CALYXES AND COROLLAS

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Abstract. A calyx (multiplicative lattice) is a complete lattice endowed with the structure of a monoid such that multiplication by an element is a left adjoint functor of complete lattices (equivalently, a left adjoint functor which preserves colimits). Calyxes are a generalization of the set of ideals of a ring, which form a complete lattice under intersection and summation; in conjunction with the natural multiplication operation (ideal multiplication), this lattice forms a calyx. One can axiomatize the submodules of a module similarly to arrive at the definition of a corolla (module lattice). We present here a notion of principal element equivalent to Dilworth's original definition of principal on a multiplicative lattice, and show that this is equivalent to weak principal. The new concept of principal element begins with a notion of morphism in the category of corollas over a given calyx. We characterize principal elements in the calyx induced by a Noetherian ring as those ideals in the lattice which are principal ideals (in the ordinary sense) after localizing at any given prime ideal. We include here w

Contents

In this section we define multiplicative lattices and module lattices from a categorical point of view, providing the necessary insight in establishing a new definition of morphism. This leads to a concept of 'perennial' morphism between module lattices, which is a notion much closer to an actual map of modules. For instance, take a multiplicative lattice E. Principal elements of an E-module lattice M are in direct correspondence with perennial maps $\phi : E \to M$, where we view the multiplicative lattice E as a module lattice over itself.

In this document, multiplicative lattices are referred to as calyxes and module lattices are referred to as corollas. This is partially to emphasize the distinction between morphisms in the category of multiplicative lattices and morphisms in the category of calyxes. Our terminology is taken from the subject of flower anatomy.

TODO

finitely generated corollas and finitely generated elements. the notation $E[a_1, ..., a_n]$. finite as in $(a_1, ..., a_n)$. need definition of quotient notation. the free calyx on n generators, free algebra calyx. etc. we write Ω_X for the structure sheaf of X. domain calyx open sets U in X change name to calical schemes change abstract tensor universal property finite module is different than a finite element. section on modularity move annihilators from Zerodivisors, the Corolla Quotient to section on associated primes fix graded corollas. remember finitely generated. check "the hom functor" fix primary decomposition localization at a principal element notation. note that it is possible to fully recover a spectrum from its caylx. compact elements

1. Intrinsic Definitions

Let $\mathcal{C}$ be a

the category set and the category of sup-lattices

-there is a left adjoint functor set to sup-lattices, injective on objects -a set can be recovered from its sup-lattice

the category monoid and the category of calyxes

-ab = xy : x in a, y in b sumd : ad ¡= c ¡- c b -¿ ab -xy in c for all x in a and y in b iff x in b implies ax ¡= c thus every monoid induces a calyx -the functor monoid to calyx is left adjoint.

the category of group actions and the category of corollas

the category of rings and the category of perianths

the category of modules and the category of sepals

2. Intrinsic Definitions

2.1. Preliminaries.

Definition 0. A lattice is a poset E for which every two elements $x, y \in E$ have a supremum and an infimum. A bounded lattice is a lattice which has two elements $0, 1$ such that $0 \leq x \leq 1 \ \forall x \in E$. A complete lattice is a lattice such that any arbitrary collection of elements has a unique infimum and supremum. Any complete lattice is bounded. Meets and joins in a lattice are automatically associative, commutative, and idempotent.

Lemma 1. *Suppose a poset P has arbitrary suprema. Then P has arbitrary infima, so that P is a complete lattice.*

Proof. Take a subset $X \subseteq P$ and consider the set $Y = \{y \in P : y \leq x \ \forall x \in X\}$ of lower bounds of X. This set has a supremum $p \in P$. Take $x \in X$. $y \leq x \ \forall y \in Y$, so $p \leq x$. Since x was chosen arbitrarily, $p \in Y$. But by definition $p \geq y \ \forall y \in Y$, so p is an infimum for X. □

Lemma 2. *Let $f : P \to Q$ be a morphism of complete lattices which preserves suprema (colimits). Then f has a right adjoint.*

Proof. Define a function $g : Q \to P$ where $g \mapsto \sup\{x \in P : f(x) \leq y\}$. If $x \in P, y \in Q$ are such that $f(x) \leq y$, then $x \leq \sup\{x \in P : f(x) \leq y\} = g(y)$. Conversely, if $x \leq g(y)$, then

$$f(x) \leq f(g(y)) = f(\sup\{x \in P : f(x) \leq y\}) = \sup f(\{x \in P : f(x) \leq y\}) \leq y$$

□

Lemma 3. *It follows from the proof above that, for an adjoint pair $f \dashv g$ of complete lattice morphisms $f : P \to Q$ and $g : Q \to P$, $f(x) = \varprojlim\{y \in Q : x \leq g(y)\}$ and $g(y) = \varinjlim\{x \in P : f(x) \leq y\}$.*

Lemma 4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a left adjoint functor with right adjoint $G : \mathcal{D} \to \mathcal{C}$. Let $\mathcal{A}_{X,Y} : Hom_{\mathcal{D}}(F(X), Y) \to Hom_{\mathcal{C}}(X, G(Y))$ be the isomorphisms testifying to the adjoint relationship $F \dashv G$ for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. We show that F preserves colimits.*

Proof. Let Λ be a small category and $\Phi : \Lambda \to \mathcal{C}$ a functor such that $\varinjlim \Phi$ exists. Let $\alpha_\lambda : \Phi(\lambda) \to \varinjlim \Phi$ be the canonical maps. We show that $\{F(\varinjlim \Phi), F(\alpha_\lambda)\}$ forms a colimit for $F \circ \Phi$. Take an object $P \in \mathcal{D}$ with morphisms $\beta_\lambda : F \circ \Phi(\lambda) \to P$ such that the following diagram commutes for each $\phi : \lambda \to \mu$ in Λ:

$$\begin{array}{ccc} F\circ\Phi(\lambda) & \xrightarrow{F(\phi)} & F\circ\Phi(\mu) \\ \downarrow{\scriptstyle \beta_\lambda} & & \downarrow{\scriptstyle \beta_\mu} \\ P & \xrightarrow{1_P} & P \end{array}$$

Since $F \dashv G$ the following diagram commutes:

$$\begin{array}{ccc} \Phi(\lambda) & \xrightarrow{\phi} & \Phi(\mu) \\ \downarrow{\scriptstyle \mathcal{A}(\beta_\lambda)} & & \downarrow{\scriptstyle \mathcal{A}(\beta_\mu)} \\ G(P) & \xrightarrow{1_{G(P)}} & G(P) \end{array}$$

By the universal property of colimit, there is $\beta : \varprojlim \Phi \to G(P)$ such that the following diagram commutes:

$$\begin{array}{ccccc} \Phi(\lambda) & \xrightarrow{\phi} & \Phi(\mu) & \xrightarrow{\alpha_\mu} & \varinjlim \Phi \\ \downarrow{\scriptstyle \mathcal{A}(\beta_\lambda)} & & \downarrow{\scriptstyle \mathcal{A}(\beta_\mu)} & & \downarrow{\scriptstyle \beta} \\ G(P) & \xrightarrow{1_{G(P)}} & G(P) & \xrightarrow{1_{G(P)}} & G(P) \end{array}$$

Since $F \dashv G$ the following diagram commutes:

$$\begin{array}{ccccc} F\circ\Phi(\lambda) & \xrightarrow{\phi} & F\circ\Phi(\mu) & \xrightarrow{\alpha_\mu} & F(\varinjlim \Phi) \\ \downarrow{\scriptstyle \beta_\lambda} & & \downarrow{\scriptstyle \beta_\mu} & & \downarrow{\scriptstyle \mathcal{A}^{-1}(\beta)} \\ P & \xrightarrow{1_{G(P)}} & P & \xrightarrow{1_P} & P \end{array}$$

We can apply this argument in reverse to get uniqueness. □

Analogously, right adjoints preserve limits.

Corollary 5. *Colimits commute and limits commute.*

Proof. Follows since colimits are left adjoint and limits are right adjoint. A proof can be found in 'A Term of Commutative Algebra' By Allen Altman and Steven Kleiman. It occurs as 6.6. □

Definition 6 (Composition of Adjoints)**.** Let $\mathfrak{f} = (\mathfrak{f}_*, \mathfrak{f}^*) : \mathcal{C} \to \mathcal{D}$ be an adjoint pair with $\mathfrak{f}_* : \mathcal{C} \to \mathcal{D}$, $\mathfrak{f}^* : \mathcal{D} \to \mathcal{C}$. Suppose $\mathfrak{g} = (\mathfrak{g}_* : \mathfrak{g}^*) : \mathcal{D} \to \mathcal{E}$ is another adjoint pair with $\mathfrak{g}_* : \mathcal{D} \to \mathcal{E}$ and $\mathfrak{g}^* : \mathcal{E} \to \mathcal{D}$. We can form the composition $\mathfrak{g} \circ \mathfrak{f} = (\mathfrak{g}_* \circ \mathfrak{f}_*, \mathfrak{f}^* \circ \mathfrak{g}^*)$ of the two adjoints, which is an adjoint in its own right. We check this:

$$Hom_{\mathcal{E}}(\mathfrak{g}_* \circ \mathfrak{f}_*(X), Y) \cong Hom_{\mathcal{D}}(\mathfrak{f}_*(X), \mathfrak{g}^*(Y)) \cong Hom_{\mathcal{C}}(X, \mathfrak{f}^*(\mathfrak{g}^*(Y)))$$

Moreover the identity functor on a category $\mathcal{C}$ composes with any adjoint pair $\mathfrak{f}$ to make $\mathfrak{f}$ again. In this way, one can replace the functors in **Cat** with adjoint pairs to obtain a category. This same process works with the category of posets viewed as a full subcategory of **Cat**.

2.2. **Lattices.** To motivate the definition of a calyx we start with the notion of a lattice induced by a monoid. We form the corresponding category of complete lattices with morphisms adjoint pairs of poset functors. From complete lattices we construct calyxes just as rings are constructed from abelian groups.

Definition 7. Let M be a (not necessarily commutative) monoid and consider the lattice $E = (E, +, \cap)$ of its submonoids, where $+$ is the join and $\cap$ is the meet. The lattice E is complete, with M and 0 the bounds. To each monoid M we assign the lattice of its submonoids by M^{lat}. Note that, while the monoid M may not be commutative in general, joins and meets in M^{lat} are still commutative.

lat can be made into a functor in a natural way, which is desirable as it vital to the other definitions to come, but this cannot happen on the ordinary category of lattices and lattice morphisms. To make E functorial, let us examine the properties of extension and contraction of submonoids under a morphism $f : M \to N$ of monoids. Write $K^e \in E(N)$ for the image (extension) of $K \in E(M)$ and $L^c \in E(M)$ for the preimage (contraction) of $L \in E(N)$. Take $K, K', \{K_i\}_{i \in I} \in E(M)$ and $L, L', \{L_i\}_{i \in I} \in E(N)$. Then

$K \subseteq K^{ec}$	$L^{ce} \subseteq L$
$K \subseteq K' \Rightarrow K^e \subseteq K'^e$	$L \subseteq L' \Rightarrow L^c \subseteq L'^c$
$\left(\sum_{i \in I} K_i\right)^e = \sum_{i \in I} K_i^e$	$\bigcap_{i \in I} L_i^c = \left(\bigcap_{i \in I} L_i\right)^c$
$(\bigcap_{i \in I} K_i)^e \subseteq \bigcap_{i \in I} K_i^e$	$\sum_{i \in I} L_i^c \subseteq \left(\sum_{i \in I} L_i\right)^c$
$0^e = 0$	$1^c = 1$

These properties can be seen to follow from the fact that extension and contraction are a pair of adjoint poset functors, where the lattices in question are viewed as poset categories. We denote by **Lat** the category of complete (bounded) lattices which has as its morphisms adjoint pairs $\phi = (\phi_*, \phi^*), \phi_* \dashv \phi^*$ with $\phi_* : M \to N$ and $\phi^* : N \to M$, written $\phi : M \to N$. An unfortunate consequence of this is that the left adjoint ϕ_* is written left to right. Note the distinction between **Lat** and the ordinary category of lattices and lattice morphisms. In the second, morphisms are not adjoint pairs but maps $f : P \to Q$ such that $x \leq y \Rightarrow f(x) \leq f(y)$. The composition of an adjoint pair is an adjoint pair and there is the identity adjoint pair on each complete lattice, so **Lat** indeed forms a category. We write ϕ_* for the left adjoint and ϕ^* for the right adjoint of a morphism pair $\phi : M \to N$ in **Lat**, with ϕ_* written left to right and ϕ^* right to left.

Example 8. Let E be a complete lattice. The set $End_{\mathbf{Lat}}(E)$ of complete lattice morphisms forms a monoid under composition. $End_{\mathbf{Lat}}(E)$ also has a canonically induced partial order where $\phi \leq \psi$ when $\phi_*(x) \leq \psi_*(x)\ \forall x \in E$ and $\phi^*(x) \geq \psi^*(x)\ \forall x \in E$. Under this order $End_{\mathbf{Lat}}(E)$ is bounded and complete. To show this it suffices to show that $End_{\mathbf{Lat}}(E)$ has arbitrary joins, by lemma 1.

Let $\{\phi_i\}_{i\in I}$ be elements of $End_{\mathbf{Lat}}(E)$. Define $\alpha_*(x) = \sum_{i\in I}(\phi_i)_*(x)$ and $\alpha^*(x) = \bigcap_{i\in I}(\phi_i)^*(x)$. α_* and α^* are poset functors $E \to E$; since colimit is a functor,

$$x \leq y \Rightarrow \sum_{i\in I}(\phi_i)_*(x) \leq \sum_{i\in I}(\phi_i)_*(y)$$

since limit is a functor,

$$x \leq y \Rightarrow \bigcap_{i\in I}(\phi_i)^*(x) \leq \bigcap_{i\in I}(\phi_i)^*(y)$$

To show $\alpha_* \dashv \alpha^*$ we must show $\sum_{i\in I}(\phi_i)_*(x) \leq y \iff x \leq \bigcap_{i\in I}(\phi_i)^*(y)$. Observe that

$$\begin{aligned} &\sum_{i\in I}(\phi_i)_*(x) \leq y \\ \Leftrightarrow\ &(\phi_i)_*(x) \leq y\ \forall i \in I \\ \Leftrightarrow\ &x \leq (\phi_i)^*(y)\ \forall i \in I \\ \Leftrightarrow\ &x \leq \bigcap_{i\in I}(\phi_i)^*(y) \end{aligned}$$

A 0 element for F is (ϕ_*, ϕ^*) where $\phi_*(x) = 0$ for each $x \in E$ and $\phi^*(x) = 1$ for each $x \in E$. A 1 element for F is (ψ_*, ψ^*) where $\psi_*(x) = x$ for each $x \in E$ and $\psi^*(x) = x$ for each $x \in E$.

2.3. **Calyxes.**

Remark. Notice that a ring A is an abelian group G with a group homomomorphism $\phi : G \to End_{\mathbf{Ab}}(G)$ such that $\phi(1) = 1$ and $\phi(\phi(x)(y))(z) = \phi(x)(\phi(y)(z))\ \forall x, y, z \in G$. We can use this as motivation in constructing the category of calyxes from the category of complete lattices.

Definition 9. A calyx is a complete lattice E along with a complete lattice morphism $\phi = (\phi_*, \phi^*) : E \to End_{\mathbf{Lat}}(E)$ such that

$$\psi(\phi_*(\mathfrak{a})(\mathfrak{b}))(\mathfrak{c}) = \psi(\mathfrak{a})(\psi(\mathfrak{b})(\mathfrak{c}))$$

where $\psi = \phi_*$ or ϕ^* (called associativity of ϕ) and $\phi(1) = 1$. We denote the join in E with summation notation and the meet with intersection notation. We denote $\phi_*(\mathfrak{a})(\mathfrak{b}) \in E$ by $\mathfrak{ab}$ for $\mathfrak{a}, \mathfrak{b} \in E$ (called product) and $\phi^*(\mathfrak{a})(\mathfrak{b})$ by $(\mathfrak{b} : \mathfrak{a})$ (called quotient), to match the notation for product and ideal quotient. It follows from the axioms that $\phi : E \to End_{\mathbf{Lat}}(E)$ is a morphism of monoids (with the product in E being $(\mathfrak{a}, \mathfrak{b}) \mapsto \mathfrak{ab}$ and the product in $End_{\mathbf{Lat}}(E)$ being composition of adjoint pairs). A calyx is called commutative if $\phi_*(\mathfrak{a})(\mathfrak{b}) = \phi_*(\mathfrak{b})(\mathfrak{a})\ \forall \mathfrak{a}, \mathfrak{b} \in E$.

In light of the following motivating example, the theory of calyxes can be viewed as a generalization of ring theory:

Example 10. The set of ideals of any commutative ring A forms an example of a complete lattice A^{cal}, whose meet is intersection and whose join is ideal sum. We define the left adjoint $\phi : (\phi_*, \phi^*) : A^{cal} \to End(A^{cal})$ where $\phi_*(\mathfrak{a})(\mathfrak{b}) = \mathfrak{a}\mathfrak{b}$ (the ideal product) and $\phi^*(\mathfrak{a})(\mathfrak{b}) = (\mathfrak{b} : \mathfrak{a})$ (the ideal quotient), and this makes E into a calyx.

Proof. Take $\mathfrak{a} \in A^{cal}$. For each $\mathfrak{b}, \mathfrak{c} \in A^{cal}$, $\mathfrak{b} \leq (\mathfrak{c} : \mathfrak{a}) \iff \mathfrak{a}\mathfrak{b} \leq \mathfrak{c}$, so $\phi_*(\mathfrak{a}) \dashv \phi^*(\mathfrak{a})$.

The map ϕ forms the left adjoint of a lattice morphism pair. We check this as follows: if $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{c} \in A^{cal}$ then $\mathfrak{a}\mathfrak{c} \leq \mathfrak{b}\mathfrak{c}$ and $(\mathfrak{c} : \mathfrak{b}) \leq (\mathfrak{c} : \mathfrak{a})$. Suppose $\{\mathfrak{a}_i\}_{i \in I}$ and $\mathfrak{b}$ are elements of A^{cal}. Then

$$\phi_* \left(\sum_{i \in I} \mathfrak{a}_i \right) (\mathfrak{b}) = \left(\sum_{i \in I} \mathfrak{a}_i \right) \mathfrak{b} = \sum_{i \in I} (\mathfrak{a}_i \mathfrak{b}) = \left(\varinjlim \phi_*(\mathfrak{a}_i) \right) (\mathfrak{b})$$

$$\phi^*(\mathfrak{a}_i)(\mathfrak{b}) = \left(\mathfrak{b} : \sum_{i \in I} \mathfrak{a}_i \right) = \bigcap_{i \in I} (\mathfrak{b} : \mathfrak{a}_i) = \left(\varinjlim \phi^*(\mathfrak{a}_i) \right) (\mathfrak{b})$$

So that ϕ preserves colimits. By lemma 2, ϕ is indeed left adjoint. □

Lemma 11. $\mathfrak{a}\mathfrak{b} = \bigcap_{\mathfrak{b} \leq (\mathfrak{c}:\mathfrak{a})} \mathfrak{c}$ *and* $(\mathfrak{a} : \mathfrak{b}) = \sum_{\mathfrak{c}\mathfrak{b} \leq \mathfrak{a}} \mathfrak{c}$ *for elements* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ *of any calyx.*

Proof. $\mathfrak{a}\mathfrak{b} \leq x \iff \mathfrak{b} \leq (x : \mathfrak{a})$. So

$$\begin{aligned} & y = \mathfrak{a}\mathfrak{b} \\ \iff & y \leq x \iff \mathfrak{a}\mathfrak{b} \leq x \\ \iff & y = \inf\{x \in E : \mathfrak{a}\mathfrak{b} \leq x\} = \inf\{x \in E : \mathfrak{b} \leq (x : \mathfrak{a})\} \end{aligned}$$

The other claim follows similarly. □

Definition 12. We often write $\mu_{\mathfrak{a}*}(\mathfrak{b})$ for $\mathfrak{a}\mathfrak{b}$ in a calyx E. We write $\mu^*_{\mathfrak{a}}(\mathfrak{b})$ for $(\mathfrak{b} : \mathfrak{a})$. This is to emphasize that $\mu_{\mathfrak{a}}$ is an element of $End_{\mathbf{Lat}}(E)$.

Lemma 13. *The following familiar properties of ideals in ring theory hold in a general for an arbitrary calyx* E *with left adjoint structure map* $\phi = (\phi_*, \phi^*) : E \to End(E)$. *Let* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \{\mathfrak{a}_i\}_{i \in I}$ *be elements of* E.

Reason	Property	Dual property
$\phi(1) = 1$	$1\mathfrak{a} = \mathfrak{a}$	$(\mathfrak{a} : 1) = \mathfrak{a}$
Associativity of ϕ	$(\mathfrak{a}\mathfrak{b})\mathfrak{c} = \mathfrak{a}(\mathfrak{b}\mathfrak{c})$	$((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c})$
ϕ is a functor	$\mathfrak{a} \leq \mathfrak{b} \Rightarrow \mathfrak{a}\mathfrak{c} \leq \mathfrak{b}\mathfrak{c}$	$\mathfrak{a} \leq \mathfrak{b} \Rightarrow (\mathfrak{c} : \mathfrak{a}) \geq (\mathfrak{c} : \mathfrak{b})$
$\phi_*(\mathfrak{a})$ and $\phi^*(\mathfrak{a})$ are functors.	$\mathfrak{b} \leq \mathfrak{c} \Rightarrow \mathfrak{a}\mathfrak{b} \leq \mathfrak{a}\mathfrak{c}$	$\mathfrak{b} \leq \mathfrak{c} \Rightarrow (\mathfrak{b} : \mathfrak{a}) \leq (\mathfrak{c} : \mathfrak{a})$
$\phi_*(\mathfrak{a})$ and $\phi^*(\mathfrak{a})$ are functors and $\mathfrak{b} \leq 1$	$\mathfrak{a}\mathfrak{b} \leq \mathfrak{a}$	$\mathfrak{a} \leq (\mathfrak{a} : \mathfrak{b})$
Since ϕ_* is left adjoint to ϕ^*, $\mathfrak{a} \leq \phi^* \circ \phi_*(\mathfrak{a})$ and $\phi_* \circ \phi^*(\mathfrak{a}) \leq \mathfrak{a}$	$(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \leq \mathfrak{a}$	$\mathfrak{a} \leq (\mathfrak{a}\mathfrak{b} : \mathfrak{b})$
$\phi(\mathfrak{a})$ is an adjoint pair so $\phi_*(\mathfrak{a})$ (resp. $\phi^*(\mathfrak{a})$) distributes over coproducts (resp. products).	$\sum_{i\in I}(\mathfrak{a}\mathfrak{a}_i) = \mathfrak{a}\left(\sum_{i\in I} \mathfrak{a}_i\right)$	$\bigcap_{i\in I}(\mathfrak{a}_i : \mathfrak{a}) = \left(\bigcap_{i\in I} \mathfrak{a}_i : \mathfrak{a}\right)$
ϕ is an adjoint pair so ϕ_* (resp. ϕ^*) preserves initial objects (resp. terminal objects)	$\mathfrak{a}0 = 0$	$(\mathfrak{a} : 1) = \mathfrak{a}$
Canonical morphism from universal property of product (resp. coproduct)	$\mathfrak{a}\left(\bigcap_{i\in I} \mathfrak{a}_i\right) \leq \bigcap_{i\in I} \mathfrak{a}\mathfrak{a}_i$	$\bigcap_{i\in I}(\mathfrak{a} : \mathfrak{a}_i) \leq \left(\mathfrak{a} : \sum_{i\in I} \mathfrak{a}_i\right)$
ϕ is itself left adjoint, and so distributes over colimits.	$\left(\sum_{i\in I} \mathfrak{a}_i\right)\mathfrak{a} = \sum_{i\in I}(\mathfrak{a}_i\mathfrak{a})$	$\left(\mathfrak{a} : \sum_{i\in I} \mathfrak{a}_i\right) = \bigcap_{i\in I}(\mathfrak{a} : \mathfrak{a}_i)$
ϕ is left adjoint and so sends initial objects to initial objects.	$0\mathfrak{a} = 0$	$(\mathfrak{a} : 0) = 1$

The following identities follow from the properties listed above:

$(\mathfrak{a}+\mathfrak{b})(\mathfrak{a}\cap\mathfrak{b}) \leq \mathfrak{a}\mathfrak{b}$	$\mathfrak{a}+\mathfrak{b} \leq (\mathfrak{a}\mathfrak{b} : \mathfrak{a}\cap\mathfrak{b})$
$(\mathfrak{a} : \mathfrak{a}+\mathfrak{b}) = (\mathfrak{a} : \mathfrak{b})$	
$\mathfrak{a}\mathfrak{b} \leq \mathfrak{a}\cap\mathfrak{b}$	
$\mathfrak{a}+\mathfrak{b} = 1 \Rightarrow \mathfrak{a}\cap\mathfrak{b} = \mathfrak{a}\mathfrak{b}$	
$\mathfrak{a} \leq \mathfrak{b} \iff (\mathfrak{b} : \mathfrak{a}) = 1$	
$\mathfrak{a}+\mathfrak{c} = 1, \mathfrak{a}+\mathfrak{b} = 1 \Rightarrow \mathfrak{a}+(\mathfrak{b}\cap\mathfrak{c}) = \mathfrak{b}+\mathfrak{c}$	
$(\mathfrak{a}\cap\mathfrak{b} : \mathfrak{b}) = (\mathfrak{a} : \mathfrak{b})$	
$\mathfrak{a}+\mathfrak{b} = 1, \mathfrak{a}+\mathfrak{c} = 1 \Rightarrow (\mathfrak{a}\mathfrak{b})+\mathfrak{c} = 1$	

The list of properties tabulated above determines the definition of a calyx, but it is clearly not minimal. To satisfy those who would like a minimal set of axioms, we show that this is equivalent to a multiplicative lattice.

Lemma 14. *Let E be a complete lattice on which there is a binary operation called product, written $(\mathfrak{a}, \mathfrak{b}) \mapsto \mathfrak{a}\mathfrak{b}$, such that the following hold for each $\mathfrak{a}, \{\mathfrak{a}_i\}_{i\in I} \in E$:*

(1) Product forms an abelian monoid.
(2) $\mathfrak{a}\sum_{i\in I} \mathfrak{a}_i = \sum_{i\in I} \mathfrak{a}\mathfrak{a}_i$
(3) $\mathfrak{a}0 = 0$

We call E a multiplicative lattice. Obviously each calyx is a multiplicative lattice. We show that each multiplicative lattice is a calyx.

Proof. For each $\mathfrak{a} \in E$ define $\mu(\mathfrak{a}) : E \to E$ to be the adjoint pair whose left adjoint $\mu(\mathfrak{a})_*$ is multiplication by $\mathfrak{a}$. Indeed, multiplication by $\mathfrak{a}$ is left adjoint by the adjoint functor theorem for posets since it distributes over sums and preserves the initial object. This defines a set map $\mu : E \to End_{\mathbf{Lat}}(E)$ which again distributes over the initial object and sums. Thus μ is itself a left adjoint. Since multiplication is associative, the left adjoint of the maps below match, so that they are equal:

$$\mu(\mu(\mathfrak{a})_*(\mathfrak{b}))(\mathfrak{c}) = \mu(\mathfrak{a})(\mu(\mathfrak{b})(\mathfrak{c})) \ \forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in E$$

Also $\mu(\mathfrak{a})_*(\mathfrak{b}) = \mu(\mathfrak{b})_*(\mathfrak{a})$, so that $\mu(\mathfrak{a})_*(\mathfrak{b}) = \mu(\mathfrak{b})_*(\mathfrak{a})$ for each $\mathfrak{a}, \mathfrak{b} \in E$. Thus E is a calyx. □

Example 15. Let $E = \mathbb{Z}^{cal}$, the calyx over the integers. Then $+$ and $\cap$ distribute over each other, so that E is a distributive lattice. This is not true in general. The best one obtains in this direction in general is the modular law, which holds for any calyx induced by a ring:

$$\text{If } \mathfrak{b} \leq \mathfrak{a} \text{ or } \mathfrak{c} \leq \mathfrak{a} \text{ then } \mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \cap \mathfrak{b} + \mathfrak{a} \cap \mathfrak{c}$$

Definition 16. A morphism of calyxes E and F is a set map $\phi : E \to F$ which is both a morphism of complete lattices on the underlying complete lattice structure and a morphism of monoids on the underlying monoid structure. We denote the category of calyxes and calyx morphisms by **Cal**.

Lemma 17. *There is a functor* ***Cal*** $\to$ ***Mon***. *For a calyx E define a monoid M with the same underlying set as E, where multiplication $\mu : M \times M \to M$ in M is defined by* $(\mathfrak{a}, \mathfrak{b}) \mapsto \mathfrak{a}\mathfrak{b}$. *Indeed, M has $1 \in E$ as an identity element and is associative by requirement on E. There is also a forgetful functor* ***Cal*** $\to$ ***Lat*** *which forgets all but the lattice structure.*

Definition 18. We call a calyx E representable if $E \cong A^{cal}$ for some ring A. Not every calyx is representable. For example, take a semiring S over the group D_8 of symmetries of a square. The lattice structure of D_8 is nonmodular, and a semiring over D_8 exists whose two sided ideal structure is nonmodular. One can construct a calyx from S in the same way as with rings, forming in particular a lattice with non-modular ideal structure. Wheras every calyx arising from a ring has modular ideal structure.

One might ask if the representable calyxes are exactly the modular ones.

