine functions $f(x_1.x_2, ..., x_n) > 0$), we use Burnside rig to classify objects:

$$(0,1) = (0, \tfrac{1}{2}) \cup \{\tfrac{1}{2}\} \cup (\tfrac{1}{2}, 1)$$

Representing the isomorphic class of finite open interval by $x$, we have $x = x + 1 + x$ which implies

[1]Schanuel S.H. (1991) Negative sets have Euler characteristic and dimension. In: Carboni A., Pedicchio M.C., Rosolini G. (eds) Category Theory. Lecture Notes in Mathematics, vol 1488. Springer, Berlin, Heidelberg

$$x\ ' =' -1$$
$$\mathbb{N}[X]/(X \approx 2X+1) \to R$$
Euler characteristic $\chi$: $R \xrightarrow{\chi} R(E)$ where

$E(R)$ is a rig with additive cancellation

Similarly, endowing isomorphic class (range of $\gamma$ isomorphic to semi open infinite interval) with Y, then we have Burnside rig of unbounded polyhedra,

$$\bar{R} = \mathbb{N}[X,Y]/(X \approx 2X+1, Y \approx X+Y+1, Y^2 \approx 2Y^2+Y) \to B(\mathbb{P}).$$

Then Euler Characteristic acts as

$$\chi(X) = (-1,-1)$$
$$\chi(Y) = (-1,0)$$

*Remark:*

· Associativity law holds and Euler characteristic only depends on $i,j,k \in [13]$, thus it's reasonable to replace intersection number $P_{ij}^k$ with Euler characteristic and coherent configuration is extended to infinite set case.

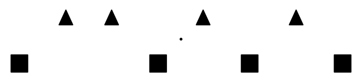
· After admitting that $\mathbb{P} = \{R_1, \cdots, R_{13}\}$ is a coherent configuration, we consider about its automorphisms and weak-automorphisms.

1 an automorphism $f$ induces a strictly increasing function over $\mathbb{R}$;

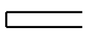
2 a weak-automorphism $g$ induces a strictly monotone function over $\mathbb{R}$.

## 2.3 Algebra generated by $\{R_1, ..., R_{13}\}$ on $\mathbb{C}$

### 2.3.1 Examples:

Compute $R_9 \cdot R_3 =$ 

By the process above, for i=1, 2,$\cdots$, 13, we look for the range of $\gamma$ represented in Burnside rig or Euler Characteristic such that $\gamma$ and $\alpha$ of $R_i$ form the pattern $R_9$, at the same time, $\gamma$ and $\beta$ of $R_i$ form the pattern $R_3$. Therefore, $R_9 \cdot R_3 =$

 $(R_1 + R_2 + R_3 + R_6 + R_9)$ {Burnside rig form}

or

(1,0)$(R_1 + R_2 + R_3 + R_6 + R_9)$ {Euler Characteristic form}

*Remark:*

① in the multiplication, those $R_i$ whose coefficient in Burnside rig form is not ø, have isomorphic coefficients;

② the number of $R_i$ whose coefficient isn't 0 (in Euler Characteristic) is odd.

③ by symmetry, $R_i \cdot R_j = \overline{R_{14-j} R_{14-i}}$

### 2.3.2 Decomposition:

For this non-commutative ring, its center is generated by $R_1 + R_2 + R_3, R_3 + R_4 + R_6, R_8 + R_{10} + R_{11}, R_{11} + R_{12} + R_{13}$ and $R_7$. The center has dimension 5, and this algebra has dimension 13. Analogues to the finite case coherent configuration, by Artin-Wedderburn theorem (13=$3^2 + 1^2 + +1^2 + 1^2 + 1^2$), we'll try to decompose this algebra into direct sum of complete matrix algebra. This, along with further expansion of the problem into three-point set relation, will be on the future working track.