Lemma 19. *The following familiar properties of extension and contraction of ideals hold in general for an arbitrary calyx morphism $\mathfrak{f} : E \to F$. Let $\mathfrak{a}, \mathfrak{a}', \mathfrak{a}'', \{\mathfrak{a}_i\}_{i \in I}$ be elements of the calyx E and let $\mathfrak{b}, \mathfrak{b}', \mathfrak{b}'', \{\mathfrak{b}_j\}_{j \in J}$ be elements of the calyx F.*

Reason	Property	Dual Property
$\mathfrak{f}$ is a monoid morphism and so sends 1 to 1	$\mathfrak{f}_*(1) = 1$	
$\mathfrak{f}$ is a monoid morphism and so distributes over product.	$\mathfrak{f}_*(\mathfrak{a}\mathfrak{a}') = \mathfrak{f}_*(\mathfrak{a})(\mathfrak{a}')$	
Lemma 20	$\mathfrak{f}_*(\mathfrak{a})\mathfrak{f}_*(\mathfrak{a}') \leq \mathfrak{f}^*(\mathfrak{a}\mathfrak{a}')$	$\mathfrak{f}^*(\mathfrak{b})\mathfrak{f}^*(\mathfrak{b}') \leq \mathfrak{f}^*(\mathfrak{b}\mathfrak{b}')$
Lemma 20	$\mathfrak{f}_*((\mathfrak{a} : \mathfrak{a}')) \leq (\mathfrak{f}_*(\mathfrak{a}) : \mathfrak{f}_*(\mathfrak{a}'))$	$\mathfrak{f}^*((\mathfrak{b} : \mathfrak{b}')) \leq (\mathfrak{f}^*(\mathfrak{b}) : \mathfrak{f}^*(\mathfrak{b}'))$
$\mathfrak{f}_*$ (resp. $\mathfrak{f}^*$) is left adjoint (resp. right adjoint) and so distributes over colimits (resp limits)	$\mathfrak{f}_*\left(\sum_{i\in I} \mathfrak{a}_i\right) = \sum_{i\in I} \mathfrak{f}_*(\mathfrak{a}_i)$	$\mathfrak{f}^*\left(\bigcap_{i\in I} \mathfrak{b}_i\right) = \bigcap_{i\in I} \mathfrak{f}^*(\mathfrak{b}_i)$
$\mathfrak{f}_*$ (resp. $\mathfrak{f}^*$) is left adjoint (resp. right adjoint) and so preserves initial objects (resp. terminal objects).	$\mathfrak{f}_*(0) = 0$	$\mathfrak{f}^*(1) = 1$
Canonical morphism from the universal property of product (resp. coproduct)	$\mathfrak{f}_*\left(\bigcap_{i\in I} \mathfrak{a}_i\right) \leq \bigcap_{i\in I} \mathfrak{f}_*(\mathfrak{a}_i)$	$\sum_{i\in I} \mathfrak{f}_*(\mathfrak{b}_i) \leq \mathfrak{f}^*\left(\sum_{i\in I} \mathfrak{b}_i\right)$
Unitor and counitor from adjoint relationship $\mathfrak{f}_* \dashv \mathfrak{f}^*$	$\mathfrak{a} \leq \mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{a}))$	$\mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{b})) \leq \mathfrak{b}$
	$\mathfrak{f}_*(\mathfrak{a}) = \mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{a})))$	$\mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{b}))) = \mathfrak{f}^*(\mathfrak{b})$

Lemma 20. *Let $\mathfrak{f} : E \to F$ be a morphism of calyxes E and F, and take elements $\mathfrak{a}, \mathfrak{a}' \in E$, and $\mathfrak{b}, \mathfrak{b}' \in F$. Then we have the following:*

(1) $\mathfrak{f}_*(\mathfrak{a})\mathfrak{f}_*(\mathfrak{a}') \leq \mathfrak{f}_*(\mathfrak{a}\mathfrak{a}')$
(2) $\mathfrak{f}^*(\mathfrak{b})\mathfrak{f}^*(\mathfrak{b}') \leq \mathfrak{f}^*(\mathfrak{b}\mathfrak{b}')$
(3) $\mathfrak{f}_*((\mathfrak{a} : \mathfrak{a}')) \leq (\mathfrak{f}_*(\mathfrak{a}) : \mathfrak{f}_*(\mathfrak{a}'))$
(4) $\mathfrak{f}^*((\mathfrak{b} : \mathfrak{b}')) \leq (\mathfrak{f}_*(\mathfrak{b}) : \mathfrak{f}_*(\mathfrak{b}'))$

Notice that in one case we have equality by requirement: $\mathfrak{f}_*(\mathfrak{a})\mathfrak{f}_*(\mathfrak{a}') = \mathfrak{f}_*(\mathfrak{a}\mathfrak{a}')$.

Proof. As stated above, (1) holds automatically. To show (2), note that

$$\mathfrak{f}^*(\mathfrak{b})\mathfrak{f}^*(\mathfrak{b}') \leq \mathfrak{f}^*(\mathfrak{b}\mathfrak{b}') \;\Leftrightarrow\; \mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{b})\mathfrak{f}^*(\mathfrak{b}')) \leq \mathfrak{b}\mathfrak{b}'$$

And

$$\mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{b})\mathfrak{f}^*(\mathfrak{b}')) = \mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{b}))\mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{b}'))) \leq \mathfrak{b}\mathfrak{b}'$$

To show (3), note that

$$\mathfrak{f}_*((\mathfrak{a} : \mathfrak{a}')) \leq (\mathfrak{f}_*(\mathfrak{a}) : \mathfrak{f}_*(\mathfrak{a}')) \;\Leftrightarrow\; \mathfrak{f}_*(\mathfrak{a}')\mathfrak{f}_*((\mathfrak{a} : \mathfrak{a}')) \leq \mathfrak{f}_*(\mathfrak{a}) \;\Leftrightarrow\; \mathfrak{f}_*(\mathfrak{a}'(\mathfrak{a} : \mathfrak{a}')) \leq \mathfrak{f}_*(\mathfrak{a})$$

To show (4), note that

$$\mathfrak{f}^*((\mathfrak{b} : \mathfrak{b}')) \leq (\mathfrak{f}^*(\mathfrak{b}) : \mathfrak{f}^*(\mathfrak{b}')) \;\Leftrightarrow\; \mathfrak{f}^*(\mathfrak{b}')\mathfrak{f}^*((\mathfrak{b} : \mathfrak{b}')) \leq \mathfrak{f}^*(\mathfrak{b})$$

and

$$\mathfrak{f}^*(\mathfrak{b}')\mathfrak{f}^*((\mathfrak{b} : \mathfrak{b}')) \leq \mathfrak{f}^*(\mathfrak{b}'(\mathfrak{b} : \mathfrak{b}')) \leq \mathfrak{f}^*(\mathfrak{b})$$

□

Lemma 21. *Every ring morphism $f : A \to B$ induces a calyx morphism $f^{cal} : A^{cal} \to B^{cal}$ where f_*^{cal} is extension of ideals under f and f_*^{cal} is contraction of ideals under f. cal : **Rng** $\to$ **Cal** is a functor sending A to A^{cal} and $f : A \to B$ to f^{cal}.*

Proof. We already know there is a functor **Rng** $\to$ **Lat** which comes as a restriction of the functor **Grp** $\to$ **Lat**. Take rings A and B and let $f : A \to B$ be a ring-homomorphism. Then $f(\mathfrak{a}\mathfrak{b}) = f(\mathfrak{a})f(\mathfrak{b})$ for ideals $\mathfrak{a}, \mathfrak{b} \in A^{cal}$. And since $f(1) = 1$, the extension of $f(A)$ is B. Thus f^{cal} is a morphism of the monoids induced by ideal product in A^{cal} and B^{cal}. Clearly $(id_A)^{cal} = id_{A^{cal}}$ and $(g \circ f)^{cal} = g^{cal} \circ f^{cal}$ for ring maps $f : A \to B$ and $g : B \to C$. □

2.4. **Corollas.**

Definition 22. For a complete lattice M, the complete lattice $E = End_{\mathbf{Lat}}(M)$ is in fact a calyx. To see this, define a map $Comp : E \to End_{\mathbf{Lat}}(E)$ where $\phi = (\phi_*, \phi^*)$ is sent to the adjoint pair whose left adjoint is composition by ϕ. Explicitly, we define $Comp(\phi)_*$ to send an adjoint pair ψ to the adjoint pair whose left adjoint is $\phi_* \circ \psi_*$ and whose right adjoint is $\phi^* \circ \psi^*$. To see that $Comp(\phi)_*$ is left adjoint, take $\{\psi_i\}_{i \in I}$ in E. Then

$$\phi_* \left(\left(\varinjlim_{i \in I} (\psi_i)_* \right) (x) \right) = \phi_* \left(\varinjlim_{i \in I} ((\psi_i)_*(x)) \right) = \varinjlim_{i \in I} \phi_* \left((\psi_i)_*(x) \right) = \left(\varinjlim_{i \in I} (\phi_* \circ (\psi_i)_*) \right) (x)$$

since ϕ_* is left adjoint, so that $Comp(\phi)_*$ distributes over colimits and is left adjoint by the adjoint functor theorem for posets, lemma 2. Define $Comp(\phi)$ to be this adjoint pair.

Next let $\{\phi_i\}_{i \in I}$ be a collection of adjoint pairs in E and take $\psi \in E$.

$$Comp(\varinjlim_{i \in I} \phi_i)_*(\psi) = (\varinjlim_{i \in I} \phi_i)_* \circ \psi = \varinjlim_{i \in I} (\phi_{i*} \circ \psi) = \varinjlim_{i \in I} Comp(\phi_{i*})_*(\psi)$$

It follows that $Comp(\varinjlim_{i \in I} \phi_i)_* = \varinjlim_{i \in I} Comp(\phi_i)_*$ so that $Comp(\varinjlim_{i \in I} \phi_i) = \varinjlim_{i \in I} Comp(\phi_i)$. Thus $Comp$ distributes over colimits, so that by lemma 2, $Comp$ is left adjoint. Lastly, we check associativity and unity as follows:

$$Comp(\phi \circ \psi)(\rho) = \phi \circ \psi \circ \rho = Comp(\phi)(Comp(\psi)(\rho))$$
$$Comp(1)(\phi) = 1 \circ \phi = \phi$$

Definition 23. Let E be a calyx. An E-corolla is a complete lattice M equipped with a calyx morphism $E \to End_{\mathbf{Lat}}(M)$. We often write M for 1 in the lattice M, an abuse of notation.

Lemma 24. *An A-module M induces an A^{cal}-corolla, called M^{cor}.*

Proof. Define $\Phi : E(A) \to End_{\mathbf{Lat}}(E(M))$ by taking

$$\Phi(\mathfrak{a})_*(x) = \mathfrak{a}x = \left\{ \sum_{i=1}^{n} a_i y_i : a_i \in \mathfrak{a}, y_i \in x \right\}$$

and

$$\Phi(\mathfrak{a})^*(x) = (x : \mathfrak{a}) = \{y \in M : y\mathfrak{a} \leq x\}$$

$\Phi(\mathfrak{a})_*$ and $\Phi(\mathfrak{a})^*$ are adjoint for each $\mathfrak{a} \in E(A)$. Indeed, $\mathfrak{a}x \leq y \iff x \leq (y : \mathfrak{a})$. Φ distributes over colimits, so that it is left adjoint by the adjoint functor theorem.

Moreover $\Phi(\mathfrak{a}\mathfrak{b})_*(x) = \mathfrak{a}\mathfrak{b}x = \Phi(\mathfrak{a})_*(\Phi(\mathfrak{b})_*(x))$ and

$$\Phi(\mathfrak{a}\mathfrak{b})^*(x) = (x : \mathfrak{a}\mathfrak{b}) = ((x : \mathfrak{a}) : \mathfrak{b}) = \Phi(\mathfrak{a})_*(\Phi(\mathfrak{b})_*(x))$$

□

Remark. A calyx E can be viewed as a corolla over itself. By definition there is a **Lat** morphism $E \to End_{\mathbf{Lat}}(E)$. This morphism becomes a morphism of calyxes under the apparent calyx structure of $End_{\mathbf{Lat}}(E)$.

Lemma 25. *The following properties hold in a general E-corolla. Notice in particular that they hold for M^{cor} for an A-module M.*

Reason	*Property*	*Dual Property*
ϕ^* is a functor.	$\mathfrak{a} \leq \mathfrak{b} \Rightarrow \mathfrak{a}x \leq \mathfrak{b}x$	$\mathfrak{a} \leq \mathfrak{b} \Rightarrow (x : \mathfrak{b}) \leq (x : \mathfrak{a})$
$\phi^*(\mathfrak{a})_*$ and $\phi^*(\mathfrak{a})_*$ are functors.	$x \leq y \Rightarrow \mathfrak{a}x \leq \mathfrak{a}y$	$x \leq y \Rightarrow (x : \mathfrak{a}) \leq (y : \mathfrak{a})$.
$\phi_*(1) = 1$.	$1x = x$	$(x : 1) = x$
ϕ_* is left adjoint and therefore preserves initial objects.	$0x = 0$	$(x : 0) = 1$
ϕ_* is left adjoint and therefore preserves colimits.	$\sum_{i \in I}(\mathfrak{a}_i x) = \left(\sum_{i \in I} \mathfrak{a}_i\right) x$	$\bigcap_{i \in I}(x : \mathfrak{a}_i) = \left(x : \sum_{i \in I} \mathfrak{a}_i\right)$
$\phi_*(\mathfrak{a})_*$ (resp. $\phi_*(\mathfrak{a})^*$ is left adjoint (resp. right adjoint) and therefore preserves colimits (resp. limits).	$\mathfrak{a} \sum_{i \in I} x_i = \sum_{i \in I} \mathfrak{a}x_i$	$\left(\bigcap_{i \in I} x_i : \mathfrak{a}\right) = \bigcap_{i \in I} (x_i : \mathfrak{a})$
Canonical morphism from universal property of colimit (resp. limit)	$\mathfrak{a} \bigcap_{i \in I} x_i \leq \bigcap_{i \in I} \mathfrak{a}x_i$	$\sum_{i \in I} (x_i : \mathfrak{a}) \leq \left(\sum_{i \in I} x_i : \mathfrak{a}\right)$
Unitor and counitor from adjoint relationship $\phi_*(\mathfrak{a})_* \dashv \phi_*(\mathfrak{a})^*$	$x \leq (\mathfrak{a}x : \mathfrak{a})$	$\mathfrak{a}(x : \mathfrak{a}) \leq x$.
$\phi_*(\mathfrak{a}\mathfrak{b}) = \phi_*(\mathfrak{a}) \circ \phi_*(\mathfrak{b})$	$\mathfrak{a}(\mathfrak{b}x) = (\mathfrak{a}\mathfrak{b})x$	$((x : \mathfrak{a}) : \mathfrak{b}) = (x : \mathfrak{a}\mathfrak{b})$
$\phi_*(\mathfrak{a}) \leq 1$	$\mathfrak{a}x \leq x$	$(x : \mathfrak{a}) \geq x$.
$\phi_*(\mathfrak{a})_*$ (resp. $\phi_*(\mathfrak{a})^*$) is left adjoint (resp. right adjoint) and so preserves initial objects (resp. terminal objects).	$\mathfrak{a}0 = 0$	$(1 : \mathfrak{a}) = 1$
$\phi_*(\mathfrak{a})_* \dashv \phi_*(\mathfrak{a})^*$	$\mathfrak{a}x \leq y \;\Leftrightarrow\; x \leq (y : \mathfrak{a})$	
$\phi_*(\mathfrak{a})_*$ is a limit, $\phi_*(\mathfrak{b})_*$ is a right adjoint.	$\mathfrak{a}(x : \mathfrak{b}) = (\mathfrak{a}x : \mathfrak{b})$	

Definition 26. Let M and N be E-corollas with structure maps $\phi : E \to End_{\mathbf{Lat}}(M)$ and $\psi : E \to End_{\mathbf{Lat}}(N)$. An E-lattice morphism is a morphism $f : M \to N$ in **Lat** such that, for each $\mathfrak{a} \in E$, the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \phi(\mathfrak{a})} & & \downarrow{\scriptstyle \psi(\mathfrak{a})} \\ M & \xrightarrow{f} & N \end{array}$$

Lemma 27. *Every A-module morphism $f : M \to N$ induces an A^{cal}-corolla morphism $f^{cor} : M^{cor} \to N^{cor}$ by extension and contraction. The resulting function $cor : A$-**mod** $\to A^{cal}$-**Cor** is a functor.*

Proof. Take $\mathfrak{a} \in M^{cor}$, $\mathfrak{b} \in N^{cor}$. $f(\mathfrak{a}) \subseteq \mathfrak{b} \iff \mathfrak{a} \subseteq f^{-1}(\mathfrak{b})$, so extension and contraction of submodules is indeed an adjoint relationship. Also extension and contraction by an identity map of modules does nothing, and

$$(f \circ g)_*^{cor}(\mathfrak{a}) = f \circ g(\mathfrak{a}) = f_*^{cor} \circ g_*^{cor}(\mathfrak{a})$$

$$(f \circ g)^{cor*}(\mathfrak{a}) = g^{-1}(f^{-1}(\mathfrak{a})) = g^{cor*} \circ f^{cor*}(\mathfrak{a})$$

□

Lemma 28. *Let $\mathfrak{f} : M \to N$ be a morphism of E-corollas. Take elements $\mathfrak{a}, \{\mathfrak{a}_i\}_{i\in I}$ in E, $x, \{x_i\}_{i\in I}$ in M, and $y, \{y_i\}_{i\in I}$ in N. Then*

Reason	*Property*	*Dual Property*
$\mathfrak{f}_*$ *(resp. $\mathfrak{f}^*$) is left adjoint (resp. right adjoint) and so distributes over colimits (resp limits)*	$\mathfrak{f}_*\left(\sum_{i\in I} \mathfrak{a}_i\right) = \sum_{i\in I} \mathfrak{f}_*(\mathfrak{a}_i)$	$\mathfrak{f}^*\left(\bigcap_{i\in I} \mathfrak{b}_i\right) = \bigcap_{i\in I} \mathfrak{f}^*(\mathfrak{b}_i)$
$\mathfrak{f}_*$ *(resp. $\mathfrak{f}^*$) is left adjoint (resp. right adjoint) and so preserves initial objects (resp. terminal objects).*	$\mathfrak{f}_*(0) = 0$	$\mathfrak{f}^*(1) = 1$
Canonical morphism from the universal property of product (resp. coproduct)	$\mathfrak{f}_*\left(\bigcap_{i\in I} \mathfrak{a}_i\right) \leq \bigcap_{i\in I} \mathfrak{f}_*(\mathfrak{a}_i)$	$\sum_{i\in I} \mathfrak{f}_*(\mathfrak{b}_i) \leq \mathfrak{f}^*\left(\sum_{i\in I} \mathfrak{b}_i\right)$
Unitor and counitor from adjoint relationship $\mathfrak{f}_* \dashv \mathfrak{f}^*$	$\mathfrak{a} \leq \mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{a}))$	$\mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{b})) \leq \mathfrak{b}$
	$\mathfrak{f}_*(\mathfrak{a}) = \mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{a})))$	$\mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{b}))) = \mathfrak{f}^*(\mathfrak{b})$

Lemma 29. *Let E be a calyx and M an E-corolla. Note that, for $\mathfrak{a} \in E$ and $x, y \in M$, $\mathfrak{a}x \leq y \iff x \leq (y : \mathfrak{a})$. This leads to a characterization of $\mathfrak{a}x$.*

$$\mathfrak{a}x = \varprojlim_{\mathfrak{a}x \leq y} y = \varprojlim_{x \leq (y:\mathfrak{a})} y$$

Likewise,

$$(x : \mathfrak{a}) = \varinjlim_{y \leq (x:\mathfrak{a})} y = \varinjlim_{\mathfrak{a}y \leq x} y$$

Lemma 30. *Let $\mathfrak{f} : M \to N$ be a morphism of E-corollas. Then*

$\mathfrak{f}_*(\mathfrak{a}x) = \mathfrak{a}\mathfrak{f}_*(x)$	$\mathfrak{a}\mathfrak{f}^*(x) \leq \mathfrak{f}^*(\mathfrak{a}x)$
$\mathfrak{f}_*((x : \mathfrak{a})) \leq (\mathfrak{f}_*(x) : \mathfrak{a})$	$\mathfrak{f}^*((x : \mathfrak{a})) \leq (\mathfrak{f}^*(x) : \mathfrak{a})$

Proof. The top left follows by definition. For the top right,

$$\mathfrak{f}^*(\mathfrak{a}x) = \mathfrak{f}^*(\varprojlim_{\mathfrak{a}x \leq y} y) = \varprojlim_{\mathfrak{a}x \leq y} \mathfrak{f}^*(y) \leq \varprojlim_{\mathfrak{f}^*(\mathfrak{a}x) \leq y} y = \mathfrak{f}^*(\mathfrak{a}x)$$

The others follow similarly. □

3. Universal Constructions in **Lat**, **Cal**, and E-**Cor**

In this section we calculate many of the limits and colimits in **Lat**, **Cal**, and E-**Cor**.

Lat The categry of complete lattices whose morphisms are adjoint pairs.

Cal The category of calyxes and calyx morphisms.

E-**Cor** The category of E-corollas with E-corolla morphisms.

For the case of corollas we also establish a notion of perennial morphism $\mathfrak{f} : M \to N$ in E-**Cor**, meaning that $\mathfrak{f}_* : M \to N$ is injective on $\mathfrak{f}^*(N)$ and $\mathfrak{f}^*$ is injective on $\mathfrak{f}_*(M)$. If $\mathfrak{f}$ is not perennial we call it annual. Perennial morphisms more closely resemble morphisms of modules, and indeed every morphism of modules induces a perennial morphism of E-corollas. The category E-**Cor** is complete with a zero object. However, only the perennial monomorphisms are normal and only the perennial epimorphisms are conormal, so E-**Cor** lacks the crucial property of being abelian. Restricting the category to perennial morphisms, the category loses existence of products and coproducts.

3.1. **The Category Lat.**

Lemma 31. *Let* $\mathfrak{f} : M \to N$ *be a morphism in* ***Lat***. *Then* $\mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{f}_*(x))) = \mathfrak{f}_*(x)\ \forall x \in M$ *and* $\mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{f}^*(x))) = \mathfrak{f}^*(x)\ \forall x \in N$.

Proof. $\mathfrak{f}_*(\mathfrak{f}^*(x)) \leq x\ \forall x \in N$, so, applying $\mathfrak{f}^*$, $\mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{f}^*(x))) \leq \mathfrak{f}^*(x)$. And $\mathfrak{f}^*(\mathfrak{f}_*(y)) \geq y\ \forall y \in M$, so taking $y = \mathfrak{f}_*(x)$, $\mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{f}^*(x))) \geq \mathfrak{f}^*(x)$. The other claim follows similarly. □

Theorem. *Let* $\mathfrak{f} : M \to N$ *be a morphism in* ***Lat***. *The following are equivalent:*

(1) $\mathfrak{f}^*(\mathfrak{f}_*(x)) = x\ \forall x \in M$
(2) $\mathfrak{f}_*$ *is injective.*
(3) $\mathfrak{f}^*$ *is surjective.*
(4) $\mathfrak{f}$ *is a monomorphism.*

Proof. We show (1) $\Leftrightarrow$ (2), (1) $\Leftrightarrow$ (3), (2) $\Leftrightarrow$ (4).

(1) $\Rightarrow$ (2). Suppose $\mathfrak{f}^*(\mathfrak{f}_*(x)) = x\ \forall x \in M$. Then $\mathfrak{f}_*(x) = \mathfrak{f}_*(y) \Rightarrow x = \mathfrak{f}^*(\mathfrak{f}_*(x)) = \mathfrak{f}^*(\mathfrak{f}_*(y)) = y$.

(2) $\Rightarrow$ (1). Suppose $\mathfrak{f}_*$ is injective. By 31, $\mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{f}_*(x))) = \mathfrak{f}_*(x)\ \forall x \in M$, so $\mathfrak{f}^*(\mathfrak{f}_*(x)) = x\ \forall x \in M$.

(1) $\Rightarrow$ (3). Suppose $\mathfrak{f}^*(\mathfrak{f}_*(x)) = x\ \forall x \in M$. It follows immediately that $\mathfrak{f}^*$ is surjective.

(3) $\Rightarrow$ (1). Suppose $\mathfrak{f}^*$ is surjective. By 31, $\mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{f}^*(y))) = \mathfrak{f}^*(y)\ \forall y \in M$. Taking $x \in M$, we can write $x = \mathfrak{f}^*(y)$ for some $x \in M$, so that

$$\mathfrak{f}^*(\mathfrak{f}_*(x)) = \mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{f}^*(y))) = \mathfrak{f}^*(y) = x$$

Clearly (2) $\Rightarrow$ (4) using the uniqueness of right adjoints for a given left adjoint. Suppose $\neg$(2). Then take $x, y \in M$ such that $\mathfrak{f}_*(x) \neq \mathfrak{f}_*(y)$. There are left adjoint morphisms $\mathfrak{g}, \mathfrak{h} : I = \{0, 1\} \to M$ where $\mathfrak{g}(1) = x$ and $\mathfrak{h}(1) = y$. $\mathfrak{f} \circ \mathfrak{g} = \mathfrak{f} \circ \mathfrak{h}$ but $\mathfrak{g} \neq \mathfrak{h}$.

□

Theorem. *Let* $\mathfrak{f} : M \to N$ *be a morphism in* ***Lat***. *The following are equivalent:*

(1) $\mathfrak{f}_*(\mathfrak{f}^*(x)) = x \ \forall x \in N$
(2) $\mathfrak{f}^*$ *is injective.*
(3) $\mathfrak{f}_*$ *is surjective.*
(4) $\mathfrak{f}$ *is an epimorphism.*

Proof. Follows in the same way as before. □

Example 32. Not all monomorphisms in **Lat** are kernels. For example take the lattice $\mathbb{Z}^{\mathbf{Lat}}$. There is a morphism $\mathfrak{f} : I \to \mathbb{Z}^{\mathbf{Lat}}$ such that $\mathfrak{f}_*(0) = 0$ and $\mathfrak{f}_*(1) = 1$. But clearly this is not a kernel.

Lemma 33. *Take a complete lattice $E \in$ **Lat**. The set*

$$\mathcal{C} = \{X \subseteq E : X \text{ is closed under sums and contains } 0\}$$

forms a poset category. Note that elements $X \in \mathcal{C}$ form complete lattices with an intersection and terminal object possibly distinct from the ones in E (a poset with arbitrary upper bounds and an initial object is complete). Morphisms $\mathfrak{f} : X \to Y$ are lattice morphisms $\mathfrak{f} : X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\mathfrak{f}} & Y \\ & \searrow & \downarrow \\ & & E \end{array}$$

The category $Sub(E)$ of subobjects of E is categorically equivalent to $\mathcal{C}$.

Proof. Define a functor $\Phi : Sub(E) \to E$ where a subobject $\mathfrak{f} : X \to E$ is sent to $\mathfrak{a} = \mathfrak{f}_*(X)$. The resulting set $\mathfrak{a}$ is closed under sums and contains the initial object 0 of the lattice E, since $\mathfrak{f}_*$ is left adjoint. To define an inverse map $\Psi : E \to Sub(E)$, send $\mathfrak{a}$ to the monomophism $\mathfrak{a} \to E$ whose left adjoint is the inclusion map $\mathfrak{a} \to E$. □

Lemma 34. *Take a complete lattice $E \in$ **Lat**. The set*

$$\mathcal{D} = \{X \subseteq E : X \text{ is closed under intersections and contains } 1\}$$

forms a poset category. Note that elements $X \in \mathcal{C}$ form complete lattices with a sum and initial object possibly distinct from the ones in E (a poset with arbitrary lower bounds and a terminal object is complete). Morphisms $\mathfrak{f} : X \to Y$ are morphisms $\mathfrak{f} : X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} E & & \\ \downarrow & \searrow & \\ X & \xrightarrow{\mathfrak{f}} & Y \end{array}$$

The category $Quot(E)$ of subobjects of E is categorically equivalent to $\mathcal{D}$.

Proof. Similar to the previous proof. □

Lemma 35. *There is a unique element 0 in **Lat** which has one element, which forms a zero object for **Lat**.*

Definition 36. Let M be a complete lattice. We define $(x) = \{y \in M : y \leq x\}$ and $[x] = \{y \in M : y \geq x\}$. There are canonical adjunctions $i = (i_*, i^*) : (x) \to M$ where

$i_*(y) = y$ and $i^*(y) = y \cap x$, and $\pi : M \to [x]$ where $\pi_*(y) = y + x$ and $\pi^*(y) = y$. To check that $i_* \dashv i^*$, take $y \in (x)$ and $z \in M$. Then

$$y \le i^*(z) \;\Leftrightarrow\; y \le z \cap x \;\Leftrightarrow\; y \le z \;\Leftrightarrow\; i^*(y) \le z$$

To check that $\pi_* \dashv \pi^*$, take $y \in M$ and $z \in [x]$. Then

$$y \le \pi^*(z) \;\Leftrightarrow\; y \le z \;\Leftrightarrow\; y + x \le z \;\Leftrightarrow\; \pi_*(y) \le z$$

Lemma 37. Lat *has kernels.*

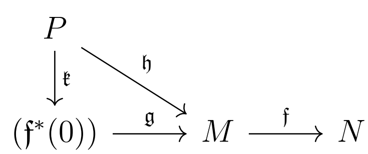
Proof. Let $\mathfrak{f} : M \to N$ be a morphism in Lat. Let $\mathfrak{g}_* : (\mathfrak{f}^*(0)) \to M$ be the embedding and let $\mathfrak{g}^* : M \to (\mathfrak{f}^*(0))$ send x to $x \cap \mathfrak{f}^*(0)$. These form an adjoint pair: take $x \le \mathfrak{f}^*(0)$ and $y \in M$. Then, since $x \le \mathfrak{f}^*$,

$$x \le \mathfrak{g}^*(y) \;\Leftrightarrow\; x \le y \cap \mathfrak{f}^*(0) \;\Leftrightarrow\; x \le y$$

$\mathfrak{g} = (\mathfrak{g}_*, \mathfrak{g}^*)$ is a kernel for $\mathfrak{f}$ in **Lat**. To show this, take a complete lattice P and a morphism $\mathfrak{h} = (\mathfrak{h}_*, \mathfrak{h}^*) : P \to M$ such that $\mathfrak{f} \circ \mathfrak{h}$ factors through the 0 object. Then $\mathfrak{f}_*(\mathfrak{h}_*(x)) = 0 \; \forall x \in P$, so that $\mathfrak{h}_*(x) \le \mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{h}_*(x))) = \mathfrak{f}^*(0)$. Thus $\mathfrak{h}_*$ factors through $(\mathfrak{f}^*(0))$ by a morphism $\mathfrak{k}_* : P \to (\mathfrak{f}^*(0))$. Define a lattice morphism $\mathfrak{k}^* : (\mathfrak{f}^*(0)) \to P$ where $\mathfrak{k}^*(x) = \mathfrak{h}^*(x)$. $\mathfrak{k}_*$ and $\mathfrak{k}^*$ form an adjoint pair: take $x \in P$ and $y \le \mathfrak{f}^*(0)$. Then

$$x \le \mathfrak{k}^*(y) \;\Leftrightarrow\; x \le \mathfrak{h}^*(y) \;\Leftrightarrow\; \mathfrak{h}^*(x) \le y \;\Leftrightarrow\; \mathfrak{k}^*(x) \le y$$

Thus $\mathfrak{g} = ker(\mathfrak{f})$ in **Lat**.



∎

Lemma 38. Lat *has cokernels.*

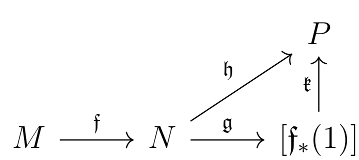
Proof. Let $\mathfrak{f} : M \to N$ be a morphism in Lat. Let $\mathfrak{g}_* : N \to [\mathfrak{f}_*(1)]$ be the quotient map sending x to $x + \mathfrak{f}_*(1)$ and let $\mathfrak{g}^* : [\mathfrak{f}_*(1)] \to N$ be the restriction sending x to x. These form an adjoint pair: take $x \in N$ and $y \ge \mathfrak{f}^*(1)$. Then, since $y \ge \mathfrak{f}^*(1)$,

$$\mathfrak{g}_*(x) \le y \;\Leftrightarrow\; x + \mathfrak{g}^*(0) \le y \;\Leftrightarrow\; x \le y$$

$\mathfrak{g} = (\mathfrak{g}_*, \mathfrak{g}^*)$ is a cokernel for $\mathfrak{f}$ in **Lat**. To show this, take a complete lattice P and a morphism $\mathfrak{h} = (\mathfrak{h}_*, \mathfrak{h}^*) : N \to P$ such that $\mathfrak{h} \circ \mathfrak{f}$ factors through the 0 object. Then $\mathfrak{h}^*(\mathfrak{f}^*(0)) = 0 \; \forall x \in P$, so that $\mathfrak{h}^*(x) = \mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{h}^*(x))) \le \mathfrak{f}_*(1)$. Thus $\mathfrak{h}^*$ factors through $[\mathfrak{f}_*(1)]$ by a morphism $\mathfrak{k}^* : [\mathfrak{f}_*(1)] \to P$. Define a lattice morphism $\mathfrak{k}_* : P \to [\mathfrak{f}_*(1)]$ where $\mathfrak{k}_*(x) = \mathfrak{h}^*(x) \le \mathfrak{f}_*(1)$. $\mathfrak{k}_*$ and $\mathfrak{k}^*$ form an adjoint pair: take $x \ge \mathfrak{f}_*(1)$ and $y \in P$. Then

$$x \le \mathfrak{f}^*(y) \;\Leftrightarrow\; x \le \mathfrak{h}^*(y) \;\Leftrightarrow\; \mathfrak{h}_*(x) \le y \;\Leftrightarrow\; \mathfrak{k}_*(x) \le y$$

Thus $\mathfrak{g} = cok(\mathfrak{f})$ in **Lat**.



□

Lemma 39. *The normal monomorphisms $\mathfrak{f} : M \to N$ in **Lat** are all equivalent (as subobjects) to $(x) \to N$ for some $x \in N$.*

Lemma 40. *The conormal morphisms $\mathfrak{f} : M \to N$ in **Lat** are all equivalent (as quotient objects) to $M \to [x]$ for some $x \in M$.*

Definition 41. Let $\{M_i\}_{i\in I}$ be complete lattices. We form the direct sum lattice $\oplus_{i\in I} M_i$ from the cartesian product $P = \{(x_i)_{i\in I} : x_i \in M_i\}$ where $(x_i)_{i\in I} \le (y_i)_{i\in I}$ when $x_i \le y_i$ $\forall i \in I$. Take $X \subseteq P$. It follows that

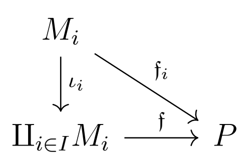
$$\cap_{x\in X} x = \left(\cap_{x=(x_j)_{j\in I}\in X} x_i\right)_{i\in I}$$

$$\sum_{x\in X} x = \left(\sum_{x=(x_j)_{j\in I}\in X} x_i\right)_{i\in I}$$

So that $\oplus_{i\in I} M_i$ is complete. There are canonical morphisms $\iota_i : M_i \to \oplus_{i\in I} M_i$. Define ι_i as follows: set $\iota_{i*}(x) = (x_j)_{j\in I}$ where $x_i = x$ and $x_j = 0$ for $j \neq i$. Set ${\iota_i}^*((x_i)_{i\in I}) = x_i$ clearly $\iota_{i*} \dashv {\iota_i}^*$.

Lemma 42. *Let $\{M_i\}_{i\in I}$ be complete lattices. $\oplus_{i\in I} M_i$ forms a coproduct of the lattices M_i.*

Proof. Let $\iota_i : M_i \to \amalg_{i\in I} M_i$ be the canonical maps. Take a complete lattice P and morphisms $\mathfrak{f}_i : M_i : P$. To make the following diagram commute we must define $\mathfrak{f} : \amalg_{i\in I} M_i \to P$ by setting $\mathfrak{f}_*((x_i)_{i\in I}) = \sum_{i\in I} \mathfrak{f}_i(x_i)$.



$\mathfrak{f}_*$ is then left adjoint by the adjoint functor theorem for posets, as it distributes over colimits. □

Lemma 43. *Let $\{M_i\}_{i\in I}$ be complete lattices. $\oplus_{i\in I} M_i$ forms a product of the lattices M_i.*

Proof. Similar to the case of coproducts. □

Definition 44. Let M and N be complete lattices. We form the lattice $Hom_{\mathbf{Lat}}(M, N)$ from the set of adjoint pairs

$$A = \{\mathfrak{f} : \mathfrak{f} = (\mathfrak{f}_*, \mathfrak{f}^*) : M \to N, \mathfrak{f}_* \dashv \mathfrak{f}^*, \mathfrak{f}_* : M \to N, \mathfrak{f}^* : N \to M\}$$

where $\mathfrak{f} \le \mathfrak{g}$ when $\mathfrak{f}_*(x) \le \mathfrak{g}_*(x)$ $\forall x \in M$ and $\mathfrak{f}^*(x) \ge \mathfrak{g}^*(x)$ $\forall x \in N$. To show that $Hom_{\mathbf{Lat}}(M, N)$ is complete it suffices to show that $Hom_{\mathbf{Lat}}(M, N)$ has arbitrary joins, by the adjoint functor theorem for posets.

Let $\{\phi_i\}_{i\in I}$ be elements of $Hom_{\mathbf{Lat}}(M, N)$. Define $\alpha_*(x) = \sum_{i\in I} (\phi_i)_*(x)$ and $\alpha^*(x) = \bigcap_{i\in I} (\phi_i)^*(x)$. α_* and α^* are poset functors $M \to N$; since colimit is a functor,

$$x \le y \Rightarrow \sum_{i\in I} (\phi_i)_*(x) \le \sum_{i\in I} (\phi_i)_*(y)$$

since limit is a functor,

$$x \leq y \Rightarrow \bigcap_{i \in I} (\phi_i)^*(x) \leq \bigcap_{i \in I} (\phi_i)^*(y)$$

To show $\alpha_* \dashv \alpha^*$ we must show $\sum_{i \in I} (\phi_i)_*(x) \leq y \iff x \leq \bigcap_{i \in I} (\phi_i)^*(y)$. Observe that

$$\begin{aligned} &\sum_{i \in I} (\phi_i)_*(x) \leq y \\ &\Leftrightarrow (\phi_i)_*(x) \leq y \; \forall i \in I \\ &\Leftrightarrow x \leq (\phi_i)^*(y) \; \forall i \in I \\ &\Leftrightarrow x \leq \bigcap_{i \in I} (\phi_i)^*(y) \end{aligned}$$

A 0 element for $Hom_{\mathbf{Lat}}(M, N)$ is (ϕ_*, ϕ^*) where $\phi_*(x) = 0$ for each $x \in M$ and $\phi^*(x) = 1$ for each $x \in N$.

Definition 45. Let $\{M_i\}_{i=1}^n$ be lattices. Let $\mathfrak{f}_* : \Pi_{i=1}^n M_i \to P$ be a set map from the cartesian product $\Pi_{i=1}^n M_i$ to a lattice P. We say $\mathfrak{f}_*$ is multilinear if, for each $1 \leq i \leq n$, and for each choice of elements $X = (x_1, ..., x_{i-1}, \hat{x_i}, x_{i+1}, ..., x_n)$ with x_i omitted, the map $\mathfrak{f}_{X*} : M_i \to P$ sending x_i to $\mathfrak{f}_*(x_1, ..., x_n)$ is left adjoint.

Definition 46. Take lattices M and N, which we can view as I-corollas. We form the tensor product of lattices $M \otimes_I N$ as follows: Take $M \otimes_I N = (\oplus_{e \in M \times N} I) / \sim$, where $M \times N$ is the cartesian product and $\sim$ is the intersection of all equivalence relations such that

$$\left(\sum_{i \in I} x_i, y \right) \sim \sum_{i \in I} (x_i, y) \text{ and } (0, y) \sim 0$$
$$\left(x, \sum_{i \in I} y_i \right) \sim \sum_{i \in I} (x, y_i) \text{ and } (x, 0) \sim 0$$
$$x_i \sim y_i \forall i \in I \Rightarrow \sum_{i \in I} x_i \sim \sum_{i \in I} y_i, \bigcap_{i \in I} x_i \sim \bigcap_{i \in I} y_i$$

Notice there is a canonical map $M \times N \to M \otimes_I N$ where (m, n) is sent to the equivalence class generated by (m, n).

Lemma 47 (Universal Property of Tensor Product)**.** *Let M and N be elements of **Lat**. Let $\phi : M \times N \to M \otimes_I N$ be the canonical map. For each multilinear map $f_* : M \times N \to P$ there is a unique lattice map $g : M \otimes_I N \to P$ such that $g \circ \phi = f$.*

Theorem. ***Lat** forms a monoidal category under tensor product.*

Definition 48. Define for each set S the lattice S^{lat} whose underlying set is the power set $\mathcal{P}(S)$ and such that $U \leq V$ in $\mathcal{P}(S)$ when $U \subseteq V$. This makes $(-)^{lat}$ into a functor from **Set** to **Lat** which has as a right adjoint the forgetful functor **Lat** $\to$ **Set**.

3.2. **The Category Cal.**

Definition 49. The unique calyx with a single element is called the 0 calyx. It is a terminal object in **Cal**.

Definition 50. We have the calyx $I = \{0,1\} = \mathbb{F}^{cal}$ for any field $\mathbb{F}$. I is an initial object in the category **Cal**.

Lemma 51. *For an E-corolla M and an element $x \in M$ there is a unique E-corolla morphism $\mathfrak{f}_x : E \to M$ such that $\mathfrak{f}_{x*}(1) = x$.*

Proof. We must have $\mathfrak{f}_{x*}(\mathfrak{a}) = \mathfrak{a}\mathfrak{f}_{x*}(1) = \mathfrak{a}x$. Defining $\mathfrak{f}_{x*}$ in this way, we have

$$\mathfrak{f}_{x*}\left(\sum_{i\in I}\mathfrak{a}_i\right) = \left(\sum_{i\in I}\mathfrak{a}_i\right)x = \sum_{i\in I}(\mathfrak{a}_i x) = \sum_{i\in I}\mathfrak{f}_{x*}\left(\sum_{i\in I}\mathfrak{a}_i\right)$$

and $\mathfrak{f}_{x*}(0) = 0$, so that $\mathfrak{f}_{x*}$ is indeed left adjoint. Clearly $\mathfrak{f}_{x*}(\mathfrak{a}\mathfrak{b}) = \mathfrak{a}\mathfrak{f}_x(\mathfrak{b})$. □

Remark. For an E-corolla M and an element $x \in M$, there is not always a perennial corolla morphism $\mathfrak{f} : E \to M$ such that $\mathfrak{f}(1) = x$. We call such elements principal elements of M. For a ring A and an A module M, the regular morphisms from A^{cal} to M^{cor} resemble elements of M up to units. Take a vector space V over a field $\mathbb{F}$. The nonzero perennial corolla morphisms from $I = \mathbb{F}^{cal}$ to V^{cor} correspond to elements of projective space over V.

Definition 52. For a morphism $\mathfrak{f} : E \to F$ of calyxes we define $ker(\mathfrak{f}) = (\mathfrak{f}^*(0))$. It is a subcorolla, a notion to be defined later. Define $im(\mathfrak{f}) = \{\mathfrak{f}_*(\mathfrak{a}) : \mathfrak{a} \in E\}$ and $coim(\mathfrak{f}) = \{\mathfrak{f}^*(\mathfrak{a}) : \mathfrak{a} \in F\}$. $im(\mathfrak{f})$ is a subcalyx of F and $coim(\mathfrak{f})$ is a subcalyx of E, a notion to be defined later.

Example 53. Let $E = \mathbb{Z}^{cal}$ and $F = \mathbb{Q}^{cal} = I$. The embedding $\mathbb{Z} \to \mathbb{Q}$ induces a morphism $\mathfrak{f} : E \to F$ of calyxes where $\mathfrak{f}_*(\mathfrak{a}) = 1$ for $\mathfrak{a} \neq 0$, $\mathfrak{f}_*(0) = 0$, $\mathfrak{f}^*(0) = 0$, $\mathfrak{f}^*(1) = 1$. $ker(\mathfrak{f}) = \{0\}$. $im(\mathfrak{f}) = I$, $coim(\mathfrak{f}) = \{0,1\}$. Even though $ker(\mathfrak{f}) = 0$, $\mathfrak{f}_*$ is not injective, and the obstruction to this is the difference between E and $coim(\mathfrak{f})$.

Lemma 54. *Let $\mathfrak{f} : E \to F$ be a calyx morphism. $\mathfrak{f}$ is a monomorphism in **Cal** if and only if it is a monomorphism in **Lat**.*

Proof. If $\mathfrak{f}$ is a monomorphism in **Lat** then a priori it is a monomorphism in **Cal**. Conversely, by lemma 3.1 it suffices to show that if $\mathfrak{f}_* : E \to F$ is not injective then $\mathfrak{f}$ is not a monomorphism in **Cal**. So take $\mathfrak{a}, \mathfrak{b}$ in E such that $\mathfrak{f}_*(\mathfrak{a}) = \mathfrak{f}_*(\mathfrak{b})$. Then by 51 there are morphisms $\mathfrak{g} : E \to E$ and $\mathfrak{h} : E \to E$ such that $\mathfrak{g}(1) = \mathfrak{a}$ and $\mathfrak{h}(1) = \mathfrak{b}$. Then $\mathfrak{f} \circ \mathfrak{g} = \mathfrak{f} \circ \mathfrak{h}$ but $\mathfrak{g} \neq \mathfrak{h}$. □

Lemma 55. *An epimorphism in **Lat** is an epimorphism in **Cal***

Proof. Follows from 3.1 □

Definition 56. Let E be a calyx. A subcalyx of E is a set $F \subseteq E$ closed under sums and products and contianing 0. For each $x \in E$, $(x) = \{y \in E : y \leq x\}$ forms a subcalyx, called the principal subcalyx with respect to x. It follows that (x), or any subcalyx of E, forms a calyx in its own right, as an adjoint pair is determined by a left adjoint. The quotient and intersection in a subcalyx of E is distinct from the quotient and intersection in E.

Lemma 57. *Take a calyx E. The category $Sub(E)$ of subobjects of E is equivalent to the category $\mathcal{C}$ of subcalyxes X of E whose morphisms $\mathfrak{f} : X \to Y$ are calyx epimorphisms $\mathfrak{f} : X \to Y$ making the following diagram commute:*

$$\begin{array}{ccc} X & \xrightarrow{\mathfrak{f}} & Y \\ & \searrow & \downarrow \\ & & E \end{array}$$

Proof. Define a functor $\Phi : Sub(E) \to \mathcal{C}$ where a subobject $\mathfrak{f} : X \to E$ is sent to $im(\mathfrak{f})$. For an inverse functor define $\Psi : \mathcal{C} \to Sub(E)$ where a subcalyx $X \subseteq E$ is sent to the morphism $\mathfrak{f} : X \to E$ whose left adjoint is the inclusion poset functor $X \to E$. □

Definition 58. Let E be a calyx. A quotient calyx of E is a set $F \subseteq E$ closed under intersections and ideal quotients, and containing 1. For each $x \in E$, $[x] = \{y \in E : y \geq x\}$ forms a quotient calyx, called the principal quotient calyx with respect to x. It follows that $[x]$, or any subcalyx of E, forms a calyx in its own right, as an adjoint pair is determined by a right adjoint. The product and sum in a quotient calyx of E is distinct from the product and sum in E.

Lemma 59. *Take a calyx E. The category $Quot(E)$ of quotient objects of E is equivalent to the category $\mathcal{D}$ of quotient calyxes of E whose morphisms $\mathfrak{f} : X \to Y$ are calyx epimorphisms $\mathfrak{f} : X \to Y$ making the following diagram commute:*

$$\begin{array}{ccc} E & & \\ \downarrow & \searrow & \\ X & \xrightarrow{\mathfrak{f}} & Y \end{array}$$

Proof. Define a functor $\Phi : Quot(E) \to \mathcal{D}$ where a quotient object $\mathfrak{f} : E \to X$ is sent to $coim(\mathfrak{f})$. For an inverse functor define $\Psi : \mathcal{D} \to Quot(E)$ where a quotient object X of E is sent to the morphism $\mathfrak{f} : E \to X$ whose right adjoint is the inclusion poset functor $X \to E$. □

Theorem. ***Cal*** *forms a complete category.*

Definition 60 (Tensor product of calyxes)**.**

Definition 61. There is a functor $\Phi : \mathbf{Lat} \to \mathbf{Cal}$ which sends a complete lattice E to the calyx $\oplus_{i=0}^{\infty} E^{\otimes n}$ where

$$(\mathfrak{a}_1 \otimes \cdots \otimes \mathfrak{a}_n)(\mathfrak{a}_{n+1} \otimes \cdots \otimes \mathfrak{a}_{n+m}) = \mathfrak{a}_1 \otimes \cdots \otimes \mathfrak{a}_{n+m}$$

3.3. The Category E-Cor.

3.3.1. *Morphisms in E-**Cor**.*

Definition 62. Let $\mathfrak{f} : M \to N$ be a (possibly annual) morphism of E-corollas. We define the following sets, called the kernel, cokernel, image, and coimage respectively:

$$ker(\mathfrak{f}) = (\mathfrak{f}^*(0))$$
$$cok(\mathfrak{f}) = [\mathfrak{f}_*(1)]$$
$$im(\mathfrak{f}) = (\mathfrak{f}_*(1))$$
$$coim(\mathfrak{f}) = [\mathfrak{f}^*(0)]$$

We will later see that $ker(\mathfrak{f})$ and $im(\mathfrak{f})$ are subcorollas, while $cok(\mathfrak{f})$ and $coim(\mathfrak{f})$ are quotient corollas.

Definition 63. Let $\mathfrak{f} : M \to N$ be a morphism of E-corollas. The following are equivalent:

(1) $im(\mathfrak{f}) = (\mathfrak{f}_*(1))$
(2) $\mathfrak{f}^*$ is injective on $(\mathfrak{f}_*(1))$

If these hold we say $\mathfrak{f}^*$ is perennial.

Definition 64. Let $\mathfrak{f} : M \to N$ be a morphism of E-corollas. The following are equivalent:

(1) $coim(\mathfrak{f}) = [\mathfrak{f}^*(0)]$
(2) $\mathfrak{f}_*$ is injective on $[\mathfrak{f}^*(0)]$

If these hold we say $\mathfrak{f}_*$ is perennial.

Definition 65. If $\mathfrak{f}_*$ and $\mathfrak{f}^*$ are perennial we say $\mathfrak{f}$ is perennial.

Lemma 66. *An A^{cal}-corolla morphism induced by an A-module morphism is perennial.*

Proof. □

Lemma 67. *Let $\mathfrak{f} : M \to N$ be a perennial E-corolla morphism. Then*

$$\begin{aligned} ker(\mathfrak{f}) &= \{x \in M : \mathfrak{f}_*(x) = 0\} \\ cok(\mathfrak{f}) &= \{y \in N : \mathfrak{f}^*(y) = 1\} \\ im(\mathfrak{f}) &= \{\mathfrak{f}_*(x) : x \in M\} \\ coim(\mathfrak{f}) &= \{\mathfrak{f}^*(y) : y \in N\} \end{aligned}$$

Proof. □

3.4. **The Category E-Cor.**

Lemma 68. *Let $\mathfrak{f} : M \to N$ be a perennial morphism of E-corollas. $ker(\mathfrak{f})$ is the obstruction to the injectivity of $\mathfrak{f}_*$ (equivalently, the surjectivity of $\mathfrak{f}^*$) and $cok(\mathfrak{f})$ is the obstruction to the surjectivity of $\mathfrak{f}_*$ (equivalently, the injectivity of $\mathfrak{f}_*$). More precisely, the following are equivalent:*

(1) $\mathfrak{f}^* \circ \mathfrak{f}_* = id_E$
(2) $\mathfrak{f}$ *is a monomorphism.*
(3) $\mathfrak{f}_*$ *is injective.*
(4) $\mathfrak{f}^*$ *is surjective.*
(5) $ker(\mathfrak{f}) = 0$
(6) $coim(\mathfrak{f}) = 1$

Proof. Obvious. □

Lemma 69. *Analogously, the following are equivalent for a perennial morphism $\mathfrak{f} : M \to N$ of E-corollas: and the following are equivalent:*

(1) $\mathfrak{f}_* \circ \mathfrak{f}^* = id_E$
(2) $\mathfrak{f}$ *is an epimorphism.*
(3) $\mathfrak{f}_*$ *is surjective.*
(4) $\mathfrak{f}^*$ *is injective.*
(5) $im(\mathfrak{f}) = 1$
(6) $cok(\mathfrak{f}) = 0$

Proof. Obvious. □

Definition 70. Let M be an E-corolla. For an element $x \in M$ let $(x) = \{y \in M : y \leq x\}$ and let $[x] = \{y \in M : y \geq x\}$. We show that (x) and $[x]$ are E-corollas in their own right.

Lemma 71. *Let M be an E-corolla with element $x \in M$. (x) can be made into an E-corolla in its own right. It is already a complete lattice. Define $\phi : E \to End_{\boldsymbol{Lat}}((x))$ where $\phi(\mathfrak{a})_*(y) = \mathfrak{a}y$, where $\mathfrak{a}y$ is multiplication in the corolla M. This determines an adjoint map $\phi(\mathfrak{a})^* : (x) \to (x)$ which is possibly distinct from the quotient operation in M. We have*

$$\phi(\mathfrak{a})^*(y) = \sup_{z \in (x), z \leq \phi(\mathfrak{a})^*(y)} z = \sup_{z \in (x), \mathfrak{a}z \leq y} z$$

Lemma 72. *Let M be an E-corolla with element $x \in M$. The quotient $(y : \mathfrak{a})$ in (x) is equal to $(y : \mathfrak{a}) \cap x$ where $\cdot$ is multiplication in M.*

Proof. By the uniqueness of adjoints it suffices to show that and $y \mapsto y\mathfrak{a}$ is left adjoint to $y \mapsto (y : \mathfrak{a}) \cap x$ in $[x]$. □

Lemma 73. *Let M be an E-corolla with element $x \in M$. $[x]$ can be made into an E-corolla in its own right. It is already a complete lattice. Define $\phi : E \to End_{\boldsymbol{Lat}}([x])$ where $\phi(\mathfrak{a})^*(y) = (y : \mathfrak{a})$, where $(y : \mathfrak{a})$ is the quotient operation in M. This determines an adjoint map $\phi(\mathfrak{a})_* : (x) \to (x)$ which is possibly distinct from the quotient operation in M. We have*

$$\phi(\mathfrak{a})_*(y) = \inf_{z \in [x], \phi(\mathfrak{a})_*(y) \leq z} z = \inf_{z \in [x], y \leq (z:\mathfrak{a})} z$$

Lemma 74. *Let M be an E-corolla with element $x \in M$. The multiplication $\mathfrak{a}y$ in $[x]$ is equal to $\mathfrak{a} \cdot y + x$ where $\cdot$ is multiplication in M*

Proof. By the uniqueness of adjoints it suffices to show that $y \mapsto \mathfrak{a} \cdot y + x$ is left adjoint to $y \mapsto (y : \mathfrak{a})$ in $[x]$. □

Lemma 75. *Let M be an E-corolla and take $x \in M$. There is a canonical E-corolla morphism $(x) \to M$ and a canonical E-corolla morphism $M \to [x]$.*

Proof. Take the canonical map $i : (x) \to M$ in **Lat** and $\pi : M \to [x]$ in **Lat** and note that $i(\mathfrak{a}y) = \mathfrak{a}y = \mathfrak{a}i(y)$ and

$$\pi(\mathfrak{a}y) = \mathfrak{a}y + x = \mathfrak{a}(y + x) + x = \mathfrak{a}y + \mathfrak{a}x + x = \mathfrak{a}(\pi(y))$$

□

Definition 76. We have the zero corolla 0, the unique E-corolla with one element.

Proof. Take an E-corolla M. It suffices to note that the unique lattice morphisms $0 \to M$ and $M \to 0$ are in fact corolla moprhisms. □

Lemma 77. *Let $\mathfrak{f} : M \to N$ be an E-corolla morphism. Then the canonical map $\mathfrak{g}$: $ker(\mathfrak{f}) \to M$ is a kernel for $\mathfrak{f}$ in the categorical sense.*

Lemma 78. *Let $\mathfrak{f} : M \to N$ be an E-corolla morphism. Then the canonical map $\mathfrak{g} : N \to cok(\mathfrak{f})$ is a cokernel for $\mathfrak{f}$ in the categorical sense.*

Lemma 79. *Let $\mathfrak{f} : M \to N$ be an E-corolla morphism. Then the canonical map $\mathfrak{g} : M \to im(\mathfrak{f})$ is an image for $\mathfrak{f}$ in the categorical sense.*

Lemma 80. *Let $\mathfrak{f} : M \to N$ be an E-corolla morphism. Then the canonical map $\mathfrak{g}$: $coim(\mathfrak{f}) \to N$ is a coimage for $\mathfrak{f}$ in the categorical sense.*

Lemma 81. *Let $\mathfrak{f} : M \to N$ be a perennial morphism of E-corollas. $\mathfrak{f}$ is normal if and only if $\mathfrak{f}$ is a monomorphism.*

Lemma 82. *Let $\mathfrak{f} : M \to N$ be a perennial morphism of E-corollas. $\mathfrak{f}$ is conormal if and only if $\mathfrak{f}$ is an epimorphism.*

Definition 83. Let $\{M_i\}_{i\in I}$ be E-corollas. We form the corolla $\oplus_{i\in I} M_i$ from the cartesian product $P = \{(x_i)_{i\in I} : x_i \in M_i\}$ where $(x_i)_{i\in I} \leq (y_i)_{i\in I}$ when $x_i \leq y_i$ $\forall i \in I$. Take $X \subseteq P$. It follows that

$$\cap_{x\in X} x = \left(\cap_{x=(x_j)_{j\in I}\in X} x_i\right)_{i\in I}$$

$$\sum_{x\in X} x = \left(\sum_{x=(x_j)_{j\in I}\in X} x_i\right)_{i\in I}$$

Take $x = (x_i)_{i\in I} \in P$. We define

$$\mathfrak{a}x = (\mathfrak{a}x_i)_{i\in I}$$

$$(x : \mathfrak{a}) = ((x_i : \mathfrak{a}))_{i\in I}$$

It is routine to check that this satisfies the requirements of an E-corolla. There are canonical monomorphisms $\iota_i : M_i \to \oplus_{i\in I} M_i$ where x_i maps to the element $(x_i)_{i\in I}$ where $x_j = 0$ for $j \neq i$. Likewise, there are canonical morphisms $\pi_i : \oplus_{i\in I} M_i \to M_i$ where $(x_i)_{i\in I}$ maps to x_i.

Lemma 84. *Let $\{M_i\}_{i\in I}$ be E-corollas. $\oplus_{i\in I} M_i$ forms a coproduct for $\{M_i\}_{i\in I}$*

Proof. Let P be an E-corolla and take E-corolla morphisms $\mathfrak{f}_i : M_i \to P$. We must define $\mathfrak{f} : \oplus_{i\in I} M_i \to P$ such that $\mathfrak{f} \circ \iota_i = \mathfrak{f}_i$. Take an element $(x_i)_{i\in I}$ in $\oplus_{i\in I} M_i$ and let X_i be the element whose ith entry is x_i and which is 0 elsewhere. Then we must have

$$\mathfrak{f}_*((x_i)_{i\in I}) = \mathfrak{f}_*\left(\sum_{i\in I} X_i\right) = \sum_{i\in I} \mathfrak{f}_*(X_i) = \sum_{i\in I} \mathfrak{f}_*(\iota_{i*}(x_i)) = \sum_{i\in I} \mathfrak{f}_{i*}(x_i)$$

Defining $\mathfrak{f}_*$ in this way produces a poset functor which distributes over coproducts and preserves the initial object 0, so that it is left adjoint. Moreover $\mathfrak{f}_*(\mathfrak{a}(x_i)_{i\in I}) = \mathfrak{a}\mathfrak{f}_*((x_i)_{i\in I})$, so that the lattice morphism $\mathfrak{f}$ induced by $\mathfrak{f}_*$ is an E-corolla morphism, as claimed. □

Lemma 85. *Let $\{M_i\}_{i\in I}$ be E-corollas. $\oplus_{i\in I} M_i$ forms a product for $\{M_i\}_{i\in I}$*

Proof. Let P be an E-corolla and take E-corolla morphisms $\mathfrak{f}_i : P \to M_i$. We must define $\mathfrak{f} : P \to \oplus_{i\in I} M_i$ such that $\pi_i \circ \mathfrak{f} = \mathfrak{f}_i$. Take an element $(x_i)_{i\in I}$ in $\oplus_{i\in I} M_i$ and let X_i be the element whose ith entry is x_i and which is 1 elsewhere. Then we must have

$$\mathfrak{f}^*((x_i)_{i\in I}) = \mathfrak{f}^*\left(\bigcap_{i\in I} X_i\right) = \bigcap_{i\in I} \mathfrak{f}^*(X_i) = \sum_{i\in I} \mathfrak{f}^*({\pi_i}^*(x_i)) = \bigcap_{i\in I} {\mathfrak{f}_i}^*(x_i)$$

Defining $\mathfrak{f}^*$ in this way produces a poset functor which distributes over products and preserves the terminal object 1, so that it is right adjoint. Moreover

$$\mathfrak{f}_*(\mathfrak{a}x) = \sum_{i\in I} \mathfrak{f}_{i*}(\mathfrak{a}x) = \sum_{i\in I} \mathfrak{a}\mathfrak{f}_{ii}(x) = \mathfrak{a}\left(\sum_{i\in I} \mathfrak{f}_{i*}(x)\right) = \mathfrak{a}\mathfrak{f}_*(x)$$

so that the lattice morphism $\mathfrak{f}$ induced by $\mathfrak{f}^*$ is an E-corolla morphism, as claimed. □

Theorem. *E-**Cor** forms a complete category.*

Definition 86. Let M and N be E-corollas. We form the corolla $Hom_E(M,N)$ as the set of E-corolla morphisms $\mathfrak{f} : M \to N$ where $\mathfrak{f} \leq \mathfrak{g}$ when $\mathfrak{f}_*(x) \leq \mathfrak{g}_*(x)$ $\forall x \in M$ and $\mathfrak{f}^*(x) \geq \mathfrak{g}^*(x)$ $\forall x \in N$. To show that $Hom_E(M,N)$ is complete it suffices to show that $Hom_E(M,N)$ has arbitrary joins, by the adjoint functor theorem for posets.

Let $\{\phi_i\}_{i\in I}$ be elements of $Hom_E(M,N)$. Define $\alpha_*(x) = \sum_{i\in I}(\phi_i)_*(x)$ and $\alpha^*(x) = \bigcap_{i\in I}(\phi_i)^*(x)$. α_* and α^* are poset functors $M \to N$; since colimit is a functor,

$$x \leq y \Rightarrow \sum_{i\in I}(\phi_i)_*(x) \leq \sum_{i\in I}(\phi_i)_*(y)$$

since limit is a functor,

$$x \leq y \Rightarrow \bigcap_{i\in I}(\phi_i)^*(x) \leq \bigcap_{i\in I}(\phi_i)^*(y)$$

Moreover,

$$\sum_{i\in I}(\phi_i)_*(\mathfrak{a}x) = \sum_{i\in I}\mathfrak{a}(\phi_i)_*(x) = \mathfrak{a}\sum_{i\in I}(\phi_i)_*(x)$$

and

$$\bigcap_{i\in I}(\phi_i)^*((x:\mathfrak{a})) = \bigcap_{i\in I}((\phi_i)^*(x):\mathfrak{a}) = \left(\bigcap_{i\in I}(\phi_i)^*(x):\mathfrak{a}\right)$$

To show $\alpha_* \dashv \alpha^*$ we must show $\sum_{i\in I}(\phi_i)_*(x) \leq y \iff x \leq \bigcap_{i\in I}(\phi_i)^*(y)$. Observe that

$$\begin{aligned}
&\sum_{i\in I}(\phi_i)_*(x) \leq y \\
&\Leftrightarrow (\phi_i)_*(x) \leq y \;\forall i \in I \\
&\Leftrightarrow x \leq (\phi_i)^*(y) \;\forall i \in I \\
&\Leftrightarrow x \leq \bigcap_{i\in I}(\phi_i)^*(y)
\end{aligned}$$

A 0 element for $Hom_E(M,N)$ is (ϕ_*,ϕ^*) where $\phi_*(x) = 0$ for each $x \in M$ and $\phi^*(x) = 1$ for each $x \in N$.

Lemma 87. *For a commutative calyx E, $Hom_E(E,M) \cong M$ as E-corollas.*

Proof. Define a lattice morphism $\Phi_* : M \to Hom_E(E,M)$ where $x \in M$ is sent to the morphism $\mathfrak{f}_x \in Hom_E(E,M)$ where $\mathfrak{a} \mapsto \mathfrak{a}x$. Then $\Phi_*(\sum_{i\in I} x_i) = \sum_{i\in I}\Phi_*(x_i)$ and $\Phi_*(\mathfrak{a}\mathfrak{b}) = \mathfrak{a}\Phi_*(\mathfrak{b})$.

Φ is surjective and injective, so $\Phi = (\Phi_*,\Phi^*)$ is an isomorphism, where Φ^* is the right adjoint of Φ_*. ∎

Definition 88. Let $\{M_i\}_{i=1}^n$ be E-corollas. Let $\mathfrak{f}_* : \Pi_{i=1}^n M_i \to P$ be a set map from the cartesian product $\Pi_{i=1}^n M_i$ to a lattice P. We say $\mathfrak{f}_*$ is multilinear if, for each $1 \leq i \leq n$, and for each choice of elements $X = (x_1, ..., x_{i-1}, \hat{x}_i, x_{i+1}, ..., x_n)$ with x_i omitted, the map $\mathfrak{f}_{X*} : M_i \to P$ sending x_i to $\mathfrak{f}_*(x_1, ..., x_n)$ is a left adjoint morphism of E-corollas.

Definition 89. Take E-corollas M and N. We form the tensor product of corollas $M \otimes_E N$ as follows: Take $M \otimes_E N = (\oplus_{e \in M \times N} E) / \sim$, where $M \times N$ is the cartesian product and $\sim$ is the intersection of all equivalence relations such that

$$\left(\sum_{i\in I} x_i, y\right) \sim \sum_{i\in I}(x_i, y), (\mathfrak{a}x, y) \sim \mathfrak{a}(x, y), \text{ and } (0, y) \sim 0$$
$$\left(x, \sum_{i\in I} y_i\right) \sim \sum_{i\in I}(x, y_i), (x, \mathfrak{a}y) \sim \mathfrak{a}(x, y), \text{ and } (x, 0) \sim 0$$
$$x_i \sim y_i \forall i \in I \Rightarrow \sum_{i\in I} x_i \sim \sum_{i\in I} y_i, \bigcap_{i\in I} x_i \sim \bigcap_{i\in I} y_i$$
$$x \sim y \Rightarrow \mathfrak{a}x \sim \mathfrak{a}y$$

Notice there is a canonical multilinear map $M \times N \to M \otimes_I N$ where (m, n) is sent to the equivalence class generated by (m, n).

Lemma 90 (Universal Property of Tensor Product)**.** *Let M and N be elements of E-**Cor**. Let $\phi : M \times N \to M \otimes_I N$ be the canonical multilinear map. For each multilinear map $f_* : M \times N \to P$ there is a unique morphism of E-corollas $g : M \otimes_I N \to P$ such that $g \circ \phi = f_*$.*

Lemma 91. *The functor $M \to M \otimes_E N$ is left adjoint to the functor $M \to Hom_E(M, N)$.*

Theorem. *E-**Cor** forms a monoidal category under tensor product.*

Remark. **Lat** is categorically equivalent to I-**Cor**. As such, **Lat** can be viewed as a monoidal category.

4. Commutative Calyxes and their Corollas

Definition 92. A calyx E is called commutative if $\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$ $\forall \mathfrak{a}, \mathfrak{b} \in E$. In this section, all calyxes are assumed to be commutative.

4.1. Maximal, Minimal, Irreducible, Co-Irreducible, and Prime Elements [-1].

Definition 93. Let E be a calyx with element $\mathfrak{p} \in E$. we make the following two symmetrical definitions:

(1) $\mathfrak{p}$ is called irreducible if $\bigcap_{i\in I} \mathfrak{a}_i \subseteq \mathfrak{p} \Rightarrow \mathfrak{a}_i \subseteq \mathfrak{p}$ for some $i \in I$.
(2) $\mathfrak{p}$ is called co-irreducible if $\mathfrak{p} \subseteq \sum_{i\in I} \mathfrak{a}_i \Rightarrow \mathfrak{p} \subseteq \mathfrak{a}_i$ for some $i \in I$.

Notice that 0 is irreducible when $\bigcap_{i\in I} \mathfrak{a}_i = 0 \Rightarrow \mathfrak{a}_i = 0$ for some $i \in I$ and that 1 is co-irreducible when $\sum_{i\in I} = 1 \Rightarrow \mathfrak{a}_i = 1$ for some $i \in I$. Notice that $\mathfrak{p}$ is irreducible in E if and only if 0 (the smallest element in $[x]$, i.e. x) is irreducible in $[x]$ (the quotient calyx) and $\mathfrak{p}$ is co-irreducible if and only if 1 (the largest element in (x), i.e. x) is co-irreducible in (x) (the subcalyx).

Example 94. In the calyx $I = \{0, 1\}$, 0 is irreducible and co-irreducible and 1 is irreducible and co-irreducible.

Definition 95. Let E be a calyx with element $x \in E$. We make the following two symmetrical definitions:

(1) x is called maximal if $x < 1$ and $x < y \Rightarrow y = 1$. Equivalently, if $[x] \cong I$.

(2) x is called minimal if $x > 0$ and $x > y \Rightarrow y = 0$. Equivalently, if $(x) \cong I$.

Lemma 96. *Notice that maximal elements are irreducible and minimal elements are co-irreducible, since $[x] \cong I$ for a maximal element and $(x) \cong I$ for a minimal element. Thus, if an element $\mathfrak{m}$ is maximal, then $\bigcap_{i \in I} \mathfrak{m} \leq x \Rightarrow y \leq x$ or $z \leq x$*

Lemma 97. *Suppose E is a calyx. Then for every element $\mathfrak{a} \in E$ there is a maximal element $\mathfrak{m} \in E$ such that $\mathfrak{a} \leq \mathfrak{m}$ and a minimal element $\zeta \in E$ such that $\zeta \leq \mathfrak{a}$. This follows from Zorn's lemma. We can form the intersection of all maximal elements containing an element and the sum of all minimal elements contained in an element.*

Lemma 98. *Suppose $\mathfrak{a} \in E$ has $\mathfrak{a} + \mathfrak{m} = 1$ for each maximal element $\mathfrak{m} \in E$. Then $\mathfrak{a} = 1$.*

Proof. By contrapositive: if $\mathfrak{a} \neq 1$ then $\mathfrak{a}$ is contained in some maximal element $\mathfrak{m}$, so that $\mathfrak{a} + \mathfrak{m} = \mathfrak{m} \subsetneq 1$. □

Lemma 99. *Suppose $\mathfrak{a} \in E$ has $\mathfrak{a} \cap \zeta = 0$ for each minimal element $\zeta \in E$. Then $\mathfrak{a} = 0$*

Proof. By contrapositive: if $\mathfrak{a} \neq 0$ then $\mathfrak{a}$ contains some minimal element ζ, so that $\mathfrak{a} \cap \zeta = \zeta \supsetneq 0$. □

We next establish the notions of prime elements.

Definition 100. Let E be a calyx. We say an element $\mathfrak{p} \in E$ is prime if $\mathfrak{ab} \subseteq \mathfrak{p} \Rightarrow \mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$. Equivalently, if we have $\mathfrak{ab} = 0 \Rightarrow \mathfrak{a} = 0$ or $\mathfrak{b} = 0$ in $[\mathfrak{p}]$.

Lemma 101. *Let E be a calyx. If $\mathfrak{p} \in E$ is prime then $\mathfrak{p}$ is irreducible.*

Proof. If $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$ then $\mathfrak{ab} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$, so that $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$. □

Lemma 102. *Let $\mathcal{S} \subseteq E - \{0\}$ be closed under multiplication. Suppose that $\forall x, y \in E, x \leq y, x \in \mathcal{S} \Rightarrow y \in \mathcal{S}$. Then there is a prime ideal $\mathfrak{p} \subseteq E$ not in $\mathcal{S}$.*

Proof. The set $E - \mathcal{S}$ is nonempty since it contains 0, and closed under suprema of tosets. By Zorn's lemma, there is a maximal ideal $\mathfrak{p}$ among those not contained in $\mathcal{S}$. Take $\mathfrak{a}, \mathfrak{b} \not\leq \mathfrak{p}$ such that $\mathfrak{ab} \subseteq \mathfrak{p}$. Since $\mathfrak{a}, \mathfrak{b} \not\subset \mathfrak{p}$, $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}$, so $\mathfrak{ab} \in \mathcal{S}$. Thus $\mathfrak{p} \geq \mathfrak{ab}$, so that $\mathfrak{p} \in \mathcal{S}$. □

Lemma 103. *Let E be a calyx. A maximal element $\mathfrak{m} \in E$ is prime.*

Proof.

$$\mathfrak{m} \text{ is maximal} \quad \Leftrightarrow \quad E/\mathfrak{m} \cong I \Rightarrow (\mathfrak{ab} = 0 \Rightarrow \mathfrak{a} = 0 \text{ or } \mathfrak{b} = 0) \text{ in } E/\mathfrak{m} \quad \Leftrightarrow \quad \mathfrak{m} \text{ is prime}$$

□

A priori a maximal element $\mathfrak{m} \in E$ is irreducible.
Thus we have the following diagram of implications in any calyx:

$$\text{Minimal} \longrightarrow \text{Co-Irreducible}$$

$$\text{Maximal} \longrightarrow \text{Prime} \longrightarrow \text{Irreducible}$$

Definition 104. We say that $x \in M$ is irreducible if $y \cap z \subseteq x \Rightarrow y \subseteq x$ or $z \subseteq x$. We say $x \in M$ is coirreducible if $x \subseteq y + z \Rightarrow x \subseteq y$ or $x \subseteq z$.

4.2. **The Isomorphism Theorems [0].** Remark: perenniality of a morphism $\mathfrak{f} : M \to N$ is exactly the condition necessary for the first isomorphism theorem to hold. That is, for a perennial map $\mathfrak{f} : M \to N$ of E-corollas M and N, $[\mathfrak{f}^*(0)] \cong (\mathfrak{f}_*(1))$.

Theorem (The Second Isomorphism Theorem)**.** *Let M be a modular E-corolla. Then $[x \cap y, y] \cong [x, x+y]$.*

Theorem (The Third Isomorphism Theorem)**.** *Let M be an E-corolla with elements $x \leq y$. $[y] \cong [z]$ where $z \in [x]$ is the image of y.*

4.3. **Duality [1].**

Definition 105. For a lattice E define E^{op} as the opposite category of E viewed as a poset category. E^{op} is complete when E is. The category **Lat** is isomorphic to $\mathbf{Lat}^{op}$. There is a contravariant functor *op* from **Lat** to $\mathbf{Lat}^{op}$.

Lemma 106. *$End_{\mathbf{Lat}}(M, N)$ and $End_{\mathbf{Lat}}(N^{op}, M^{op})$ are isomorphic as lattices.*

Proof. Define $\Phi_* : End_{\mathbf{Lat}}(M, N) \to End_{\mathbf{Lat}}(N^{op}, M^{op})$ as follows. Take an element $\mathfrak{f} \in End_{\mathbf{Lat}}(M, N)$. Define $\mathfrak{g}_* : N^{op} \to M^{op}$ by $\mathfrak{g}_*(x^{op}) = \mathfrak{f}^*(x)^{op}$. Then, for $x, y \in N$,

$$x^{op} \leq y^{op} \Rightarrow y \leq x \Rightarrow \mathfrak{f}^*(y) \leq \mathfrak{f}^*(x) \Rightarrow \mathfrak{f}^*(x)^{op} \leq \mathfrak{f}^*(y)^{op} \Rightarrow \mathfrak{g}_*(x^{op}) \leq \mathfrak{g}_*(y^{op})$$

Define $\mathfrak{g}^* : M^{op} \to N^{op}$ by $\mathfrak{g}^*(x^{op}) = \mathfrak{f}_*(x)^{op}$. Then, for $x, y \in M$,

$$x^{op} \leq y^{op} \Rightarrow y \leq x \Rightarrow \mathfrak{f}_*(y) \leq \mathfrak{f}_*(x) \Rightarrow \mathfrak{f}_*(x)^{op} \leq \mathfrak{f}_*(y)^{op} \Rightarrow \mathfrak{g}^*(x^{op}) \leq \mathfrak{g}^*(y^{op})$$

To check that $\mathfrak{g}_* \dashv \mathfrak{g}^*$, take $x \in M$ and $y \in N$.

$$\begin{aligned}
&\mathfrak{g}_*(y^{op}) \leq x^{op} \\
\Leftrightarrow\ & x \leq \mathfrak{g}_*(y^{op})^{op} = \mathfrak{f}^*(y) \\
\Leftrightarrow\ & \mathfrak{f}_*(x) \leq y \\
\Leftrightarrow\ & x \leq \mathfrak{f}^*(y) \\
\Leftrightarrow\ & \mathfrak{f}_*(x) \leq y \\
\Leftrightarrow\ & \mathfrak{g}^*(x^{op})^{op} \leq y \\
\Leftrightarrow\ & y^{op} \leq \mathfrak{g}^*(x^{op})
\end{aligned}$$

Φ_* is bijective. Thus it is left adjoint to a morphism $\Phi^* : End_{\mathbf{Lat}}(N^{op}, M^{op}) \to End_{\mathbf{Lat}}(M, N)$, and the pair $\Phi = (\Phi_*, \Phi^*)$ is an isomorphism of lattices. □

Lemma 107. *$End_{\mathbf{Lat}}(M, M)$ and $End_{\mathbf{Lat}}(M^{op}, M^{op})$ are isomorphic as calyxes.*

Proof. Define the lattice morphism $\Phi_* : End_{\mathbf{Lat}}(M, M) \to End_{\mathbf{Lat}}(M^{op}, M^{op})$ as before, where for an element $\mathfrak{f} \in End_{\mathbf{Lat}}(M, M)$, $\Phi(\mathfrak{f})_* = \mathfrak{g}_* : N^{op} \to M^{op}$ is given by $\mathfrak{g}_*(x^{op}) = \mathfrak{f}^*(x)^{op}$. Define $\Phi(\mathfrak{f})_* = \mathfrak{g}^* : M^{op} \to M^{op}$ by $\mathfrak{g}^*(x^{op}) = \mathfrak{f}_*(x)^{op}$. To see that Φ is an isomorphism of calyxes it suffices to show that Φ preserves composition. Take lattice morphisms

$\mathfrak{f}, \mathfrak{g} : M \to M$

$$\begin{aligned} &\Phi(\mathfrak{f} \circ \mathfrak{g})_*(x^{op}) \\ =&((\mathfrak{f} \circ \mathfrak{g})^*(x))^{op} \\ =&(\mathfrak{f}^* \circ \mathfrak{g}^*(x))^{op} \\ =&\Phi(\mathfrak{f})_*(\mathfrak{g}^*(x)^{op}) \\ =&\Phi(\mathfrak{f})_* \circ \Phi(\mathfrak{g})_*(x^{op}) \end{aligned}$$

□

Definition 108. An E-corolla M induces an E-corolla M^{op} with sturcture morphism $E \to End_{\mathbf{Lat}}(M) \to End_{\mathbf{Lat}}(M^{op})$. There is a contravariant functor $op : E--\mathbf{Cor} \to E--\mathbf{Cor}$ where M mapsto M^{op} with the mentioned structure map (it should be clear what it does to morphisms). It is an involution. If M is an E-corolla with structure map $\mu : E \to M$, then we write $\mu^{op} : E \to M^{op}$ for the opposite corolla. Notice that $\mu^{op}(\mathfrak{a})_*(x)^{op} = (x : \mathfrak{a})$ and that $\mu^{op}(\mathfrak{a})^*(x)^{op} = \mathfrak{a}x$.

replacement...?

4.4. **Principal Elements [2].** Reminder: A morphism $\mathfrak{f} : M \to N$ of E-corollas is called perennial if $\mathfrak{f}_*$ is surjective onto $(\mathfrak{f}_*(1))$ and $\mathfrak{f}^*$ is surjective onto $[\mathfrak{f}^*(0)]$.

Definition 109. Let M be an E-corolla. An element $x \in M$ is called principal if the canonical map $E \to M$ sending 1 to x is perennial. We often write '$\zeta \in \mathfrak{a}$' for '$\zeta \leq \mathfrak{a}, \mathfrak{a} \in E$, ζ principal'. Note the distinction between principal ideals of a realizable calyx and principal elements.

Lemma 110. *If $\zeta \in E$ is principal and $x \in M$ is principal then ζx is principal.*

Proof. To see this, let μ_x be the E-corolla map sending 1 to x. Since composition of perennial corolla maps is perennial, the map $E \overset{\mu(\zeta)_*}{\to} E \overset{\mu_x}{M}$ is perennial, so that ζx is principal. □

Lemma 111. *If $\zeta \in E$ is principal and $x \in M$ is principal then $(x : \zeta)$ is principal.*

Definition 112. Let $\mathfrak{f} : M \to N$ be a morphism of E-corollas. We say $\mathfrak{f}$ is perennial if any of the equivalent statements are true:

1. The canonically induced morphism $\mathfrak{g} : [\mathfrak{f}^*(0)] \cong (\mathfrak{f}_*(1))$ is an isomorphism of E-corollas.
2. $\mathfrak{f}_*$ is injective on $[\mathfrak{f}^*(0)]$ and $\mathfrak{f}^*$ is injective on $(\mathfrak{f}_*(1))$.
3. $\mathfrak{f}_*$ is surjective onto $(\mathfrak{f}^*(1))$ and $\mathfrak{f}^*$ is surjective onto $[\mathfrak{f}^*(0)]$.
4. $(\mathfrak{f}_* \circ \mathfrak{f}^*)_{(\mathfrak{f}_*(1))} = id|_{(\mathfrak{f}_*(1))}$ and $(\mathfrak{f}^* \circ \mathfrak{f}_*)_{[\mathfrak{f}^*(0)]} = id|_{[\mathfrak{f}^*(0)]}$

Proof. Clearly (1) ⇒ (2), and (4) ⇒ (1).

To show (2) ⇒ (3), take $x \in [\mathfrak{f}^*(0)]$. $\mathfrak{f}_*(x) = \mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{f}_*(x)))$ so $x = \mathfrak{f}^*(\mathfrak{f}_*(x))$. Thus $\mathfrak{f}^*$ is surjective onto $[\mathfrak{f}^*(0)]$. Conversely, take $x \in (\mathfrak{f}_*(1))$. $\mathfrak{f}^*(x) = \mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{f}^*(x)))$ so $x = \mathfrak{f}_*(\mathfrak{f}^*(x))$. Thus $\mathfrak{f}_*$ is surjective onto $(\mathfrak{f}_*(1))$.

To show (3) ⇒ (4), take $x \in (\mathfrak{f}_*(1))$. Then $x = \mathfrak{f}_*(y)$ for $y \in N$. Then

$$\mathfrak{f}_* \circ \mathfrak{f}^*(x) = \mathfrak{f}_* \circ \mathfrak{f}^* \circ \mathfrak{f}_*(y) = \mathfrak{f}_*(y) = x$$

Thus $(\mathfrak{f}_* \circ \mathfrak{f}^*)_{(\mathfrak{f}_*(1))} = id|_{(\mathfrak{f}_*(1))}$. Similarly, taking $x \in [\mathfrak{f}^*(0)]$, write $x = \mathfrak{f}^*(y)$. Then

$$\mathfrak{f}^* \circ \mathfrak{f}_*(x) = \mathfrak{f}^* \circ \mathfrak{f}_* \circ \mathfrak{f}^*(y) = \mathfrak{f}^*(y) = x$$

Thus $(\mathfrak{f}^* \circ \mathfrak{f}_*)_{[\mathfrak{f}^*(0)]} = id|_{[\mathfrak{f}^*(0)]}$.

□

Definition 113. Let E be a calyx with structure map $\mu : E \to End_{\mathbf{Lat}}(E)$. We say $\zeta \in E$ is principal if $\mu(\zeta)$ is a perennial E-corolla morphism. In other words, we say an element ζ of a calyx E is principal if the (unique) E-corolla map $E \to E$ whose left adjoint sends 1 to $\mathfrak{a}$ is perennial. The set of perennial maps in $End_{E\text{-}\mathbf{Cor}}(E)$ forms a submonoid under multiplication, and multiplication of principal elements $\zeta, \eta \in E$ corresponds to composition of maps $\mu(\zeta) \circ \mu(\eta)$. We say an element $\mathfrak{a}$ of a calyx E is coprincipal if the (unique) E-corolla map $E \to E^{op}$ whose left adjoint is multiplication by $\mathfrak{a}^{op}$ is perennial.

Lemma 114. *Let E be a calyx. For a minimal element $\mathfrak{a} \in E$, $I_\mathfrak{a}\left(\sum_{i\in I} \mathfrak{a}_i\right) = \sum_{i \in I} I_\mathfrak{a}(\mathfrak{a}_i)$ and $I_\mathfrak{a}(0) = 0$, so that $I_\mathfrak{a}$ is (both right and) left adjoint.*

Proof. Define $I_{\mathfrak{a}*} : E \to E$ where $x \mapsto x \cap \mathfrak{a}$. To show that $\mathfrak{a}$ is co-regular, it suffices to show that $I_{\mathfrak{a}*}$ preserves colimits. Take $\{\mathfrak{a}_i\}_{i\in I}$. $\sum_{i\in I} \mathfrak{a} \cap \mathfrak{a}_i \le \mathfrak{a} \cap \sum_{i \in I} \mathfrak{a}_i \le \mathfrak{a}$. Either $\mathfrak{a} \cap \sum_{i\in I} \mathfrak{a}_i \le \mathfrak{a} = 0$ or $\mathfrak{a} \cap \sum_{i\in I} \mathfrak{a}_i \le \mathfrak{a} = \mathfrak{a}$. If $\mathfrak{a} \cap \sum_{i\in I} \mathfrak{a}_i \le \mathfrak{a} = 0$ then $\sum_{i\in I} \mathfrak{a} \cap \mathfrak{a}_i = 0$. Otherwise, $\mathfrak{a} \subseteq \sum_{i\in I} \mathfrak{a}_i$, so that $\mathfrak{a} \subseteq \mathfrak{a}_i$ for some $i \in I$ since a minimal element is co-irreducible by lemma 96. So $\mathfrak{a} \le \mathfrak{a} \cap \mathfrak{a}_i \le \sum_{i\in I} \mathfrak{a} \cap \mathfrak{a}_i \le \mathfrak{a}$, so that

$$\sum_{i\in I} \mathfrak{a} \cap \mathfrak{a}_i \le \mathfrak{a} \cap \sum_{i\in I} \mathfrak{a}_i \le \mathfrak{a}$$

Moreover $I_\mathfrak{a}(0) = 0$, so that $I_\mathfrak{a}$ preserves colimits. □

Lemma 115. *Let E be a calyx. For a maximal element $\mathfrak{m} \in E$, $P_\mathfrak{m}$ is (both left and) right adjoint.*

Definition 116. Let E be a calyx. For any $\mathfrak{a} \in E$ let $I_\mathfrak{a}$ be the right adjoint lattice morphism sending $\mathfrak{b} \in E$ to $\mathfrak{b} \cap \mathfrak{a}$. For each $\mathfrak{a} \in E$, let $P_\mathfrak{a}$ be the left adjoint lattice morphism sending $\mathfrak{b} \in E$ to $\mathfrak{b} + \mathfrak{a}$.

Definition 117. Let E be a calyx with structure map $\mu : E \to End_{\mathbf{Lat}}(E)$. $\zeta \in E$ is principal in Dilworth's sense if for each $\mathfrak{a} \in E$, the following two diagrams commute:

$$\begin{array}{ccc} E & \xrightarrow{\mu(\zeta)_*} & E \\ \downarrow{\scriptstyle I_{(\mathfrak{a}:\zeta)}} & & \downarrow{\scriptstyle I_\mathfrak{a}} \\ E & \xrightarrow{\mu(\zeta)_*} & E \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{\mu(\zeta)^*} & E \\ \downarrow{\scriptstyle P_{\zeta\mathfrak{a}}} & & \downarrow{\scriptstyle P_\mathfrak{a}} \\ E & \xrightarrow{\mu(\zeta)^*} & E \end{array}$$

In other words, the following diagram of adjoint pairs commutes for each $\mathfrak{a} \in E$:

$$\begin{array}{ccc} E & \xrightarrow{\quad\mu(\zeta)\quad} & E \\ \downarrow{\scriptstyle (P_{\zeta\mathfrak{a}}, I_{(\mathfrak{a}:\zeta)})} & & \downarrow{\scriptstyle (P_\mathfrak{a}, I_\mathfrak{a})} \\ E & \xrightarrow{\quad\mu(\zeta)\quad} & E \end{array}$$

Lemma 118. *Let E be a calyx. $\zeta \in E$ is principal if and only if it is principal in Dilworth's original sense.*

Proof. Suppose $\zeta \in E$ is principal. Let $I_{\mathfrak{a}} : E \to E$ be the right adjoint map sending x to $x \cap \mathfrak{a}$. $I_{\mathfrak{a}}$ can be viewed as a map $[x] \to [x]$ for each $x \leq \mathfrak{a}$. Let $P_{\mathfrak{a}} : E \to E$ be the left adjoint map sending x to $x + \mathfrak{a}$. $P_{\mathfrak{a}}$ can be viewed as a map $(x) \to (x)$ for each $x \geq \mathfrak{a}$. The map $\mu(\zeta)_* : E \to (\zeta)$ factors through $[(0:\zeta)]$ as $E \overset{P_{(0:\zeta)}}{\to} [(0:\zeta)] \overset{\theta}{\to} (\zeta)$ by the first isomorphism theorem. Since ζ is principal, $\mu(\zeta)$ is perennial, so that θ is an isomorphism. Take $\mathfrak{a} \in E$. Since $(0:\zeta) \leq (\mathfrak{a}:\zeta)$, the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{P_{(0:\zeta)}} & [(0:\zeta)] \\ \downarrow{\scriptstyle I_{(\mathfrak{a}:\zeta)}} & & \downarrow{\scriptstyle I_{(\mathfrak{a}:\zeta)}} \\ E & \xrightarrow{P_{(0:\zeta)}} & [(0:\zeta)] \end{array}$$

The following diagram also commutes since θ is an isomorphism:

$$\begin{array}{ccc} [(0:\zeta)] & \xrightarrow{\theta} & (\zeta) \\ \downarrow{\scriptstyle I_{\theta^{-1}(\mathfrak{a})}} & & \downarrow{\scriptstyle I_{\mathfrak{a}}} \\ [(0:\zeta)] & \xrightarrow{\theta} & (\zeta) \end{array}$$

Since $\theta^{-1}(\mathfrak{a}) = (\mathfrak{a}:\zeta)$, the following diagram commutes for each $\mathfrak{a} \in E$:

$$\begin{array}{ccccc} E & \xrightarrow{P_{(0:\zeta)}} & [(0:\zeta)] & \xrightarrow{\theta} & (\zeta) \\ \downarrow{\scriptstyle I_{(\mathfrak{a}:\zeta)}} & & \downarrow{\scriptstyle I_{(\mathfrak{a}:\zeta)}} & & \downarrow{\scriptstyle I_{\mathfrak{a}}} \\ E & \xrightarrow{P_{(0:\zeta)}} & [(0:\zeta)] & \xrightarrow{\theta} & (\zeta) \end{array}$$

Thus the following diagram commutes for each $\mathfrak{a} \in E$:

$$\begin{array}{ccc} E & \xrightarrow{\mu(\zeta)_*} & E \\ \downarrow{\scriptstyle I_{(\mathfrak{a}:\zeta)}} & & \downarrow{\scriptstyle I_{\mathfrak{a}}} \\ E & \xrightarrow{\mu(\zeta)_*} & E \end{array}$$

Commutativity of the dual diagram is shown similarly.

Suppose next that ζ is Dilworth principal. We show that ζ is principal. To show that $\mu(\zeta)_*$ is surjective onto $(\mu(\zeta)_*(1))$, take $\mathfrak{a} \leq \zeta$. Since ζ is dilworth principal, the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\mu(\zeta)_*} & (\zeta) \\ \downarrow{\scriptstyle I_{(\mathfrak{a}:\zeta)}} & & \downarrow{\scriptstyle I_{\mathfrak{a}}} \\ E & \xrightarrow{\mu(\zeta)_*} & (\zeta) \end{array}$$

Therefore

$$\mu(\zeta)_*((\mathfrak{a}:\zeta)) = \mu(\zeta)_*(I_{(\mathfrak{a}:\zeta)})(1) = I_{\mathfrak{a}}(\mu(\zeta)_*(1)) = I_{\mathfrak{a}}(\zeta) = \mathfrak{a}$$

So that $\mu(\zeta)_*$ is surjective onto $(\mu(\zeta)_*(1))$. The dual aspect of principality is shown similarly.

□

Lemma 119. *$\mathfrak{f}$ is principal if and only if $\mathfrak{f}_* \circ \mathfrak{f}^* = I_{\mathfrak{f}_*(1)}$ and $\mathfrak{f}^* \circ \mathfrak{f}_* = I_{\mathfrak{f}^*(0)}$. This is establishes that principal is equivalent to weakly principal.*

Proof. If $\mathfrak{f}_* \circ \mathfrak{f}^* = I_{\mathfrak{f}_*(1)}$ and $\mathfrak{f}^* \circ \mathfrak{f}_* = I_{\mathfrak{f}^*(0)}$ then $\mathfrak{f}_* \circ \mathfrak{f}^*|_{(\mathfrak{f}_*(1))} = id_{(\mathfrak{f}_*(1))}$ and $\mathfrak{f}^* \circ \mathfrak{f}_*|_{[\mathfrak{f}^*(0)]} = id_{[\mathfrak{f}^*(0)]}$, which is one of the equivalent statements in definition 112. Conversely, suppose $\mathfrak{f}$ is principal. By lemma 118, the following diagrams commute

$$\begin{array}{ccc} E & \xrightarrow{\mu(\zeta)_*} & E \\ \downarrow{\scriptstyle I_{(\mathfrak{a}:\zeta)}} & & \downarrow{\scriptstyle I_{\mathfrak{a}}} \\ E & \xrightarrow{\mu(\zeta)_*} & E \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{\mu(\zeta)^*} & E \\ \downarrow{\scriptstyle P_{\mathfrak{a}}} & & \downarrow{\scriptstyle P_{\zeta\mathfrak{a}}} \\ E & \xrightarrow{\mu(\zeta)^*} & E \end{array}$$

Evalutating at 1 in the first diagram, we see that

$$\begin{array}{ccc} 1 & \longmapsto & \zeta \\ \downarrow & & \downarrow \\ (\mathfrak{a} : \zeta) & \longmapsto & \zeta(\mathfrak{a} : \zeta) = \zeta \cap \mathfrak{a} \end{array}$$

Evaluating at 0 in the second diagram, we see that

$$\begin{array}{ccc} 0 & \longmapsto & ann(\zeta) \\ \downarrow & & \downarrow \\ \mathfrak{a}\zeta & \longmapsto & (\zeta\mathfrak{a} : \zeta) = \mathfrak{a} + ann(\zeta) \end{array}$$

These imply that ζ is weakly principal. ☐

Lemma 120. *Let M be an E-corolla. Suppose $\zeta \in M$ is principal and $\mathfrak{f} : M \to N$ is a perennial map of E-corollas. Then $\mathfrak{f}_*(\zeta)$ is principal.*

Proof. Let $\mu : E \to M$ be the perennial map sending 1 to x. $\mathfrak{f} \circ \mu$ is the composition of perennial maps and therefore perennial. ☐

4.5. **Exact Sequences [3].**

Lemma 121. *Let M, N, P be E-corollas and let $\mathfrak{f} : M \to N$ and $\mathfrak{g} : N \to P$ be perennial maps. We say $M \xrightarrow{\mathfrak{f}} N \xrightarrow{\mathfrak{g}} P$ is exact at N if the following equivalent conditions hold:*

(1) $im(\mathfrak{f}) = ker(\mathfrak{g})$
(2) $coker(\mathfrak{f}) = coim(\mathfrak{g})$

Proof.

$$im(\mathfrak{f}) = ker(\mathfrak{g}) \;\Leftrightarrow\; \mathfrak{f}_*(1) = \mathfrak{g}^*(0) \;\Leftrightarrow\; cok(\mathfrak{f}) = coim(\mathfrak{g})$$

☐

Definition 122. We say a sequence

$$\cdots \to M_{i-1} \to M_i \to M_{i+1} \to \cdots$$

of perennial E-corolla morphisms is exact if it is exact at every object.

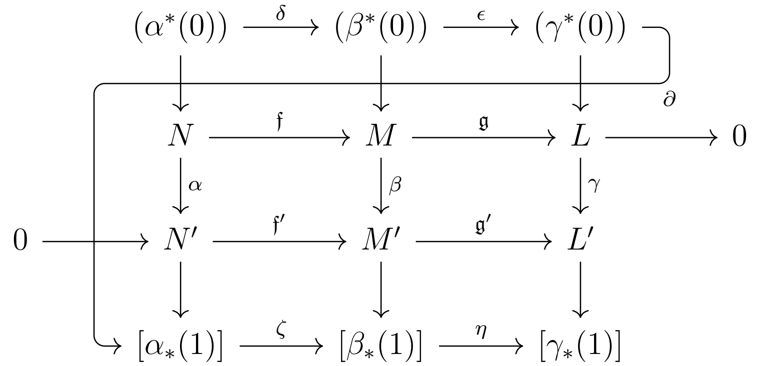
Lemma 123 (Snake Lemma)**.** *Take a commutative diagram of E-corollas as below*

$$\begin{array}{ccccccccc} & & N & \xrightarrow{\mathfrak{f}} & M & \xrightarrow{\mathfrak{g}} & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & N' & \xrightarrow{\mathfrak{f}'} & M' & \xrightarrow{\mathfrak{g}'} & L' & & \end{array}$$

There is an exact sequence

$$(\alpha^*(0)) \xrightarrow{\delta} (\beta^*(0)) \xrightarrow{\epsilon} (\gamma^*(0)) \xrightarrow{\partial} [\alpha_*(1)] \xrightarrow{\zeta} [\beta_*(1)] \xrightarrow{\eta} [\gamma_*(1)]$$

so that the following diagram commutes and has exact rows and collumns:



Proof. We construct the left adjoint in ∂ as follows: for $x \in L$ such that $\gamma(x) = 0$, write $x = \mathfrak{g}(y)$ for $y \in M$. $\mathfrak{g}'(\beta(y)) = 0$. Write $\beta(y) = \mathfrak{f}'(z)$. Viewing z as an element of N', take $\partial_*(x) = z$. ∂_* distributes over colimits and is therefore left adjoint.

Exactness at M: clearly $= 0$. Take $x \in (\beta^*(0))$ such that $\epsilon_*(x) = 0$. Then $\mathfrak{g}'_* \circ \beta_*(x) = 0$.
Exactness at L:
Exactness at N':
Exactness at M':
Exactness at L': □

4.6. **Hom and Tensor [10, 11].**

Definition 124 (The Hom Functor)**.** Let $\mathfrak{f} : M' \to M$ and $\mathfrak{g} : N \to N'$ be E-corolla morphisms. There are morphisms of E-corollas $Hom_E(\mathfrak{f}, N) : Hom_E(M, N) \to Hom_E(M', N)$ sending $\mathfrak{h}$ to $\mathfrak{h} \circ \mathfrak{f}$ and $Hom_E(M, \mathfrak{g}) : Hom_E(M, N) \to Hom_E(M, N')$ sending $\mathfrak{h}$ to $\mathfrak{g} \circ \mathfrak{h}$.

Lemma 125 (not done)**.**

4.7. **Deciduous, Coniferous, and Cupulate Calyxes [4].**

Definition 126. A corolla is called deciduous if every element is the sum of principal elements. Every realizable corolla is deciduous. A corolla is called coniferous if every element is the intersection of coprincipal elements. Note the existence of deciduous coniferous trees in biology. A corolla is deciduous if and only if its opposite corolla is coniferous.

Example 127. Not every corolla is deciduous or coniferous. For example take the following I-corolla:

$$\begin{array}{c} \cdot \\ \uparrow \\ \cdot \\ \uparrow \\ \cdot \end{array}$$

Does every module induce a coniferous corolla?

Lemma 128. *Let M be a deciduous noetherian corolla. Then every element is the sum of finitely many principal elements.*

Proof. Suppose $x \in M$ is not the sum of finitely many principal elements. Then we can construct a strictly increasing chain $x_0 < x_1 < x_2 < \cdots$ where $x_{i+1} = x_i + \zeta_i$ for some principal element $\zeta_i \leq x$, $\zeta_i \not\leq x_i$. □

Lemma 129. *Let M be a coniferous artinian corolla. Then every element is the intersetion of finitely many coprincipal elements.*

Proof. Suppose $x \in M$ is not the intersection of finitely many coprincipal elements. Then we can construct a strictly decreasing chain $x_0 > x_1 > x_2 > \cdots$ where $x_{i+1} = x_i \cap \zeta_i$ for some coprincipal element $\zeta_i \geq x$, $\zeta_i \not\geq x_i$. □

Definition 130. Take a finite collection of elements $\{x_1, ..., x_n\} \subseteq M$ for an E-corolla M. Let E^n be the coproduct of n copies of E, with e_i the element which is 1 in the ith slot and 0 elsewhere. We say $\{x_1, ..., x_n\}$ is a generating set if the map $E^n \to E$ where $e_i \mapsto x_i$ is surjective.

Lemma 131. *Suppose M is a Noetherian deciduous E-corolla over a calyx E. For every element $x \in M$, (x) has a finite generating set.*

Proof. □

4.8. **Modularity.**

Definition 132. Let E be a lattice. The following conditions are equivalent:

(1) $\mathfrak{a} \leq \mathfrak{a}'$ implies $\mathfrak{a} + (\mathfrak{b} \cap \mathfrak{a}') = (\mathfrak{a} + \mathfrak{b}) \cap \mathfrak{a}'$ $\forall \mathfrak{a}, \mathfrak{a}', \mathfrak{b} \in E$.
(2) $[\mathfrak{a} \cap \mathfrak{b}, \mathfrak{b}] \cong [\mathfrak{a}, \mathfrak{a} + \mathfrak{b}]$ as latices for each $\mathfrak{a}, \mathfrak{b} \in E$.
(3) $\mathfrak{a} \leq \mathfrak{a}', \mathfrak{a} + \mathfrak{b} = \mathfrak{a}' + \mathfrak{b}, \mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}' \cap \mathfrak{b} \Rightarrow \mathfrak{a} = \mathfrak{a}'$ $\forall \mathfrak{a}, \mathfrak{a}', \mathfrak{b} \in E$.

If any of the three above conditions hold we say E is modular. Note that every ring induces a modular calyx and every A-module induces a modular A^{cal}-corolla.

Example 133. The calyx induced by a semiring is not necessarily modular

4.9. **Perianths.**

Definition 134. Let E be a calyx. An E-perianth is a calyx F with a calyx morphism $E \to F$. We often write simply F when the calyx morphism $E \to F$ is understood. A morphism of perianths $E \to F$ and $E \to G$ is a morphism of calyxes $F \to G$ such that $E \to F \to G = E \to G$. There is a functor from A-algebras to A^{cal}-perianths for any ring A, a forgetful functor from E-perianths to E-corollas for any calyx E.

Lemma 135. *There is a left adjoint functor $\Phi : \boldsymbol{Cal} \to E - \boldsymbol{Per}$ which sends a calyx*

Lemma 136. *By composing the left adjoints* $\textbf{\textit{Set}} \to \textbf{\textit{Lat}} \to \textbf{\textit{Cal}} \to E - \textbf{\textit{Per}}$ *we get the* E*-algebra* $E[S]$*. We write* $E[x_1, ..., x_n]$ *for free* E*-perianth in* n *variables. In the case of* $n = 1$ *we have the free perianth in a singe variable:*

$$E[x] \cong E \otimes_I \oplus_{n \in \mathbb{N}_{\geq 0}} \otimes_{i=1}^n I$$

Definition 137. We say an E-perianth F is finitely generated if there is a perennial map $E[x_1, ..., x_n] \to F$ such that $E \to E[x_1, ..., x_n] \to F$ is the canonical structure map.

4.10. **Finitely Generated Perianths, Finitely Generated Elements.**

4.11. **Finite Corollas, Finite Elements, and Finitely Presented Corollas [5].** from stacks project

Definition 138. An E-corolla M is said to be finite over M if it is the finite sum of principal elements $\zeta_1, ..., \zeta_n$. An element $x \in M$ is said to be finite if it is the finite sum of principal elements.

4.12. **Calyx Maps of Finite Type and Finite Presentation.**

4.13. **Finite Calyx Maps.**

4.14. **Compact Elements and Cupulate Calyxes.**

Definition 139. An element x of an E-corolla M is said to be compact if $x \leq \sum_{i \in I} x_i$ implies that $x \leq \sum_{i \in F} x_i$ for some finite subset $F \subseteq I$. In particular, we say $\mathfrak{a} \in E$ is compact if $\mathfrak{a} \leq \sum_{i \in I} \mathfrak{a}_i$ implies that $\mathfrak{a} \leq \sum_{i \in F} \mathfrak{a}_i$ for some finite subset $F \subseteq I$.

Lemma 140. *The sum of finitely many compact elements is compact.*

Proof. Let $\zeta_1, ..., \zeta_n$ be compact elements and take $\{\mathfrak{a}_i\}_{i \in I}$ in E such that $\sum_{i=1}^n \zeta_i \leq \sum_{i \in I} \mathfrak{a}_i \leq E$. Then there are finite subsets $\{F_i\}_{i=1}^n$ of I such that $\mathfrak{a}_i \leq \sum_{i \in F_i} \mathfrak{a}_i$. Take $F = \cup_{i=1}^n F_i$. $\sum_{i=1}^n \zeta_i \leq \sum_{i \in I} \mathfrak{a}_i$. □

Lemma 141. *Let* M *be an* E*-corolla in which* 1 *is compact. Then if an element* $\zeta \in E$ *is principal and* $\zeta = \sum_{i \in I} x_i$*, then* $\zeta = \sum_{i \in F} x_i$ *for a finite subset* $F \subseteq I$*.*

Proof. Take a principal element $\zeta \in M$ with perennial map $\mu : E \to M$ such that $\mu(1) = \zeta$. Suppose $\zeta = \sum_{i \in I} x_i$ for $\{x_i\}_{i \in I}$ in M.

$x_i \leq \zeta$, so we can write $x_i = \mu_*(\mathfrak{a}_i)$ for $\mathfrak{a}_i \in E$. $\mu_* \left(\sum_{i \in I} \mathfrak{a}_i\right) = \sum_{i \in I} \mu_*(\mathfrak{a}_i) = \zeta$, so $\sum_{i \in I} \mathfrak{a}_i = 1$. Thus $\sum_{i \in F} \mathfrak{a}_i = 1$ for some finite set $F \subseteq I$. Therefore $\zeta = \mu_* \left(\sum_{i \in F} \mathfrak{a}_i\right) = \sum_{i \in F} \mu_* (\mathfrak{a}_i) = \sum_{i \in F} x_i$. □

Definition 142. We say an element x of an E-corolla M is finitely generated if $x = \sum_{i=1}^n \zeta_i$ for finitely many principal elements $\zeta_1, ..., \zeta_n$ in M. We say M is finitely generated if 1 is finitely generated in M.

Definition 143. A calyx is called cupulate if it satisfies the following:

(1) Every element is the sum of principal elements.
(2) Every element is the sum of compact elements.
(3) 1 is compact.

A realizable calyx is always cupulate.

Lemma 144. *If E is cupulate, then compact is equivalent to finitely generated.;*

Proof. Let $\mathfrak{a} \in E$ be compact. $\mathfrak{a} = \sum_{i \in I} \zeta_i$ for principal elements ζ_i. By compactness $A = \sum_{i \in F} \zeta_i$ for a finite set $F \subseteq I$.

Conversely, suppose $\mathfrak{a} \in E$ is principal. Express $\mathfrak{a} = \sum_{i \in I} \mathfrak{c}_i$ for compact elements $\{\mathfrak{c}_i\}_{i \in I}$. By lemma 141, $\mathfrak{a} = \sum_{i \in F} \mathfrak{c}_i$ for a finite subset $F \subseteq I$, which must be compact by lemma 140. Thus $\mathfrak{a}$ is compact, so that any principal element is compact. Thus the sum of finitely many principal elements is compact. □

Corollary 145. *The product of compact elements is compact in a cupulate calyx.*

Theorem. *Let E be a calyx in which every element is the sum of principal elements and every element is compact. Further suppose that $\mathfrak{a} \leq \mathfrak{b} \Rightarrow \mathfrak{a} = \mathfrak{c}\mathfrak{b}$ for some $\mathfrak{c} \in E$. Then E is realizable as a semiring.*

Proof. □

homework 2 problem 1.

direct sum preserves joins direct sum is a monad direct sum is a direct sum in the opposite category

4.15. **Localization [9].** For the purposes of this section all calyxes are assumed to be commutative.

Definition 146. For each $\mathfrak{a} \in E$, and each localization set $S \subseteq E$, we define the saturation $\mathfrak{a}^S = \sum_{\sigma \in S} (x : \sigma)$.

Definition 147. Let E be a calyx. A localization set S is a set $S \subseteq E$ of principal elements which is multiplicatively closed and which contains 1.

Lemma 148. *Let E be a calyx with localization set S. If $\mathfrak{a}, \mathfrak{b}, \mathfrak{p} \in E$ with $\mathfrak{p}$ prime and not contained in S, then the following hold:*

(1) $\mathfrak{a} \leq \mathfrak{b} \Rightarrow \mathfrak{a}^S \leq \mathfrak{b}^S$
(2) $\mathfrak{a} \leq \mathfrak{a}^S$
(3) $\mathfrak{p} = \mathfrak{p}^S$
(4) $(\mathfrak{a}^S : \sigma) = \mathfrak{a}^S)$ *for each* $\sigma \in S$.
(5) $\mathfrak{a}(\mathfrak{b} : \sigma) = (\mathfrak{a}\mathfrak{b} : \sigma)$ *for* $\sigma \in S$ *and* $\mathfrak{a} \in E$ *such that* $(0 : \sigma) \subseteq \mathfrak{a}$.
(6) $(\mathfrak{a}^S)^S = \mathfrak{a}^S$

Proof. (1) Suppose $\mathfrak{a} \leq \mathfrak{b}$ and take $\mathfrak{c}$ such that $\mathfrak{c}\sigma \leq \mathfrak{a}$ for some $\sigma \in S$. Then $\mathfrak{c}\sigma \leq \mathfrak{b}$ trivially.

(2) $\mathfrak{a}\sigma \leq \mathfrak{a}$ for any $\sigma \in S$, so that $\mathfrak{a} \leq (\mathfrak{a} : \sigma)$ for any $\sigma \in S$, so that $\mathfrak{a} \leq \sum_{\sigma \in S} (\mathfrak{a} : \sigma)$.

(3) Suppose $\mathfrak{b} \leq \mathfrak{p}^S$. Then $\mathfrak{b}\sigma \leq \mathfrak{p}$ for some $\sigma \in S$. Thus $\mathfrak{b} \leq \mathfrak{p}$ since $\mathfrak{p}$ is prime and $\sigma \not\leq \mathfrak{p}$.

(4) Suppose $x\sigma \leq \mathfrak{a}^S$. Then $x\sigma\tau \leq \mathfrak{a}$ for some $\tau \in S$. So $x \in \mathfrak{a}^S$.

(5) This can be seen as a consequence of (4). Alternatively, suppose $\mathfrak{b} \leq (\mathfrak{a}^S)^S$. Then $\mathfrak{b}\sigma\tau \leq \mathfrak{a}$ for some $\sigma, \tau \in S$. So $\mathfrak{b} \in \mathfrak{a}^S$ since $\sigma\tau \in S$.

(6) Follows since σ is principal.
(7)

$$\mu(\mathfrak{a}^S)_*(\varinjlim_{s\in S}(\mathfrak{b}:s)) = \varinjlim_{s\in S}\mu(\mathfrak{a}^S)_*(\mathfrak{b}:s) = \varinjlim_{s\in S}(\mathfrak{a}^S\mathfrak{b}:s) = (\mathfrak{a}^S\mathfrak{b})^S$$

□

Lemma 149. *Let E be a calyx with localization set S and structure map $\mu : E \to End_{\mathbf{Lat}}(E)$. The set $S^{-1}E = \{\mathfrak{a}^S : \mathfrak{a} \in E\}$ forms a calyx.*

Proof. To show that $S^{-1}E$ forms a complete lattice, it suffices to show that $S^{-1}E$ has arbitrary joins (note that the join of elements $\mathfrak{a}_i{}^S$ will possibly be distinct from the join $\sum_{i\in I}\mathfrak{a}_i{}^S$ in E). We show that $\left(\sum_{i\in I}\mathfrak{a}_i\right)^S$ is a colimit for $\mathfrak{a}_i{}^S$ in $S^{-1}E$. Clearly $\mathfrak{a}_i^S \leq \left(\sum_{i\in I}\mathfrak{a}_i\right)^S$. Suppose that for some $\mathfrak{b} \in E$, $\mathfrak{a}_i^S \leq \mathfrak{b}^S$ for each $i \in I$. Take x such that $x\sigma \leq \sum_{i\in I}\mathfrak{a}_i$. Then $x\sigma \leq \sum_{i\in I}\mathfrak{a}_i \leq \mathfrak{b}^S$, so that $x\sigma\tau \in \mathfrak{b}$ for some $\tau \in S$, so that $x \in \mathfrak{b}^S$. Thus $\left(\sum_{i\in I}\mathfrak{a}_i\right)^S \leq \mathfrak{b}^S$. Note that 0^S acts as an initial object. Note also that intersection in $S^{-1}E$ is possibly distinct from intersection in E.

Define a map $\nu : S^{-1}E \to End_{\mathbf{Lat}}(S^{-1}E)$ where $\mathfrak{a}^S$ is sent to the map whose left adjoint $\nu(\mathfrak{a}^S)_*$ has $\nu(\mathfrak{a}^S)_*(x^S) = (\mathfrak{a}^S x^S)^S$. Clearly $\nu(\mathfrak{a}^S)_*$ preserves initial objects. To show that it preserves all colimits, take $\{\mathfrak{a}_i\}_{i\in I}$ in E. Then

$$\begin{aligned}
&\nu(\mathfrak{a}^S)_*\left(\varinjlim_{i\in I} x_i^S\right) \\
&= \left(\mathfrak{a}^S\left(\varinjlim_{i\in I} x_i^S\right)^S\right)^S \\
&= \left(\left(\mathfrak{a}^S \varinjlim_{i\in I} x_i^S\right)^S\right)^S \\
&= \left(\left(\varinjlim_{i\in I} \mathfrak{a}^S x_i^S\right)^S\right)^S \\
&= \varinjlim_{i\in I} \mathfrak{a}^S x_i^S \\
&= \varinjlim_{i\in I} \nu(\mathfrak{a}^S)(x_i^S)
\end{aligned}$$

So $\nu(\mathfrak{a}^S)_*$ is indeed left adjoint. To show that ν itself is left adjoint, take $\{\mathfrak{a}_i\}_{i\in I}$ in E. For each $\mathfrak{b} \in E$, we can use commutativity to note that

$$\nu\left(\varinjlim_{i\in I}\mathfrak{a}_i^S\right)_*(\mathfrak{b}) = \nu(\mathfrak{b})_*(\varinjlim_{i\in I}\mathfrak{a}_i^S) = \varinjlim_{i\in I}\nu(\mathfrak{b})_*(\mathfrak{a}_i^S) = \varinjlim_{i\in I}\nu\left(\mathfrak{a}_i^S\right)_*(\mathfrak{b}) = \left(\varinjlim_{i\in I}\nu\left(\mathfrak{a}_i^S\right)\right)_*(\mathfrak{b})$$

so that $\nu\left(\varinjlim_{i\in I}\mathfrak{a}_i^S\right)_* = \left(\varinjlim_{i\in I}\nu\left(\mathfrak{a}_i^S\right)\right)_*$, so that $\nu\left(\varinjlim_{i\in I}\mathfrak{a}_i^S\right) = \left(\varinjlim_{i\in I}\nu\left(\mathfrak{a}_i^S\right)\right)$.

Note that this method of proof automatically gives a calyx morphism $\phi_S : E \to S^{-1}E$, which we call the structure map of the localization. □

Lemma 150. *Let M be an E-corolla (recall that this means that M is the finite sum of principal elements). Let $S \subseteq E$ be a multiplicative set. Then for any finitely generated element $x \in M$, $ann(x)^{\mathfrak{p}} = ann(x^{\mathfrak{p}})$.*

Proof. Write $x = \sum_{i=1}^{n} \zeta_i$ for principal elements $\{\zeta_i\}_{i=1}^{n}$. For each $1 \leq i \leq n$, we have an exact sequence of perennial morphisms $0 \to ann(\zeta_i) \to E \to (\zeta_i) \to 0$. Since localization preserves exactness, we have that $0 \to ann(\zeta_i)^S \to S^{-1}E \to (\zeta_i^S)_{S^{-1}M} \to 0$ is exact. Thus $ann(\zeta_i^S) = ann(\zeta_i)^S$. Now

$$\begin{aligned} & S^{-1}\left(ann\left(\sum_{i=1}^{n}\zeta_i\right)\right) \\ =& S^{-1}\left(\bigcap_{i=1}^{n} ann(\zeta_i)\right) \\ =& \bigcap_{i=1}^{n} S^{-1}(ann(\zeta_i)) \\ =& \bigcap_{i=1}^{n} ann(S^{-1}\zeta_i) \\ =& ann\left(\sum_{i=1}^{n} S^{-1}\zeta_i\right) \end{aligned}$$

□

***

Corollary 151. *Suppose $\mathfrak{a}, \mathfrak{b} \in M$ are finitely generated elements of an E-corolla M and $\mathfrak{b}$ is finitely generated. Then $(\mathfrak{a}^{\mathfrak{p}} : \mathfrak{b}^{\mathfrak{p}}) = (\mathfrak{a} : \mathfrak{b})^{\mathfrak{p}}$ for each $\mathfrak{p} \in Spec(E)$.*

Proof. $(\mathfrak{a} : \mathfrak{b}) = ann([\mathfrak{b}, \mathfrak{a} + \mathfrak{b}])$ so we can apply lemma 150. □

Lemma 152. *Let $\mathfrak{f} : E \to F$ be a calyx morphism and $S \subseteq E$ a localization set. Then $\mathfrak{f}_*(\mathfrak{a}^S) = \mathfrak{f}_*(\mathfrak{a})^{\mathfrak{f}_*(S)}$.*

Proof. Using 151, we have

$$\mathfrak{f}_*(\mathfrak{a}^S) = \mathfrak{f}_*\left(\sum_{\sigma \in S}(\mathfrak{a} : \sigma)\right) = \sum_{\sigma \in S} \mathfrak{f}_*\left((\mathfrak{a} : \sigma)\right) = \sum_{\sigma \in S}(\mathfrak{f}_*(\mathfrak{a}) : \mathfrak{f}_*(\sigma)) = \sum_{\sigma \in \mathfrak{f}_*(S)} (\mathfrak{f}_*(\mathfrak{a}) : \sigma)$$

□

Lemma 153 (UMP of Localization)**.** *Let S be a localization set of a calyx E. The calyx $S^{-1}E$ with the structure map $\phi_S : E \to S^{-1}E$ is universal among calyxes F with morphisms $\mathfrak{g} : E \to F$ such that $\mathfrak{f}_*(\mathfrak{a}) = 1$ for each $\mathfrak{a} \in S$.*

Proof. Let F be a calyx with a morphism $\mathfrak{g} : E \to F$ such that $\mathfrak{f}_*(\mathfrak{a}) = 1$ for each $\mathfrak{a} \in S$. Define a map $\mathfrak{h} : S^{-1}E \to F$ where $\mathfrak{a}^S \mapsto \mathfrak{g}(\mathfrak{a})$. □

Lemma 154. *Let S and T be multiplicative subsets of a calyx E with $S \subseteq T$. Let $\phi_S : E \to S^{-1}E$ and $\phi_T : E \to T^{-1}E$ be the localization maps. Then $\phi_S(T)^{-1}S^{-1}E \cong T^{-1}E$ as calyxes.*

Proof. Follows from the universal property of localization. □

Lemma 155. *Let E be a calyx with localization set S. If $\mathfrak{a}, \mathfrak{b}, \mathfrak{p} \in E$ with $\mathfrak{p}$ prime and not contained in S, then The following hold:*

(1) $x \leq y \Rightarrow x^S \leq x^S$
(2) $x \leq x^S$
(3) $(x^S : \sigma) = x^S$ *for each* $\sigma \in S$.
(4) $x(y : \sigma) = (xt : \sigma)$ *for* $\sigma \in S$ *and* $x \in E$ *such that* $(0 : \sigma) \subseteq x$.
(5) $(x^S)^S = x^S$
(6) $x^S y^S = (x^S y)^S$

Lemma 156. *Let E be a calyx with localization set S and structure map $\mu : E \to End_{\boldsymbol{Lat}}(E)$. Suppose that every element in E is the sum of principal elements. The set $S^{-1}E = \{\mathfrak{a}^S : \mathfrak{a} \in E\}$ forms a calyx.*

Definition 157. Let M be an E-corolla and let $S \subseteq E$ be a localization set. For an element $x \in M$ we form $x^S = \sum_{\sigma \in S}(x : \sigma)$ and $S^{-1}M = \{x^S : x \in M\}$ We say $x^S \leq y^S$ in $S^{-1}M$ when $x^S \leq y^S$ in M.

Theorem. *$S^{-1}M$ forms an $S^{-1}E$ corolla.*

Proof. First we show that $S^{-1}M$ forms a complete lattice, for which it suffices to show that $S^{-1}M$ has arbitrary joins (note that the join of elements $\{{x_i}^S\}_{i \in I}$ will possibly be distinct from the join $\sum_{i \in I} {x_i}^S$ in M). We show that $\left(\sum_{i \in I} x_i\right)^S$ is a join for ${x_i}^S$ in $S^{-1}M$. Clearly $x_i^S \leq \left(\sum_{i \in I} x_i\right)^S$. Suppose that for some $y \in E$, $x_i^S \leq y^S$ for each $i \in I$. Take x such that $x\sigma \leq \sum_{i \in I} x_i$. Then $x\sigma \leq \sum_{i \in I} x_i \leq y^S$, so that $x \leq (y^S)^S = y^S$. Thus $\left(\sum_{i \in I} x_i\right)^S \leq y^S$. Note that 0^S acts as an initial object. Note also that intersection in $S^{-1}M$ is possibly distinct from intersection in M.

Define a map $\nu : S^{-1}E \to End_{\mathbf{Lat}}(S^{-1}M)$ where $\mathfrak{a}^S$ is sent to the map whose left adjoint $\nu(\mathfrak{a}^S)_*$ has $\nu(\mathfrak{a}^S)_*(x^S) = (\mathfrak{a}^S x^S)^S$. Clearly $\nu(\mathfrak{a}^S)_*$ preserves initial objects. To show that it preserves all colimits, take $\{x_i\}_{i \in I}$ in M. Then

$$\nu(\mathfrak{a}^S)_*\left(\sum_{i \in I} x_i^S\right) = \left(\mathfrak{a}^S\left(\sum_{i \in I} x_i^S\right)^S\right)^S = \left(\left(\mathfrak{a}^S \sum_{i \in I} x_i^S\right)^S\right)^S = \left(\left(\sum_{i \in I} \mathfrak{a}^S x_i^S\right)^S\right)^S = \left(\sum_{i \in I} (\mathfrak{a}^S x_i)^S\right)^S$$

Thus $\nu(\mathfrak{a}^S)_*$ is indeed left adjoint. To show that ν itself is left adjoint, take $\{\mathfrak{a}_i\}_{i \in I}$ in E. For each $\mathfrak{b} \in E$, we can use commutativity to note that

$$\nu\left(\varinjlim_{i \in I} \mathfrak{a}_i^S\right)_*(\mathfrak{b}) = \nu(\mathfrak{b})_*\left(\varinjlim_{i \in I} \mathfrak{a}_i^S\right) = \varinjlim_{i \in I} \nu(\mathfrak{b})_*(\mathfrak{a}_i^S) = \varinjlim_{i \in I} \nu\left(\mathfrak{a}_i^S\right)_*(\mathfrak{b}) = \left(\varinjlim_{i \in I} \nu\left(\mathfrak{a}_i^S\right)\right)_*(\mathfrak{b})$$

so that $\nu\left(\varinjlim_{i \in I} \mathfrak{a}_i^S\right)_* = \left(\varinjlim_{i \in I} \nu\left(\mathfrak{a}_i^S\right)\right)_*$, so that $\nu\left(\varinjlim_{i \in I} \mathfrak{a}_i^S\right) = \left(\varinjlim_{i \in I} \nu\left(\mathfrak{a}_i^S\right)\right)$.

Note that this method of proof automatically gives a calyx morphism $\phi_S : E \to S^{-1}E$, which we call the structure map of the localization. □

Lemma 158. *$S^{-1}M$ is the universal $S^{-1}E$-corolla with a E-corolla morphism $M \to S^{-1}M$. Note that $S^{-1}E$-corolla is in particular an E-corolla by restriction of scalars.*

Proof. Define the map $\phi_* : M \to S^{-1}M$ where $x \mapsto x^S$. ϕ_* distributes over coproducts and is therefore left adjoint. ϕ_* induces a corolla map $\phi : M \to S^{-1}M$ whose left adjoint is ϕ_*. Take an $S^{-1}E$ corolla N with an E-corolla morphism $\psi : M \to N$. To show that ψ factors through $S^{-1}M$ it suffices to show that $\psi(x) = \psi(x^S)$ for each $x \in M$. Clearly $\psi(x) \leq \psi(x^S)$. Suppose $\sigma y \leq x$ for some $\sigma \in s$. Then $\psi(y) = \psi(\sigma y) \leq \psi(x)$. Thus $\psi(x^S) = \psi(x)$. □

Lemma 159. *Suppose $0 \to M \xrightarrow{\mathfrak{f}} N \xrightarrow{\mathfrak{g}} L \to 0$ is an exact sequence of E-corollas. Then $0 \to S^{-1}M \to S^{-1}N \to S^{-1}L \to 0$ is exact.*

Definition 160 (The Localization Functor)**.** Take a calyx E with localization set S. The map $S^{-1}(-)$ on E-corollas sending M to $S^{-1}M$ can be made into a functor E-**Cor** $\to$ $S^{-1}E$-**Cor**: a map $\phi : M \to N$ induces a map $M \to N \to S^{-1}N$, which, by the universal property in lemma 158, induces a morphism $S^{-1}\phi : S^{-1}M \to S^{-1}N$. Thus $S^{-1}\phi_*(x^S) = (\phi(x)_*)^S$ and $S^{-1}\phi^*(x^S) = (\phi(x)^*)^S$ for a morphism $\phi : M \to N$. It should be clear that this makes $S^{-1}(-)$ into a functor.

Lemma 161. *Localization is an exact functor. Also, localization commutes with taking subrings and quotient ring.*

Proof. Let E be a calyx. Take an exact sequence $0 \to (x) \to M \to [x] \to 0$, where M is an E-corolla and $x \in M$ is an element. We show that the induced map $0 \to S^{-1}(x) \to S^{-1}M \to S^{-1}[x] \to 0$ is perennial and exact. It suffices to note that this sequence is naturally isomorphic to the exact sequence $0 \to (x^S) \to S^{-1}M \to [x^s] \to 0$. In particular, there are natural isomorphisms $S^{-1}(x) \to (x^S)$ and $S^{-1}[y] \to [y^S]$ making the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^{-1}(x) & \longrightarrow & S^{-1}M & \longrightarrow & S^{-1}[y] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (x^S)_M & \longrightarrow & S^{-1}M & \longrightarrow & \left[y^S\right]_M & \longrightarrow & 0
\end{array}$$

We show that the map $S^{-1}(x) \to (x^S)_M$, which sends $\sum_{\sigma \in S}(y : \sigma)_{(x)}$ to $\sum_{\sigma \in S}(y : \sigma)_M$, is an isomorphism. □

Lemma 162. *Being 0 is a local property. That is, if $M_\mathfrak{p} = 0$ for each prime ideal $\mathfrak{p} \in E$, then $M = 0$, and vice versa.*

Proof. Clearly if $M = 0$ then $M_\mathfrak{p} = 0$ for each prime $\mathfrak{p} \in E$. Suppose $M_\mathfrak{p} = 0$ for each prime $\mathfrak{p} \in E$, and suppose for a contradiction that there is $x \neq 0$ in M. Then $ann(x) \neq E$, so that it is contained in a maximal element $\mathfrak{m}$. Now $M_\mathfrak{m} = 0$, so that $x^S = 0^S$. So $\sum_{\sigma \in S}(x : \sigma) \leq \sum_{\sigma \in S}(0 : \sigma)$. So, for each $\tau \in S$,

$$\begin{aligned}
\tau x \leq \tau(x : \tau) \leq \sum_{\sigma \in S} \sigma(x : \sigma) = \sigma \sum_{\sigma \in S}(x : \sigma) \\
= \sigma \sum_{\sigma \in S}(0 : \sigma) = \sum_{\sigma \in S} \sigma(0 : \sigma) = 0
\end{aligned}$$

Thus $\sigma x = 0$ for each $\sigma \in S$. But $\sigma \not\leq ann(x)$ for each $\sigma \in S$, a contradiction. □

Lemma 163. *Having* 0 *kernel is a local property of corollas. That is, if* $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ *has zero kernel for each prime element* $\mathfrak{p} \in E$*, then* ϕ *has zero kernel, and vice versa.*

Proof. One direction is clear. Suppose $\phi : M \to N$ is locally injective, so that $\phi_{\mathfrak{m}}$ is injective for each maximal ideal $\mathfrak{m}$ (we only need this for the maximal ideals). Let $\psi : L \to M$ be the kernel of ϕ. Then $L_{\mathfrak{m}} = 0$ for each maximal ideal $\mathfrak{m}$ since localization is exact by lemma 161, so that $L = 0$. Thus ϕ has zero kernel. □

Lemma 164. *Having* 0 *cokernel is a local property of corollas. That is, if* $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ *has zero cokernel for each prime element* $\mathfrak{p} \in E$*, then* ϕ *has zero cokernel, and vice versa.*

Proof. One direction is clear. Suppose $\phi : M \to N$ is locally surjective, so that $\phi_{\mathfrak{m}}$ is surjective for each maximal ideal $\mathfrak{m}$ (we only need this for the maximal ideals). Let $\psi : N \to P$ be the cokernel of ϕ. Then $P_{\mathfrak{m}} = 0$ for each maximal ideal $\mathfrak{m}$ since localization is exact by lemma 161, so that $P = 0$. Thus ϕ has zero cokernel. □

Lemma 165. *Let* E *be a calyx. For each prime* $\mathfrak{p}$*, write* $\mathfrak{a}^{\mathfrak{p}}$ *for* $\mathfrak{a}^{\{\sigma:\sigma\not\subseteq\mathfrak{p}\}}$*. We show that if* $\mathfrak{a}^{\mathfrak{p}} = 1$ *for each prime* $\mathfrak{p}$ *then* $\mathfrak{a} = 1$*.*

Proof. Take an element $\mathfrak{a} \in E$ such that $\mathfrak{a}^{\mathfrak{p}} = 1$ for each prime $\mathfrak{p}$. Suppose for a contradiction that $\mathfrak{a} \neq 1$. Then $\mathfrak{a} \subseteq \mathfrak{m}$ for some maximal element $\mathfrak{m}$. By assumption $\mathfrak{a}^{\mathfrak{m}} = 1$. Recall that $\mathfrak{a}^{\mathfrak{m}} = \sum_{\sigma\not\subseteq\mathfrak{m}}(\mathfrak{a} : \sigma)$. But, for each $\mathfrak{b} \in E$,

$$\mathfrak{b} \leq (\mathfrak{a} : \sigma) \;\Leftrightarrow\; \mathfrak{b}\sigma \leq \mathfrak{a} \Rightarrow \mathfrak{b}\sigma \leq \mathfrak{m} \Rightarrow \mathfrak{b} \leq \mathfrak{m}$$

So $\sum_{\sigma\not\subseteq\mathfrak{m}}(\mathfrak{a} : \sigma) \leq \mathfrak{m}$, a contradiction. □

Lemma 166 (Not Done)**.** *Let* E *be a calyx and* M *an* E*-corolla. For each prime* $\mathfrak{p}$*, write* $x^{\mathfrak{p}}$ *for* $x^{\{\sigma:\sigma\not\subseteq\mathfrak{p}\}}$*. We show that 'Being* 1 *is a local property,' now for modules i.e. if* $x^{\mathfrak{p}} = 1$ *for each prime* $\mathfrak{p}$ *then* $x = 1$*.*

Proof. Take an element $x \in M$ such that $x^{\mathfrak{p}} = 1$ for each prime $\mathfrak{p}$. Suppose for a contradiction that $x \neq 1$. Then $ann^{op}(x) \subseteq \mathfrak{m}$ for some maximal element $\mathfrak{m}$. Recall that $ann^{op}(x) = \{\mathfrak{a} \in E : (x : \mathfrak{a}) = 1\}$. By assumption $x^{\mathfrak{m}} = 1$. Recall that $x^{\mathfrak{m}} = \sum_{\sigma\not\subseteq\mathfrak{m}}(x : \sigma)$.

$ann^{op}\left(\sum_{\sigma\not\subseteq\mathfrak{m}}(x : \sigma)\right) = \sum_{\sigma\not\subseteq\mathfrak{m}} ann^{op}\left((x : \sigma)\right)$ for each $\sigma \not\subseteq \mathfrak{m}$, $ann^{op}((x : \sigma)) \leq \mathfrak{m}$. So $ann^{op}\left(\sum_{\sigma\not\subseteq\mathfrak{m}}(x : \sigma)\right) \leq \mathfrak{m}$,

so that $\sum_{\sigma\not\subseteq\mathfrak{m}}(x : \sigma) \neq 1$, a contradiction. □

Lemma 167. *Let* M *be an* E*-corolla with finitely generated elements* x *and* y*. Suppose* $x^{\mathfrak{p}} = y^{\mathfrak{p}}$ *for each* $\mathfrak{p} \in E$*. Then* $x = y$*.*

Proof. Without loss of generality, we show that $x \leq y$, so that it suffices to show $(y : x) = 1$. It follows from corollary 151 that $(y : x)^{\mathfrak{p}} = 1$ for each $\mathfrak{p} \in Spec(E)$. By lemma 165, $(y : x) = 1$. □

Corollary 168. *Let* M *and* N *be* E*-corollas, with* M *deciduous. Suppose* $\mathfrak{f}, \mathfrak{g} : M \to N$ *are* E*-corolla morphisms such that* $\mathfrak{f}^{\mathfrak{p}} = \mathfrak{g}^{\mathfrak{p}}$ *for each prime* $\mathfrak{p} \in E$*.* $\mathfrak{f} = \mathfrak{g}$*.*

Proof. Certainly by lemma 167, $\mathfrak{f}_*(\zeta) = \mathfrak{g}_*(\zeta)$ for principal elements $\zeta \in M$. Since all elements in M are the sum of principal elements, $\mathfrak{f}_* = \mathfrak{g}_*$ en □

Corollary 169 (Not Done)**.** *If* $\mathfrak{f}^{\mathfrak{p}}$ *is an isomorphism for each* $\mathfrak{p} \in Spec(E)$ *then* $\mathfrak{f}$ *is an isomorphism.*

Proof. Take a map $\mathfrak{f} : M \to N$ such that $\mathfrak{f} : M$ □

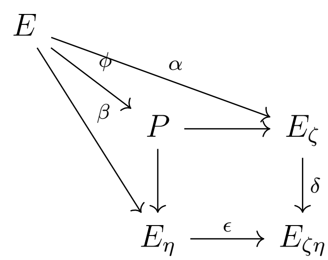
Lemma 170. *Let E be a calyx and $\zeta, \eta \in E$ principal elements such that $\zeta + \eta = 1$. Let $\alpha : E \to E_\zeta$ and $\beta : E \to E_\eta$ be the canonical localization maps. The canonical localization map $\gamma : E \to E_{\zeta\eta}$ factors through E_ζ and E_η as $E \xrightarrow{\alpha} E_\zeta \xrightarrow{\delta} E_{\zeta\eta}$ and $E \xrightarrow{\beta} E_\eta \xrightarrow{\epsilon} E_{\zeta\eta}$. We show that the following diagram of canonical maps is a pullback:*

$$\begin{CD} E @>\alpha>> E_\zeta \\ @V\beta VV @VV\delta V \\ E_\eta @>\epsilon>> E_{\zeta\eta} \end{CD}$$

Proof. Note that $\delta(\mathfrak{a}) = (\mathfrak{a} : \eta)$ for $\mathfrak{a} \in E_\zeta$ and $\epsilon(\mathfrak{a}) = (\mathfrak{a} : \zeta)$ for $\mathfrak{a} \in E_\eta$. Let P be the pullback of the following diagram:

$$\begin{CD} @. E_\zeta \\ @. @VV\delta V \\ E_\eta @>\epsilon>> E_{\zeta\eta} \end{CD}$$

The pullback is constructed from the set $E_\eta \times E_\zeta$ as all the elements $(\mathfrak{a}, \mathfrak{b})$ such that $\delta(\mathfrak{a}) = \epsilon(\mathfrak{b})$. There is a map $\phi : E \to P$ such that the following diagram commutes:



We show that ϕ has an inverse map. Define the inverse map $\psi : P \to E$ where $(\mathfrak{a}, \mathfrak{b}) \mapsto \mathfrak{a} \cap \mathfrak{b}$.

To show $\phi \circ \psi = id$, take $\mathfrak{a} \in E_\zeta$ and $\mathfrak{b} \in E_\eta$ such that $(\mathfrak{a} : \eta) = (\mathfrak{b} : \zeta)$. Then

$$(\mathfrak{a} \cap \mathfrak{b} : \zeta) = (\mathfrak{a} : \zeta) \cap (\mathfrak{b} : \zeta) = (\mathfrak{a} : \zeta) \cap (\mathfrak{a} : \eta) = (\mathfrak{a} : \zeta + \eta) = \mathfrak{a}$$

and

$$(\mathfrak{a} \cap \mathfrak{b} : \eta) = (\mathfrak{a} : \eta) \cap (\mathfrak{b} : \eta) = (\mathfrak{b} : \zeta) \cap (\mathfrak{b} : \eta) = (\mathfrak{b} : \zeta + \eta) = \mathfrak{b}$$

so that $\phi \circ \psi([(\mathfrak{a}, \mathfrak{b})]) = [(\mathfrak{a}, \mathfrak{b})]$.

To show $\psi \circ \phi = id$, take $\mathfrak{a} \in E$. $(\mathfrak{a} : \zeta) \cap (\mathfrak{a} : \eta) = (\mathfrak{a} : \zeta + \eta) = \mathfrak{a}$. □

Corollary 171. *Let E be a calyx with principal elements $\zeta, \eta \in E$. The following diagram is a pullback:*

$$\begin{CD} E_{\zeta+\eta} @>>> E_\zeta \\ @VVV @VVV \\ E_\eta @>>> E_{\zeta\eta} \end{CD}$$

Proof. Passing to $E_{\zeta+\eta}$ we see that $\zeta + \eta = 1$, so that we can apply lemma 170. □

Lemma 172. *Let E be a calyx and $\{\zeta_i\}_{i\in I} \in E$ principal elements. $E_{\sum_{i\in I}\zeta_i} = \varprojlim_{F\subseteq I} E_{\sum_{i\in F}\zeta_i}$.*

Proof. For each finite subset $S \subseteq I$ write E_S for $E_{\sum_{i\in S}\zeta_i}$. When $S \subseteq T$ there is a canonical map $E_T \to E_S$ which localizes at the element $\sum_{i\in S}\zeta_i$ (the canonical map $E \to E_S$ factors through E_T). Let $\mathcal{C}$ be the poset category of finite subsets of I. There is thus a contravariant functor $\Phi : \mathcal{C} \to \mathbf{Cal}$. We show that $E_I \cong \varprojlim \Phi$. Passing to E_I, we see that it suffices to assume $\sum_{i\in I}\zeta_i = 1$ and prove that $E \cong \varprojlim \Phi$. The canonical localization maps $E \to E_S$ make the following diagram commute for each S and T:

$$\begin{array}{ccc} E & \longrightarrow & E_S \\ \downarrow & & \downarrow \\ E_T & \longrightarrow & E_{S\cap T} \end{array}$$

This induces a map $\alpha : E \to \varprojlim \Phi$. Define a set map $\beta : \varprojlim \Phi \to E$ as follows: for an element $\{\mathfrak{a}_F\}_{F\in Obj(\mathcal{C})}$ in $\varprojlim \Phi$ take $\beta(\{\mathfrak{a}_F\}_{F\in Obj(\mathcal{C})}) = \bigcap_{\forall\in Obj(\mathcal{C})} \mathfrak{a}_F$. We check that this constitutes an inverse function for α.

Take an equivalence class $[\{\mathfrak{a}_F\}_{F\in Obj(\mathcal{C})}] \in Obj(\mathcal{C})$. $\alpha\left(\bigcap_{F\subseteq I \text{ finite}} \mathfrak{a}_F\right) = [(\bigcap_{G\subseteq I \text{ finite}} \mathfrak{a}_G : \sum_{i\in F}\zeta_i)_{F\in Obj(\mathcal{C})}]$. It suffices to show that $\left(\bigcap_{F\subseteq I \text{ finite}} \mathfrak{a}_F : \zeta_i\right) = \mathfrak{a}_{\zeta_i}$. Thus it suffices to show that $(\mathfrak{a}_G : \zeta_i) \geq \mathfrak{a}_{\zeta_i}$ $\forall G \in Obj(\mathcal{C})$. But this is obvious.

Conversely, to show $\beta\circ\alpha = id$, take $\mathfrak{a} \in E$. $\mathfrak{a} = \left(\mathfrak{a} : \sum_{F\in Obj(\mathcal{C})}\sum_{i\in F}\zeta_i\right) = \bigcap_{F\in Obj(\mathcal{C})}\left(\mathfrak{a} : \sum_{i\in F}\zeta_i\right)$. □

Lemma 173. *Let E be a calyx and let S be the set of principal elements not contained in a given prime $\mathfrak{p}$. S is a multiplicative set. View S as a category whose morphisms are pairs (λ, s) written $\lambda : s \to \lambda s$ of elements $\lambda, s \in S$ with source s and target λs. Let $\Phi : S \to$ **Cal** be the functor which on objects has $\Phi(\zeta) = E_\zeta$ and on morphisms $\lambda : s \to \lambda s$ gives the canonical localization map $E_s \to (E_s)_\lambda \cong E_{s\lambda}$. $\varinjlim \Phi \cong E_{\mathfrak{p}}$.*

Proof. It is clear that $\varinjlim \Phi$ has the an equivalent universal property to $E_{\mathfrak{p}}$. □

Lemma 174 (Not finished)**.** *Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime elements in a deciduous calyx E neither of which is less than or equal to the other. Let $S = \{\zeta \in E : \zeta \not\leq \mathfrak{p}, \zeta \not\leq \mathfrak{q}\}$. $\mathfrak{p}^S$ and $\mathfrak{q}^S$ are maximal in $S^{-1}E$.*

Proof. Take a nonzero element $\mathfrak{a} \in S^{-1}E$. $\mathfrak{a} + \mathfrak{p} + \mathfrak{q}$ □

Theorem (Not finished)**.** *Let $\{\mathfrak{p}_i\}_{i=1}^n$ be prime elements in a deciduous calyx E. Let $S = \{\zeta \in E : \zeta \not\leq \mathfrak{p}_i\ \forall 1 \leq i \leq n\}$. Maximal elements of the set $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ are maximal in $S^{-1}E$.*

Proof. Clearly the claim holds for $n = 1$. Suppose $n = 2$. Let $\phi_S : E \to S^{-1}E$ be the canonical map. It suffices to show that $I \cong [\phi_{S*}(\mathfrak{p}_1)]$.
$\leq \phi_{S*}(\mathfrak{a})$, $\mathfrak{p}$ □

Theorem (Prime Avoidance)**.** *[Not finished] Let E be a deciduous calyx and take $\mathfrak{a} \in E$. Let $\{\mathfrak{p}_i\}_{i\in I}$ be a finite set of prime ideals in E. If for each principal element $\zeta \in \mathfrak{a}$ there is $i \in I$ such that $\zeta \leq \mathfrak{p}_i$, then $\mathfrak{a} \leq \mathfrak{p}_i$ for some $i \in I$.*

Proof. We argue by contrapositive. Suppose $\mathfrak{a} \not\leq \mathfrak{p}_i\ \forall i \in I$. Let $S = \{\zeta \in E : \zeta \not\leq \mathfrak{p}_i\ \forall i \in I\}$. We induct on $|I| \in \mathbb{N}_{\geq 1}$ to show that $\mathfrak{a}$ is a unit in $S^{-1}E$. It will follow that there is a principal element $\zeta \in \mathfrak{a}$ such that $\zeta \not\leq \mathfrak{p}_i\ \forall i \in I$. The case for $|I| = 1$ is clear.

The case for $|I| = 2$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime elements. Take $\mathfrak{a}$ such that $\mathfrak{a} \not\leq \mathfrak{p}$ and $\mathfrak{a} \not\leq \mathfrak{q}$. Let $S = \{\zeta \in E : \zeta \not\leq \mathfrak{p}_i\ \forall i \in I\}$.

For the induction step, take $n \in \mathbb{N}$ and say $|I| = n + 1$. Take $i \in I$ arbitrary.

For each $j \in I - \{i\}$ take a principal element $\zeta \leq \mathfrak{a}\ \zeta \not\leq \mathfrak{p}_i$, $\zeta \leq \mathfrak{p}_j$.

Since $\mathfrak{a} \not\leq \mathfrak{p}_i$

$\mathfrak{a} + \mathfrak{p}_i \leq \mathfrak{p}_j + \mathfrak{p}_i\ \mathfrak{a} \leq \mathfrak{p}_j$. $(\mathfrak{a} + \mathfrak{p}_i) \cap \mathfrak{p}_i \leq (\mathfrak{p}_j + \mathfrak{p}_i) \cap \mathfrak{p}_i$ □

Theorem. *Let A be a ring. The functor $F : A$-**mod** $\to A^{cal}$-**Cor** commutes with localization.*

Proof. Follows from the characterization of ideals in a localization. □

Theorem (Not Done)**.** *Being perennial is a local property.*

Proof. Take a map $\mathfrak{f} : M \to N$ of E-corollas and suppose that $\mathfrak{f}_\mathfrak{p}$ is perennial for each prime $\mathfrak{p} \in E$. Then each map $\mathfrak{f}_\mathfrak{p}$ induces an isomorphism $\mathfrak{g}_\mathfrak{p} : [\mathfrak{f}_\mathfrak{p}^*(0)] \to (\mathfrak{f}_\mathfrak{p}^*(1))$ on calyxes. The map $\mathfrak{f}$ induces a map $\mathfrak{g} : [\mathfrak{f}^*(0)] \to (\mathfrak{f}_*(1))$ whose localization at $\mathfrak{p}$ is $\mathfrak{g}_\mathfrak{p}$ since localization commutes with taking subcalyxes and quotient calyxes. Since isomorphism of corollas is a local property and $\mathfrak{g}_\mathfrak{p}$ is an isomorphism for each $\mathfrak{p} \in Spec(E)$, $\mathfrak{g}$ is an isomorphism. □

Theorem (Not Done)**.** *If an ideal $\mathfrak{a}$ of a ring A is locally principal then it is Dilworth principal. Note that this is not true for typical principal elements. In any Dedekind domain, any ideal is locally principal but not principal.*

Proof. Suppose an ideal $\mathfrak{a} \in E$ is locally principal. Then $\mathfrak{a}^\mathfrak{p}$ is principal in $A^{cal}_\mathfrak{p}$ for each $\mathfrak{p} \in Spec(E)$. Thus it is locally Dilworth principal. That is, for each $\mathfrak{p}$, there is a perennial map $\mathfrak{f}_\mathfrak{p} : A^{cal} \to A^{cal}$ such that $\mathfrak{f}_\mathfrak{p}$ induces an isomorphism $[\mathfrak{f}_\mathfrak{p}^*(0)] \cong (\mathfrak{a}_{\mathfrak{p}*}(1))$. □

Lemma 175 (Not Done)**.** *Let A be a ring. There is a faithful exact functor $F : A^{cal}$-**Cor** $\to \Pi_{\mathfrak{p} \in Spec(A)} R^{cal}_\mathfrak{p}$-**Cor**.*

4.16. **Base Change [13].** c.f. stacks project.

Let $\mathfrak{f} : E \to F$ be a map of calyxes. Let M be an F-corolla. Let M be an F-corolla. Let $\phi : E \to E'$ be a map of calyxes. The base change of $\mathfrak{f}$ by ϕ is the map $E' \to F \otimes_E E'$. The base change of the F-corolla M is the F'-corolla $M \otimes_E E'$.

4.17. **The Chinese Remainder Theorem [14].**

Lemma 176. *Let E be a calyx with elements $\mathfrak{a}, \mathfrak{b} \in E$ such that $\mathfrak{a} + \mathfrak{b} = 1$.* $[\mathfrak{a} \cap \mathfrak{b}] \cong [\mathfrak{a}]\,\Pi\,[\mathfrak{b}]$.

Proof. Passing to $[\mathfrak{a} \cap \mathfrak{b}]$, it suffices to show that $E \cong [\mathfrak{a}]\ \Pi\ [\mathfrak{b}]$. We show that the canonical map $\phi : E \to [\mathfrak{a}]\ \Pi\ [\mathfrak{b}]$ is an isomorphism. Define a set map $\psi : [\mathfrak{a}]\ \Pi\ [\mathfrak{b}] \to E$ where

$(x, y) \mapsto y\mathfrak{a} + x\mathfrak{b}$. To show $\psi \circ \phi = 1$, observe that

$$\begin{aligned} &\psi \circ \phi(\mathfrak{c}, \mathfrak{d}) \\ &= (y\mathfrak{a} + x\mathfrak{b} + \mathfrak{a}, y\mathfrak{a} + x\mathfrak{b} + \mathfrak{b}) \\ &= (x\mathfrak{b} + \mathfrak{a}, y\mathfrak{a} + \mathfrak{b}) \\ &= (x\mathfrak{b} + x\mathfrak{a} + \mathfrak{a}, y\mathfrak{a} + y\mathfrak{b} + \mathfrak{b}) \\ &= (x(\mathfrak{a} + \mathfrak{b}) + \mathfrak{a}, y(\mathfrak{a} + \mathfrak{b}) + \mathfrak{b}) \\ &= (x + \mathfrak{a}, y + \mathfrak{b}) \\ &= (x, y) \end{aligned}$$

And for the other direction,

$$\phi(\psi(x)) = \phi(x + \mathfrak{a}, x + \mathfrak{b}) = (x + \mathfrak{a})\mathfrak{b} + (x + \mathfrak{b})\mathfrak{a} = x\mathfrak{b} + x\mathfrak{a} = x$$

□

Lemma 177. *Let E be a calyx with elements $\{\mathfrak{a}_i\}_{i=1}^n$ in E such that $\mathfrak{a}_i + \mathfrak{a}_j = 1$ for $1 \leq i \neq j \leq n$. $[\cap_{i=1}^n \mathfrak{a}_i] \cong \Pi_{i=1}^n [\mathfrak{a}_i]$.*

Proof. We show this by induction on n. For $n = 1$ the claim is trivial. Suppose the claim holds for some $n \in \mathbb{N}$ and take $\{\mathfrak{a}_i\}_{i=1}^n$ such that $\mathfrak{a}_i + \mathfrak{a}_j = 1$ for $i \neq j$. Then

$$[\cap_{i=1}^n \mathfrak{a}_i] = [\mathfrak{a}_n \cap \bigcap_{i=1}^{n-1} \mathfrak{a}_i] = [\mathfrak{a}_n] \,\Pi\, [\bigcap_{i=1}^{n-1} \mathfrak{a}_i] = [\mathfrak{a}_n] \,\Pi\, \left(\Pi_{i=1}^{n-1} [\mathfrak{a}_i]\right) \cong \Pi_{i=1}^n [\mathfrak{a}_i]$$

□

4.18. The Spectrum of a Calyx [16].

Definition 178. For a calyx E with element $\mathfrak{a} \in E$, let $V(\mathfrak{a})$ be the set of prime ideals of E.

Lemma 179. *Take elements $\mathfrak{a}, \mathfrak{b}$ of a clayx E. $rad(\mathfrak{a}) \leq rad(\mathfrak{b})$ if and only if $V(\mathfrak{b}) \leq V(\mathfrak{a})$.*

Proof. Note that $\bigcap V(\mathfrak{a}) = rad(\mathfrak{a})$. If $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ then $rad(\mathfrak{b}) = \bigcap V(\mathfrak{b}) \leq \bigcap V(\mathfrak{a}) = rad(\mathfrak{a})$. Conversely, if $rad(\mathfrak{b}) \leq rad(\mathfrak{a})$ and $\mathfrak{p} \in V(\mathfrak{a})$, then $\mathfrak{p} \geq rad(\mathfrak{b}) \geq \mathfrak{b}$, so that $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$. □

Lemma 180. *Let $\mathfrak{a}, \mathfrak{b}, \{\mathfrak{a}_i\}_{i \in I}$ be elements of a calyx E. Then*

(1) $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$
(2) $V\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} V(\mathfrak{a}_i)$

Proof. Clearly $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$. And if a prime $\mathfrak{p}$ has $\mathfrak{p} \not\geq \mathfrak{a}$, $\mathfrak{p} \not\geq \mathfrak{b}$, then $\mathfrak{p} \not\geq \mathfrak{a}\mathfrak{b}$. So $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.

For the second claim, note that

$$\begin{aligned} & \mathfrak{p} \in V\left(\sum_{i\in I}\mathfrak{a}_i\right) \\ \Leftrightarrow\ & \mathfrak{p} \geq \sum_{i\in I}\mathfrak{a}_i \\ \Leftrightarrow\ & \mathfrak{p} \geq \mathfrak{a}_i\ \forall i \in I \\ \Leftrightarrow\ & \mathfrak{p} \in V(\mathfrak{a}_i)\ \forall i \in I \\ \Leftrightarrow\ & \mathfrak{p} \in \bigcap_{i\in I} V(\mathfrak{a}_i) \end{aligned}$$

□

Corollary 181. *The sets of the form $V(\mathfrak{a}) \subseteq Spec(E)$ form a topology for a calyx E.*

Definition 182. For a calyx E with principal element $\zeta \in E$ define $D(\zeta) = Spec(E) - V(\zeta)$. The open sets $\{D(\zeta) : \zeta \text{ is principal}\}$ form a basis for $Spec(E)$. We refer to them as the principal open sets of $Spec(E)$.

For an element $\mathfrak{a} \in E$ we write $D(\mathfrak{a}) = Spec(E) - V(\mathfrak{a})$. Every open set $U \subseteq Spec(E)$ is of this form.

4.19. **Completions.** c.f. atiyah and macdonald.

4.20. **Graded Calyxes and Corollas [55, 57]!!!**

Definition 183. Let $\{E_i\}_{i\in\mathbb{N}_{\geq 0}}$ be lattices. A graded calyx E over E_0-corollas $\{E_i\}_{i\in\mathbb{N}_{\geq 0}}$ is a calyx E which has a direct sum decomposition as a corolla over E_0 as $\oplus_{i\in\mathbb{N}_{\geq 0}}E_i$, such that $\mathfrak{a}_i\mathfrak{a}_j \in E_{i+j}$ for $\mathfrak{a}_i \in E_i$ and $\mathfrak{a}_j \in E_j$, for each $i, j \in \mathbb{N}_{\geq 0}$. We can form the irrelevant ideal as $\sum_{i\in\mathbb{N}_{\geq 1}} E_i$, denoted E^+. We write $\mathfrak{a}_i$ for $(\mathfrak{a}_j)_{j\in\mathbb{N}_{\geq 0}}$ when $\mathfrak{a}_j = 0$ for $j \neq i$. Such elements are called homogeneous.

A graded E-corolla is a E_0-corolla endowed with a structure map $\phi_M : E\otimes_{E_0} M \to M$. We write $(\mathfrak{a}_n)_{n=0}^{\infty}(x_n)_{n=0}^{\infty}$ for $\phi_M((\mathfrak{a}_n)_{n=0}^{\infty} \otimes (x_n)_{n=0}^{\infty})$. An element $(x_j)_{j=0}^{\infty}$ is called homogeneous if there is an $i \in \mathbb{N}_{\geq 1}$ such that $x_j = 0$ for $i \neq j$.

Clearly E forms a graded corolla over itself.

Definition 184. For a graded E-corolla M and $i \in \mathbb{N}_{\geq 0}$ let $M[i]$ be the shifted graded E-corolla, i.e. where $M[i]_n = M_{i+n}$.

Definition 185. For a graded calyx E, set $E(d) = \oplus_{i\geq 0}M_{ni}$, which forms a graded calyx in its own right.

Definition 186. Let E be a graded calyx. For graded E-corollas M and N let $Hom_{gr}(M, N)_0$ be the E_0-corolla of graded E-corolla morphisms from M to N. We define a graded E-corolla $Hom_{gr}(M, N)$ by endowing the E_0-corolla $\oplus_{i\in\mathbb{N}_{\geq 0}}Hom_{gr}(M, N[i])$ with the structure of a graded E-corolla.

Lemma 187. *Let E be a graded calyx with homogeneous elements $\{\mathfrak{a}_i\}$ in E^+. We show that*

$$(\{\mathfrak{a}_i\}_{i=1}^n) = E^+ \ \Leftrightarrow \ E_0[\{\mathfrak{a}_i\}_{i\in I}] = E$$

Proof. One direction is obvious. So suppose that $\sum_{i\in I} \mathfrak{a}_i = E^+$. We claim that $E[\{\mathfrak{a}_i\}_{i\in I}] = E$. Replacing each $\mathfrak{a}_i$ with homogeneous elements $\mathfrak{a}_{i1}, ..., \mathfrak{a}_{im}$ such that $\sum_{j=1}^m \mathfrak{a}_{ij} = \mathfrak{a}_i$, it is clear that no generality is lost in assuming $\mathfrak{a}_i$ to be homogeneous. Take a homogeneous element $\mathfrak{a} \in E_m$. We show by induction on m that $\mathfrak{a} \in E_0[\{\mathfrak{a}_i\}_{i\in I}]$. Obviously if $\mathfrak{a} \in E_0$ then we are done. Otherwise, we can write $\mathfrak{a} = \sum_{i=1}^n \mathfrak{b}_i\mathfrak{a}_i$ with $\mathfrak{b}_i$ homogeneous of lower degree and apply the induction hypothesis. □

Definition 188. A graded E-corolla M is said to be finite over E (or just finite) if there are principal elements $\zeta_1, ..., \zeta_n$ in M (where we view M as an E_0 corolla) and elements $\mathfrak{a}_1, ..., \mathfrak{a}_n$ of E such that $\sum_{i=1}^n \mathfrak{a}_i\zeta_i = 1$ in M.

Lemma 189. *Let F be a modular graded calyx and set $E = F_0$. Suppose that E is a Noetherian calyx. Suppose there are elements $\zeta_1, ..., \zeta_n$ of F which are principal (where we view F as an E corolla) and $\{\mathfrak{a}_i\}_{i=1}^n$ in F such that $\sum_{i=1}^n \mathfrak{a}_i\zeta_i = F$.*

Proof. Write $\mathfrak{b}_0 = 0$ and □

Lemma 190. *Let E be a graded calyx and take a collection $\{\zeta_i\}_{i\in I}$ of homogeneous principal elements in E^+. The following are equivalent:*

1. *The E_0-corolla map $E_0[\{x_i\}_{i\in I}] \to E$ where $x_i \mapsto \zeta_i$ sends 1 to 1 (has trivial cokernel).*
2. *There are elements $\{\mathfrak{a}_i\}_{i\in I}$ in E^+ such that $\sum_{i\in I} \mathfrak{a}_i\zeta_i = E^+$.*

4.21. **Noetherian Calyxes [30]!!!**

Definition 191. A calyx R is Noetherian if every element is the sum of principal elements.

Lemma 192. *Let E be a deciduous calyx. The following are equivalent:*

1. *Every ascending chain $\mathfrak{a}_1 \leq \mathfrak{a}_2 \leq \cdots \leq \mathfrak{a}_n \leq \cdots$ stabilizes (ACC).*
2. *Every element is the sum of principal elements.*
3. *Every collection of elements has a maximal element.*

Lemma 193. *If E is a Noetherian calyx then $[x]$ is noetherian for each $x \in E$.*

Lemma 194. *A finitely generated perianth over a noetherian calyx is noetherian as a calyx.*

Proof. It suffices to show that $E[x]$ is noetherian for a calyx E. And for this it suffices to show that x is principal. The E-corolla map $\mu_x : E \to E[x]$ where $\mu_x(\mathfrak{a}) = \mathfrak{a}x$. This map is clearly perennial, so that x is principal. □

In light of the above lemma, we can think of $E[x_1, ..., x_n]$ as adjoining principal elements to E in the most general way.

Lemma 195. *Any finitely generated calyx over a Noetherian calyx is Noetherian.*

Proof. Follows from lemma 193 and lemma 194. □

Lemma 196. *Any localization of a Noetherian calyx is Noetherian.*

Corollary 197. *A finitely generated algebra over I is Noetherian.*

The following is a key lemma in the use of Noetherian calyxes.

Lemma 198. *Let E be a Noetherian calyx and M a modular E-corolla, finitely generated by elements $\zeta_1, ..., \zeta_n$. Take $x_0 = 0$, $x_i = \sum_{j=1}^{i} \zeta_j$. Take $x, y \in M$ with $x \le y$. If $x = y$ in $[x_i, x_{i+1}]$ for each $0 \le i < n$ then $x = y$.*

Proof. We show this by induction on $n \in \mathbb{N}_{\ge 0}$. For $n = 0$ the claim is trivial. For the induction step, take $n \in \mathbb{N}_{\ge 0}$ and $x \le y$ in M with $x = y$ in $[x_i, x_{i+1}]$. By the induction hypothesis $x = y$ in (x_{n-1}). Thus $x \cap x_{n-1} = y \cap x_{n-1}$ and $x + x_{n-1} = y + x_{n-1}$, so that $x = y$ by definition 132. □

Lemma 199. *Let E be a Noetherian calyx. Let M be a finitely generated deciduous modular E-corolla. Then M is Noetherian.*

Proof. Take principal elements $\zeta_1, ..., \zeta_n$ and let $x_0 = 0$, $x_i = \sum_{j=1}^{i} \zeta_j$. By lemma 192, it suffices to show that every ascending chain $y_0 \le y_1 \le y_2 \le \cdots$ terminates. By lemma 198, it suffices to show that every ascending chain $y_0 \le y_1 \le y_2 \le \cdots$ terminates in $[x_i, x_{i+1}]$ for each $0 \le i < n$. But this is true since $[x_i, x_{i+1}]$ is noetherian as there is a perennial map $E \to [x_{i+1}]$. □

4.22. **Local Calyxes [17].**

Definition 200. Define a local calyx is a pair $(E, \mathfrak{m})$ where $\mathfrak{m}$ is the unique maximal element of E. We often write E for $(E, \mathfrak{m})$. The quotient $[\mathfrak{m}]$ is always I. A local morphism of local calyxes $(E, \mathfrak{m})$ and $(F, \mathfrak{n})$ is a morphism $\phi : E \to F$ such that $\phi_*(\mathfrak{m}) \le \mathfrak{n}$. Equivalently, a local morphism of local calyxes $(E, \mathfrak{m})$ and $(F, \mathfrak{n})$ is a morphism $\phi : E \to F$ such that $\phi^*(\mathfrak{n}) = \mathfrak{m}$ Note that I is a local ring and that $E_\mathfrak{p}$ is local. The unique maximal element of $E_\mathfrak{p}$ is the extension of $\mathfrak{p}$ by the canonical map $E \to E_\mathfrak{p}$.

Lemma 201. *For a calyx E, the following are equivalent:*

(1) E is local.
(2) E has a maximal ideal $\mathfrak{m}$ such that $\mathfrak{a} \not\le \mathfrak{m} \Rightarrow \mathfrak{a} = 1$
(3) Spec(E) has exactly one closed point.

Proof. Omitted. □

Lemma 202. *Let $\phi : E \to F$ be a morphism of calyxes. Take an element $\mathfrak{p} \in F$. The canonically induced map $E_{\phi^*(\mathfrak{p})} \to F_\mathfrak{p}$ is a local morphism.*

Theorem. *Suppose A is a local ring. If $\mathfrak{a} \in A^{cal}$ is Dilworth principal then it is principal.*

Proof. Take a perennial corolla map $\phi : A^{cal} \to A^{cal}$. $\phi_*(1)$ is an ideal $\mathfrak{a}$ in A. There is a calyx isomorphism $[\phi^*(0)] \to (\phi_*(1))$. $\{(\phi^*((a)) : a \in \mathfrak{a}\}$ generates the local calyx $[\phi^*(0)]$. By Nakayama's lemma, it has a single generator as a minimal generating set, call it $\phi^*((a))$. (a) is then a generator of $(\phi_*(1))$ by the correspondence $[\phi^*(0)] \to (\phi_*(1))$. □

Definition 203. We say a set map of abelian monoids $f : G \to H$ is a homomorphism up to units if for each $g, h \in G$, $f(g)f(h) = uf(gh)$ for some invertible element $u \in H$. We say f is surjective up to units if for each $h \in H$ there is $g \in G$ and a unit u in H such that $f(g)u = h$. We say a set map of commutative rings $f : A \to B$ is a homomorphims up to units if for each $a, b \in A$, $f(ab) = uf(a)f(b)$ for some unit $u \in S$ and $f(a + b) = v(f(a) + f(b))$ for some unit $v \in S$. We say f is surjective up to units if for each $b \in B$, there is a unit u and an element $a \in A$ such that $f(a)u = b$. We can still take kernels of such maps.

Construction: Let P be the monoid of nonzero principal ideals of some domain A. Let M be the multiplicative monoid corresponding to A, without the zero element. There is a set map $\alpha : P \to M$ which is a homomorphism up to units. α induces a setmap of rings $\beta : \mathbb{Z}P \to \mathbb{Z}M$ which is a homomorphism up to units. There is a surjective homomorphism $\gamma : \mathbb{Z}M \to A$ with kernel $\mathfrak{a} \subseteq \mathbb{Z}M$, so that $\mathbb{Z}M/\mathfrak{a}$ is isomorphic to A by the map induced by γ. β induces a map $\delta : \mathbb{Z}P/\mathfrak{b} \to \mathbb{Z}M/\mathfrak{a}$ which is an injective homomorphism up to units. Every element in $\mathbb{Z}M/\mathfrak{a}$ is of the form $a + \mathfrak{a}$ for $a \in M$, so δ is surjective up to units. Thus δ induces an isomorphism of calyxes $\mathfrak{f} : (\mathbb{Z}P/\mathfrak{b})^{cal} \to (\mathbb{Z}M/\mathfrak{a})^{cal}$. Since $(\mathbb{Z}M/\mathfrak{a}) \cong A$, $(\mathbb{Z}P/\mathfrak{b})^{cal} \cong A^{cal}$. Thus $A^{cal} \cong [\mathfrak{b}]$ for some $\mathfrak{b} \in (\mathbb{Z}P)^{cal}$.

$dil(A^{cal}) \cong dil((\mathbb{Z}P/\mathfrak{b})^{cal})$?
$dil((\mathbb{Z}P/\mathfrak{b})^{cal}) \cong P \Leftrightarrow \mathbb{Z}P$ is local with maximal ideal

Theorem. *For which $\mathfrak{a}$ is P the set of Dilworth principal ideals of $(\mathbb{Z}P/\mathfrak{a})^{cal}$?*

$R^{cal} \cong S^{cal}$ and the group of units in R is isomorphic to the group of units in S implies the corresponding group rings are isomorphic up to units. $(\mathbb{Z}P/\mathfrak{a}) \cong (\mathbb{Z}M/\mathfrak{b})$ up

R and S are the same quotient of the same group ring.

Let R be a commutative domain whose units form an abelian group $\mathcal{U}$. Let $\mathfrak{M}$ be the multiplicative monoid of R with the same underlying set, but with 0 taken out. Let $\mathcal{P}$ be the abelian monoid of nonzero principal ideals of R. Let $\mathcal{F} : \mathbf{Cmon} \to \mathbf{Grp}$ be the functor from group monoids with cancelation to groups. $\mathfrak{M}$ is an abelian monoid with cancellation, and so is $\mathcal{P}$. There is a canonical exact sequence of $\mathbb{Z}$-modules $0 \to \mathcal{U} \xrightarrow{\iota} \mathcal{F}(\mathfrak{M}) \xrightarrow{\pi} \mathcal{F}(\mathcal{P}) \to 0$. Thus, with $\phi : \mathcal{P} \to \mathcal{F}(\mathcal{P})$ the canonical morphism, $\mathfrak{M} = (\phi \circ \pi)^{-1}(\mathcal{P})$. Thus, given the units $\mathcal{U}$ and principal ideals $\mathcal{P}$ of a domain, we see the monoid of the ring is an element of

$$\{\mathfrak{M} \cap (\phi \circ \pi)^{-1}(\mathcal{P}) : \mathfrak{M} \in ext_1(\mathcal{U}, \mathcal{F}(\mathcal{P}))\}$$

Take a local noetherian ring R whose calyx is E. E^{princ} is the set of principal elements of R. Let $\mathfrak{M}$ be the abelian monoid of R, let $\mathcal{U}$ be the units of R, and let $\mathcal{P} = E^{princ}$. $\mathfrak{M}$ is in

$$\{\mathfrak{N} \cap (\phi \circ \pi)^{-1}(\mathcal{P}) : \mathfrak{N} \in ext_1(\mathcal{U}, \mathcal{F}(\mathcal{P}))\}$$

The map $M \to P$ induces a map $\mathbb{Z}M \to \mathbb{Z}P$

$A^{cal} \cong B^{cal}$, $A^{cal}/\mathfrak{m}A^{cal}$

Theorem (Krull's Intersection Theorem)**.** *Let $\mathfrak{a}$ be an element of a deciduous Noetherian calyx E. Let $\mathfrak{b} = \bigcap_{n \in \mathbb{N}} \mathfrak{a}^n$. Then $\mathfrak{a}\mathfrak{b} = \mathfrak{b}$, so that $\mathfrak{b} = 0$ when $\mathfrak{b} \leq Jac(E)$ by lemma 213.*

Proof. If $\mathfrak{a}\mathfrak{b} = 1$, then $\mathfrak{b} = 1 = \mathfrak{a}\mathfrak{b}$. Otherwise, we can take a primary decomposition $\mathfrak{a}\mathfrak{b} = \bigcap_{i=1}^{n} \mathfrak{a}_i\mathfrak{b}$ where each $\mathfrak{a}_i$ is $\mathfrak{p}_i$ primary.

Take $1 \leq i \leq n$. If $\mathfrak{a} \leq \mathfrak{p}_i$. Since E is noetherian, there is $n \in \mathbb{N}_{\geq 0}$ such that $\mathfrak{p}_i^n \leq \mathfrak{a}_i$, so that

$$\mathfrak{b} = \bigcap_{m \in \mathbb{N}_{\geq 0}} \mathfrak{a}^m \leq \mathfrak{a}^n \leq \mathfrak{p}_i^n \leq \mathfrak{a}_i$$

Suppose $\mathfrak{a} \not\leq \mathfrak{p}_i$. Then there $\zeta \in \mathfrak{a}$, $\zeta \not\leq \mathfrak{p}_i$. If $\mathfrak{b} \not\leq \mathfrak{a}_i$ then take $\eta \leq \mathfrak{b}$, $\eta \not\leq \mathfrak{a}_i$. Since $\eta\zeta \leq \mathfrak{a}\mathfrak{b} \leq \mathfrak{a}_i$, $\eta \not\leq \mathfrak{b}$, we must have $\zeta^n \leq \mathfrak{a}_i$ for some $n \in \mathbb{N}_{\geq 0}$. Thus $\zeta \in rad(\mathfrak{a}_i) = \mathfrak{p}_i$. Thus $\mathfrak{b} \leq \mathfrak{a}_i$ $\forall 1 \leq i \leq n$, so that $\mathfrak{b} \leq \mathfrak{a}\mathfrak{b}$. $\square$

4.23. **The Nilradical and the Jacobson Radical [18].**

Definition 204. The nilradical $nil(E) \in E$ of a calyx E is defined as the intersection of all the prime elements in E. If $nil(E) = 0$ we say E is reduced.

Lemma 205. $(nil(E)) = \{\mathfrak{a} \in E : \mathfrak{a}^n = 0\}$

Proof. Suppose $\mathfrak{a}^n = 0$. Then $\mathfrak{a} \leq \mathfrak{p}$ for every prime $\mathfrak{p} \in E$, so that $\mathfrak{a} \leq nil(P)$.

On the other hand, suppose $\mathfrak{a}^n \neq 0\ \forall n \in \mathbb{N}$. Localize at $S = \{\mathfrak{a}^i : i \in \mathbb{N}\}$ and find a maximal element in P_S. Take its image $\mathfrak{p}$ under the monomorphism $P_S \to P$. $\mathfrak{p}$ is prime as it is the image of a prime under a ring-lattice morphism. So $\mathfrak{a} \not\leq nil(P)$. □

Definition 206. Define the radical of an ideal $\mathfrak{a} \in A$ as $\cap_{\mathfrak{p}\in[\mathfrak{p}]}\mathfrak{p}$. So $nil([\mathfrak{a}]) = rad(A)$.

Lemma 207. *The following are properties of radical ideals:*

(1) $rad(rad(\mathfrak{a})) = rad(\mathfrak{a})$
(2) $rad(\mathfrak{a}\mathfrak{b}) = rad(\mathfrak{a} \cap \mathfrak{b}) = rad(\mathfrak{a}) \cap rad(\mathfrak{b})$
(3) $rad(\mathfrak{a}) = (1) \iff \mathfrak{a} = (1)$.
(4) $rad(\mathfrak{a} + \mathfrak{b}) = rad(rad(\mathfrak{a}) + rad(\mathfrak{b}))$
(5) *If* $\mathfrak{p}$ *is prime then* $rad(\mathfrak{p}^n) = \mathfrak{p}$.

Lemma 208. *Let* E *be a calyx and* $\mathfrak{a} \in E$ *an element. Let* $\phi : E \to [\mathfrak{a}]$ *be the canonical map.* $rad(\phi(\mathfrak{b})) = \phi(rad(\mathfrak{b}))$

Lemma 209. *Let* $\mathfrak{a}$ *be an element of a calyx* E. $rad(\mathfrak{a}) = \{\mathfrak{b} \in E : \mathfrak{b}^n \leq \mathfrak{a}\}$

Proof. Follows from the previous claim. □

Definition 210. For a ring-lattice P define $Jac(E) = \cap_{\mathfrak{m}\text{ maximal}}\mathfrak{m}$.

Lemma 211. *Let* E *be a calyx and* $\mathfrak{a} \in E$ *an element. Let* $\phi : E \to [\mathfrak{a}]$ *be the canonical map.* $Jac([\mathfrak{a}]) = \phi(Jac(E))$

4.24. **Nakayama's Lemma [19].**

Lemma 212. *Let* $(E, \mathfrak{m})$ *be a local calyx with finitely generated* E*-corolla* M*. If* $\mathfrak{m}M = M$ *then* $M = 0$.

Proof. Suppose for a contradiction that $M \neq 0$, while $\mathfrak{m}M = M$. Let $\zeta_1, ..., \zeta_n \in M$ be principal elements such that $M = \sum_{i=1}^n \zeta_i$ in M, with $n \in \mathbb{N}_{\geq 1}$ minimal such that this is possible. Since $M = \mathfrak{m}M$, $\sum_{i=1}^n \zeta_i = \zeta_1\mathfrak{m} + \sum_{i=2}^n \zeta_i$. Let $y = \sum_{i=2}^n \zeta_i$. We pass to $[y]$, in which $1\overline{\zeta_1} = \mathfrak{m}\overline{\zeta_1}$. $\overline{\zeta_1}$ is principal in $[y]$. So $\overline{\zeta_1} = \overline{0}$. Thus $\zeta_1 + y = y$, so $\zeta_1 \leq \sum_{i=2}^n \zeta_i$, contradicting the minimality of n. Thus $\mathfrak{a} = 0$. □

Lemma 213 (Nakayama's Lemma)**.** *[not done] Let* E *be a calyx and let* $\mathfrak{M}$ *be the set of maximal elements in* E*. Let* M *be a finitely generated* E*-corolla. If* $Jac(E)M = M$ *then* $M = 0$.

Proof. Suppose $\mathfrak{m}\mathfrak{a} = \mathfrak{a}$ for each $\mathfrak{m} \in \mathfrak{M}$. Thus, $\mathfrak{a}^{\mathfrak{m}}\mathfrak{m}^{\mathfrak{m}} = \mathfrak{a}^{\mathfrak{m}}$ for each $\mathfrak{m} \in \mathfrak{M}$, so that $\mathfrak{a}^{\mathfrak{m}} = 0$ for each $\mathfrak{m} \in \mathfrak{M}$ since $\mathfrak{a}^{\mathfrak{m}}$ is finitely generated. By lemma 162 $\mathfrak{a} = 0$. □

Corollary 214. *Let* E *be a calyx and let* M *be a finitely generated* E*-corolla with element* $x \in M$*. If* $M = x + Jac(E)M$*, then* $x = M$.

Proof. If $M = x + Jac(E)M$ then $x + M = x + Jac(E)M$, so that $[x] = Jac(E)[x]$. Thus $[x] = 0$. We conclude $x = M$. □

Corollary 215. *If M is a finitely-generated corolla over a calyx E with principal elements $\{x_i\}_{i\in I}$. For each $i \in I$, let $y_i = x_i + Jac(E)M$ be the image of x_i in $[Jac(E)M]$. If $\sum_{i\in I} y_i = 1$ in $[Jac(E)M]$, then $\sum_{i\in I} x_i = 1$ in M.*

Proof. Suppose $\sum_{i\in I} y_i = 1$ in $[Jac(E)M]$. Then $\sum_{i\in I} x_i + Jac(E)M = \sum_{i\in I}(x_i + Jac(E)M) = \sum_{i\in I}(x_i + Jac(E)M) = M$ so $\sum_{i\in I} x_i = M$ by corollary 215. □

4.25. **Zerodivisors, the Corolla Quotient [24].**

Definition 216. Let E be a calyx. $\zeta \in E$ is called a zerodivisor if $\zeta\mathfrak{a} = 0$ for some nonzero $\mathfrak{a} \in E$. Denote the set of zerodivisors and 0 in E by $Z(E)$.

Definition 217. Let E be a calyx. $\mathfrak{a} \in E$ is called nilpotent if $\mathfrak{a}^n = 0$ for some $n \in \mathbb{N}$. A nonzero nilpotent is a zerodivisor. We write $nil(E) = \{\mathfrak{a} \in E : \mathfrak{a}^n = 0\}$ and $nil_{princ.}(E) = \{\zeta \in E \text{ principal } : \zeta^n = 0\}$.

Definition 218. For elements $x, y \in M$ we define $(x : y) = \{\mathfrak{a} \in E : \mathfrak{a}y \leq x\}$. The map $(x : -) : M \to E^{op}$ is left adjoint to the map $(x : -)E^{op} \to M$.

Lemma 219. *$E \cong E^{op}$.*

Proof. Consider the E-corolla map $\theta : E^{op} \to E$ where $\mathfrak{a} \mapsto (0^{op} : \mathfrak{a})$. Take $\sigma : E \to E^{op}$, $x \mapsto x^{op}$ and $\tau : E^{op} \to E$, $x^{op} \mapsto x$.

$$\tau \circ \theta \circ \sigma(x) = (0^{op} : \mathfrak{a}^{op})^{op} = \mathfrak{a}1 = \mathfrak{a}$$

Thus $\tau \circ \theta \circ \sigma$ is a lattice isomorphism, so that θ is a lattice isomorphism, so that it is surjective and injective. A surjective and injective corolla morphism is an isomorphism of corollas. □

Definition 220. Let M be an E-corolla with structure map $\mu : E \to M$. An element $\mathfrak{a} \in E$ is said to be a zerodivisor of M if $\mathfrak{a} \neq 0$ and $\mu(\mathfrak{a})_*$ is injective. Let the set of zerodivisors of M and zero itself be denoted by $Z(M)$.

Definition 221. An element $\mathfrak{a} \in E$ is said to be a co-zerodivisor of a corolla M with structure map $\mu : E \to M$ if it is a zerodivisor in the opposite module M^{op}. To see this in terms of requirements on M, observe:

$$\Leftrightarrow \mu(\mathfrak{a})_*(x^{op}) = \mu(\mathfrak{a})_*(y^{op}) \Rightarrow x^{op} = y^{op}$$
$$\Leftrightarrow \mu(\mathfrak{a})_*(x^{op})^{op} = \mu(\mathfrak{a})_*(y^{op})^{op} \Rightarrow x = y$$
$$\Leftrightarrow (x : \mathfrak{a}) = (y : \mathfrak{a}) \Rightarrow x = y$$

Thus a nonzero element $\mathfrak{a} \in E$ is a co-zerodivisor of a corolla M if and only if $\mu(\mathfrak{a})^*$ is injective. Let the set of co-zerodivisors of M and 0 itself be denoted by $Z^{op}(M)$.

Definition 222. Let E be a calyx and M an E-corolla. We say $\mathfrak{a} \in E$ is M-nilpotent if $\mathfrak{a}^n 1 = 0 \in M$. The set of M-nilpotent elements of E is $rad(ann(M))$. In other words $\mu(\mathfrak{a})^n = 0$ for some $n \in \mathbb{N}$. Clearly if $\mathfrak{a}$ is nilpotent then it is M-nilpotent.

Lemma 223. *Nonzero M-nilpotent elements of E are zerodivisors.*

Proof. Suppose $\mathfrak{a} \in E$ is M-nilpotent. Then $\mu(\mathfrak{a})_* \circ \mu(\mathfrak{a})_* \circ \cdots \circ \mu(\mathfrak{a})_* = 0$ so $\mu(\mathfrak{a})_*$ must send some nonzero element to 0. Thus $\mathfrak{a}$ is a zerodivisor. □

Lemma 224. *Nonzero nilpotent elements of E are co-zerodivisors.*

Proof. Suppose $\mathfrak{a} \in E$ is nilpotent. Then $\mu(\mathfrak{a})^* \circ \mu(\mathfrak{a})^* \circ \cdots \circ \mu(\mathfrak{a})^* = 1$ so $\mu(\mathfrak{a})^*$ must send some element $x \neq 1$ to 1. □

Definition 225. For a corolla M, define

$$Z(M) = \{\mathfrak{a} \in E : \mathfrak{a}x = 0 \text{ for some nonzero } x \in M\}$$

4.26. **Supports and Annihilators [39].**

4.27. **Valuation Rings [49].** discrete valuation rings and dedekind domains (example and classification)

Lemma 226. *Let A be a valuation ring with surjective valuation $v : A \to \Omega$. Take ideals $\mathfrak{a}, \mathfrak{b} \in A$.*

$$v(\mathfrak{a}\mathfrak{b}) = \{v(x) + v(y) : x \in v(\mathfrak{a}), y \in v(\mathfrak{b})\} = v(\mathfrak{a}) + v(\mathfrak{b})$$

Proof. Suppose $t \in v(\mathfrak{a}\mathfrak{b})$. Then $t = v\left(\sum_{i=1}^n x_i y_i\right) \geq \min\{v(x_i y_i) : 1 \leq i \leq n\} = \min\{v(x_i) + v(y_i) : 1 \leq i \leq$ for $x_i \in \mathfrak{a}$, $y_i \in \mathfrak{b}$, so that $t \in v(\mathfrak{a}) + v(\mathfrak{b})$. On the other hand, suppose $t \in v(\mathfrak{a}) + v(\mathfrak{b})$. Then $t = v(x) + v(y) = v(xy) \in v(\mathfrak{a}\mathfrak{b})$ for $x \in \mathfrak{a}$, $y \in \mathfrak{b}$. □

Lemma 227. *Let A be a valuation ring with surjective valuation $v : A \to \Omega$. Take ideals $\mathfrak{a}, \mathfrak{b} \in A$. Let H be the upper closure of $\{\max(v(x) - v(y), 0) : x \in v(\mathfrak{a}), y \in v(\mathfrak{b})\}$. Then $v((\mathfrak{a} : \mathfrak{b})) = H$.*

Proof. Suppose $t \in v((\mathfrak{a} : \mathfrak{b}))$. Then $t = v(x)$ for $x \in A$ such that $x \subseteq (\mathfrak{a} : \mathfrak{b})$. For each $y \in \mathfrak{b}$, there is $z \in \mathfrak{a}$ such that $v(xy) \geq v(z)$. Then for each $y \in \mathfrak{b}$, there is $z \in \mathfrak{a}$ such that $t + v(y) = v(xy) \geq v(z)$, so that $t \geq v(z) - v(y)$, so that $t \geq \max\{v(z) - v(y), 0\}$, so that $t \in H$.

Conversely, suppose $t \in H$. Then $t + v(y) \geq v(x)$ for some $y \in \mathfrak{b}, x \in \mathfrak{a}$.

So $t \geq$. for $x_i \in \mathfrak{a}$, $y_i \in \mathfrak{b}$, so that $t \in v(\mathfrak{a}) + v(\mathfrak{b})$. On the other hand, suppose $t \in v(\mathfrak{a}) + v(\mathfrak{b})$. Then $t = v(x) + v(y) = v(xy) \in v(\mathfrak{a}\mathfrak{b})$ for $x \in \mathfrak{a}$, $y \in \mathfrak{b}$. □

Theorem. *Suppose A is a valuation ring with valuation $v : A \to \Omega$, with Ω a totally ordered abelian group. Then, for any $a \in A$, $aA \in A^{cal}$ is Dilworth principal.*

Proof. Let $\zeta = aA$ and put $v(a) = t$. We show that $\mu(\zeta)_*$ is injective (note that A is a domain so that $(0 : \zeta) = 0$). Take ideals $\mathfrak{a}$ and $\mathfrak{b}$ in A such that $\zeta\mathfrak{a} = \zeta\mathfrak{b}$. Then

$$\{t + v(x) : x \in \mathfrak{a}\} = v(\zeta\mathfrak{a}) = v(\zeta\mathfrak{b}) = \{t + v(y) : y \in \mathfrak{b}\}$$

so that $v(\mathfrak{a}) = v(\mathfrak{b})$, so that $\mathfrak{a} = \mathfrak{b}$.

Next we show that $\mu(\zeta)^*$ is injective on the set of ideals contained in $\zeta = \zeta A$. Take ideals $\mathfrak{a}$ and $\mathfrak{b}$ in A such that $(\mathfrak{a} : \zeta) = (\mathfrak{b} : \zeta)$ and such that $\mathfrak{a}, \mathfrak{b} \subseteq \zeta$. Then

$$\{\max(v(x) - t, 0) : x \in \mathfrak{a}\} = v((\mathfrak{a} : \zeta)) = v((\mathfrak{b} : \zeta)) = \{\max(v(y) - t, 0) : y \in \mathfrak{b}\}$$

Since $\mathfrak{a}, \mathfrak{b} \subseteq \zeta$, $\max(v(x) - t, 0) = v(x) - t$ for each $x \in \mathfrak{a}$ and $\max(v(y) - t, 0) = v(y) - t$ for each $y \in \mathfrak{b}$. This gives

$$\{v(x) - t : x \in \mathfrak{a}\} = \{v(y) - t : y \in \mathfrak{b}\}$$

so that

$$\{v(x) : x \in \mathfrak{a}\} = \{v(y) : y \in \mathfrak{b}\}$$

i.e. $v(\mathfrak{a}) = v(\mathfrak{b})$. Thus $\mathfrak{a} = \mathfrak{b}$. □

4.28. **Primary Decomposition.** Credit to http://www.math.uiuc.edu/ r-ash/ComAlg/ComAlg1.pdf for the proofs I analogized.

Set a calyx E with corolla M throughout.

Definition 228. Let M be an E-corolla. M is primary if $Z(M) \subseteq rad(ann(M))$. In other words, M is primary if for every $\mathfrak{a} \in E$ greater than 0 such that $\mathfrak{a}x = \mathfrak{a}y$ for some $x \neq y \in M$, $\mathfrak{a}^n 1 = 0$ in M. We say $x \in M$ is $\mathfrak{p}$-primary if $x < 1$ and $[x]$ is a primary E-module and $rad(ann_M((x))) = \mathfrak{p}$, where $ann_{[x]}$ is the annihilator of x in the modu. Note that the operations in $[x]$ are possibly distinct from the operations in M.

Specializing to the case where $M = E$, we say E is primary if $Z(E) \subseteq rad(E)$. In other words, E is primary if for every $\mathfrak{a} \in E$ greater than 0 such that $\mathfrak{a}\mathfrak{b} = \mathfrak{a}\mathfrak{c}$ for some $\mathfrak{b} \neq \mathfrak{c} \in E$, $\mathfrak{a}^n E = 0$ in E. We say $\mathfrak{a} \in M$ is $\mathfrak{p}$-primary if $\mathfrak{a} < 1$ and $[\mathfrak{a}]$ is a primary E-module and $rad(ann(\mathfrak{a})) = \mathfrak{p}$.

M is secondary if $Z^{op}(M) \subseteq rad^{op}_M(M)$. In other words, M is secondary if for every $\mathfrak{a} > 0$ in E such that $(x : \mathfrak{a}) = (y : \mathfrak{a})$ for $x, y \in M$ with $x \neq y$, $\mathfrak{a}^n 1 = 0$ in M. We say $x \in M$ is $\mathfrak{p}$-secondary if $x > 0$ and (x) is a primary E-module and $rad(ann^{op}(x)) = \mathfrak{p}$. Note that the operations in $[x]$ are possibly distinct from the operations in M.

E is secondary if $Z^{op}(E) \subseteq rad(E)$. In other words, E is secondary if for every $\mathfrak{a} > 0$ in E such that $(\mathfrak{b} : \mathfrak{a}) = (\mathfrak{c} : \mathfrak{a})$ for $\mathfrak{b}, \mathfrak{c} \in M$ with $\mathfrak{b} \neq \mathfrak{c}$, $\mathfrak{a}^n 1 = 0$ in M. We say $\mathfrak{a} \in M$ is $\mathfrak{p}$-secondary if $\mathfrak{a} > 0$ and $(\mathfrak{a})$ is a primary E-module and $rad(ann^{op}(\mathfrak{a})) = \mathfrak{p}$.

Lemma 229. *Take $\mathfrak{a} \in E$ and suppose $\mathfrak{m} = rad(\mathfrak{a})$ is a maximal element in E. Then $\mathfrak{a}$ is $\mathfrak{m}$-primary.*

Proof. Take $x \in E$ with $x \notin \mathfrak{m}$ and suppose $xy + \mathfrak{a} = xz + \mathfrak{a}$ for $y, z \geq \mathfrak{a}$. Let $h = xy + \mathfrak{a} = xz + \mathfrak{a}$. $x + \mathfrak{m} = 1$. Take $k \in \mathbb{N}$ such that $\mathfrak{m}^k \subseteq \mathfrak{a}$. So

$$y = 1^k y = (\mathfrak{m} + x)^k y \leq (\mathfrak{m}^k + x)y \leq (\mathfrak{a} + x)y \leq \mathfrak{a} + xy = h$$

and similarly $z \leq h$. But $h = x \cdot y \subseteq y$ and $h = x \cdot z \subseteq z$ where $\cdot$ is multiplication in $[\mathfrak{a}]$. So $y = z$. □

Corollary 230. *If $\mathfrak{m}$ is a maximal ideal then $\mathfrak{m}^n$ is $\mathfrak{m}$ primary for each $n \in \mathbb{N}_{\geq 1}$.*

Definition 231. A primary decomposition of an element $x \in M$ is given by $x = \cap_{i=1}^n x_i$ where x_i are $\mathfrak{p}_i$-primary elements. The decomposition is reduced if $\mathfrak{p}_i$ are distinct and x is not the intersection of any proper subcollection of $\{x_i\}$. A secondary decomposition of an element $x \in M$ is given by $x = \sum_{i=1}^n x_i$ where x_i are $\mathfrak{p}_i$-secondary. Again, the decomposition is reduced if $\mathfrak{p}_i$ are distinct and x is not the sum of any proper subcollection of $\{x_i\}$.

Lemma 232. *If x, y are $\mathfrak{p}$-primary, then $x \cap y$ is $\mathfrak{p}$-primary.*

Proof. Let $z = x \cap y$. $\mathfrak{p} = rad(ann(x)) = rad(ann(y))$.

To show that $rad(ann(z)) = \mathfrak{p}$, we show that $\mathfrak{p} \subseteq rad(ann(z))$ and $rad(ann(z)) \subseteq \mathfrak{p}$. The second is clear. Take $\mathfrak{a} \in \mathfrak{p}$. There are $n, m \in \mathbb{N}$ such that $\mathfrak{a}^n 1 \subseteq x$ and $\mathfrak{a}^m 1 \subseteq y$. Thus $\mathfrak{a}^{n+m} \subseteq x \cap y = z$, so that $\mathfrak{a} \in rad_M(z)$.

To show that z is primary, note that

$$Z([z]) = Z([x \cap y]) \subseteq Z([x]) \cap Z([y]) \subseteq rad(ann(x)) \cap rad(ann(y)) = rad(ann(z))$$

□

Lemma 233. *If x, y are $\mathfrak{p}$-secondary, then $x + y$ is $\mathfrak{p}$-secondary.*

Lemma 234. *If $x \in M$ is irreducible and M is Noetherian then x is primary.*

Proof. Suppose x is not primary. Then for some $\mathfrak{a} \in E$, the canonical map $\mu : [x] \to [x]$ which sends y to $\mathfrak{a}y + x$ is not injective. Consider the ascending chain

$$ker(id) \subseteq ker(\mu) \subseteq ker(\mu^2) \subseteq \cdots \subseteq ker(\mu^n) \subseteq \cdots$$

Since M is noetherian the sequence terminates, say at $ker(\mu^n)$ for some $n \in \mathbb{N}$. Let $\phi = \mu^n$. $ker(\phi) = ker(\phi^2)$. If $x \in ker(\phi) \cap im(\phi)$ then $x = \phi(y)$ for some $y \in M$ $\phi^2(y) = \phi(x) = 0$ so $y \in ker(\phi^2) = ker(\phi)$, so $x = \phi(y) = 0$. $ker(\phi)$ is not injective, so $ker(\phi) \neq 0$, and μ is not nilpotent, so $im(\phi)$ is nonzero. Thus $[x]$ is irreducible. □

Lemma 235. *If $x \in M$ is coirreducible and M is artinian then x is secondary.*

Theorem (Existence of Primary Decompositions)**.** *By lemma 234 and lemma 232, every element $x \in M$ of a noetherian corolla has a reduced primary decomposition.*

Theorem (Existence of Secondary Decompositions)**.** *By lemma 235 and lemma 233, every element $x \in M$ of an artinian corolla has a reduced secondary decomposition.*

Definition 236. A prime $\mathfrak{p}$ of E is called an associated prime of M if $\mathfrak{p} = ann(x)$ for some principal element $x \in M$. Equivalently, if there is a perennial corolla map $E \to M$ whose kernel is $\mathfrak{p}$. We denote the set of associated primes by $Ass(M)$.

Definition 237. A prime element $\mathfrak{p}$ of E is called a co-associated prime of M if $\mathfrak{p} = ann^{op}(x)$ for some coprincipal element $x \in M$. Reminder: $ann^{op}(x)$ is defined as $(x : 1)$. Equivalently, if there is a perennial corolla map $M \to E^{op}$ whose cokernel is $\mathfrak{p}$. Equivalently, if there is a perennial corolla map $E \to M^{op}$ whose kernel is $\mathfrak{p}$. We denote the set of co-associated primes by $Ass^{op}(M)$.

Lemma 238. *A maximal element of $\{ann(x) : x \in M, x \neq 0\}$ is a prime ideal. If E is noetherian and nonzero, then $Ass(M) \neq \varnothing$.*

Proof. Suppose $ann(x)$ is maximal and take $\mathfrak{a}, \mathfrak{b} \in E$ such that $\mathfrak{a}\mathfrak{b} \leq ann(x)$. Suppose $\mathfrak{a}, \mathfrak{b} \not\leq ann(x)$. Then $\mathfrak{a}x \neq 0$. But then $ann(\mathfrak{a}x) \neq 1$ and $ann(\mathfrak{a}x)$ contains $\mathfrak{b}$ and $ann(x)$, contradicting the maximality of $ann(x)$. □

Lemma 239. *A maximal element of $\{ann^{op}(x) : x \in M, x \neq 1\}$ is a prime ideal. If E is noetherian and nonzero, then Ass^{op} is nonempty.*

Proof. Suppose $ann^{op}(x)$ is maximal and take $\mathfrak{a}, \mathfrak{b} \in E$ such that $\mathfrak{a}\mathfrak{b} \leq ann^{op}(x)$. Suppose $\mathfrak{a}, \mathfrak{b} \not\leq ann^{op}(x)$. Then $(x : \mathfrak{a}) \neq 1$. But then $ann^{op}((x : \mathfrak{a})) \neq 1$ and $ann^{op}((x : \mathfrak{a}))$ continas $\mathfrak{b}$ and $ann^{op}(x)$, contradicting the maximality of $ann^{op}(x)$. □

Lemma 240. *For any element $x \in M$, $Ass((x)) \subseteq Ass(M) \subseteq Ass((x)) \cup Ass([x])$.*

Proof. Clearly $Ass((x)) \subseteq Ass(M)$. Suppose $\mathfrak{p} = ann(y)$ for $y \in M$ but $ann(y) \notin Ass((x))$. We show that $ann(y) = (x : y) \in Ass([x])$. Clearly $ann(y) \subseteq (x : y)$ and if $\mathfrak{a} \in A$ has $\mathfrak{a}y \leq x$, $\mathfrak{a}y \neq 0$, then $\mathfrak{a} \notin ann(x)$. Since $ann(x)$ is prime,

$$\mathfrak{b} \in ann(\mathfrak{a}x) \;\Leftrightarrow\; \mathfrak{b}\mathfrak{b} \in ann(x) \;\Leftrightarrow\; \mathfrak{b} \in ann(x)$$

Thus $ann(x) = ann(\mathfrak{a}x) \in Ass((x))$, a contradiction. So $ass(M) \subseteq Ass((x)) \cup Ass([x])$. □

Lemma 241. *For any element $x \in M$, $Ass^{op}((x)) \subseteq Ass^{op}(M) \subseteq Ass^{op}((x)) \cup Ass^{op}([x])$.*

Lemma 242. *Suppose $0 = x_0 \leq x_1 \leq \cdots \leq x_n = M$ is a chain of elements in an E-corolla M. Then $Ass(M) \subseteq \cup_{i=1}^{n} Ass([x_i, x_{i+1}])$ and $Ass^{op}(M) \subseteq \cup_{i=1}^{n} Ass^{op}([x_i, x_{i+1}])$. Suppose $M_1, ..., M_n$ are E-corollas with $M = \oplus_{i=1}^{n} M_i$. Then $Ass(M) = \cup_{i=1}^{n} Ass(M_i)$ and $Ass^{op}(M) = \cup_{i=1}^{n} Ass^{op}(M_i)$.*

Proof. Follows from induction on the previous claim. □

Lemma 243. *If $\mathfrak{p} \in E$ is prime then $Ass_E([\mathfrak{p}]) = \{\mathfrak{p}\}$.*

Proof. Clearly $\mathfrak{p} \in Ass_E([\mathfrak{p}])$. For the other direction it suffices to show that $ann(x) = \mathfrak{p}$ for each nonzero $x \in [\mathfrak{p}]$. Take nonzero $x \in [\mathfrak{p}]$ and suppose $yx = 0$. Then $y \leq \mathfrak{p}$ since $x \not\leq \mathfrak{p}$. □

Lemma 244. *If $\mathfrak{p} \in E$ is prime then $Ass_E^{op}((\mathfrak{p})) = \{\mathfrak{p}\}$.*

Remark. For any prime $\mathfrak{p}$, $\mathfrak{p} \in Z(M)$ and $\mathfrak{p} \in Z^{op}(M)$.

Theorem. *Let M be a nonzero deciduous Noetherian corolla. Express the zero module 0 as the reduced intersection of primary elements $\{x_i\}_{i=1}^{n}$ in M with $rad(ann(x_i)) = \mathfrak{p}_i$ for primes $\mathfrak{p}_i$. $Ass(M) = \{\mathfrak{p}_i\}_{i=1}^{n}$.*

Proof. Let $\mathfrak{p} \in Ass_E(M)$ and take a principal element $x \in M$ such that $x \neq 0$, $\mathfrak{p} = ann(x)$. Reorder the x_i such that $x \leq x_i$ for $1 \leq i \leq j$ and $x \leq x_i$ for $j + 1 \leq i \leq n$. Take integers $\{n_i\}_{i=1}^{n}$ such that $\mathfrak{p}_i^n \leq ann(x_i)$. Then $\mathfrak{p}_i^n 1 \leq x_i$ so that $\left(\bigcap_{i=1}^{j} \mathfrak{p}_i^{n_i}\right) x \leq \bigcap_{i=1}^{n} (\mathfrak{p}_i^{n_i} x) \leq \bigcap_{i=1}^{n} x_i = 0$. Thus $\bigcap_{i=1}^{j} \mathfrak{p}_i^{n_i} \leq ann(x) = \mathfrak{p}$, so that $\mathfrak{p}_i \leq \mathfrak{p}$ for some $1 \leq i \leq j$. We show that $\mathfrak{p} \leq \mathfrak{p}_i$. Take $\mathfrak{a} \leq \mathfrak{p}$. Then $\mathfrak{a}x = 0$ and $x \not\leq x_i$, so that the map $\phi : [x_i] \to [x_i]$ which multiplies by $\mathfrak{a}$ is not injective. Since x_i is primary, ϕ is nilpotent. So $\mathfrak{a} \leq rad(ann(x_i)) = \mathfrak{p}_i$.

Conversely, each $\mathfrak{p}_i$ is an associated prime. Without loss of generality we take $i = 1$. Since the chosen primary decomposition of 0 is minimal, $\bigcap_{i=2}^{n} x_i \not\leq x_1$. Choose $x \leq \bigcap_{i=2}^{n} x_i$ with $x \not\leq x_1$. Take $n \in \mathbb{N}$ such that $\mathfrak{p}_1^n x \subseteq x_1$ but $\mathfrak{p}_1^{n-1} x \not\subseteq x_1$ (take $\mathfrak{p}_1^0 = E$). Take a principal element $y \leq \mathfrak{p}_1^{n-1} x$ such that $y \not\leq x_1$. We show that $\mathfrak{p}_1 = ann(y)$. $\mathfrak{p}_1 y \subseteq \mathfrak{p}_1 \mathfrak{p}_1^{n-1} x = \mathfrak{p}_1^n x$. Since $x \leq \bigcap_{i=2}^{n} x_i$, so $\mathfrak{p}_1 y \subseteq \bigcap_{i=2}^{n} x_i$. Thus $\mathfrak{p}_1 y \subseteq \bigcap_{i=1}^{n} x_i = 0$, so $\mathfrak{p}_1 \subseteq ann(y)$. On the other hand, if $\mathfrak{a} \in E$ and $\mathfrak{a}y = 0$, then $\mathfrak{a}y \in x_1$ but $y \notin x_1$, so that the map $\phi : [x_1] \to [x_1]$ which multiplies by $\mathfrak{a}$ is not injective and therefore is nilpotent. Thus $\mathfrak{a} \in rad(ann(x_1)) = \mathfrak{p}_1$. □

Corollary 245. *Let M be a Noetherian deciduous corolla. If $x = \bigcap_{i \in I} x_i$ is a reduced primary decomposition of x, and x_i is $\mathfrak{p}_i$-primary, then $\mathfrak{p}_i$ are uniquely determined by x.*

Lemma 246. *Suppose every element of E is the sum of finitely many principal elements and S is a localization set in E. Then*

$$Ass_{S^{-1}E}(S^{-1}M) = \{S^{-1}(\mathfrak{p}) : \mathfrak{p} \in Ass(M) : \zeta \notin \mathfrak{p}\ \forall \zeta \in S\}$$

Proof. Suppose $\mathfrak{p} \in Ass(M)$ and $\zeta \not\leq \mathfrak{p}\ \forall \zeta \in S$. Then there is a perennial exact sequence $0 \to \mathfrak{p} \to E \to M$ of E-corollas. Thus $0 \to S^{-1}\mathfrak{p} \to S^{-1}E \to S^{-1}M$ is an exact sequence of $S^{-1}E$ corollas. Since $\zeta \not\leq \mathfrak{p}\ \forall \zeta \in S$, $S^{-1}\mathfrak{p}$ is a prime of $S^{-1}E$. Thus $S^{-1}\mathfrak{p}$ is an associated prime of $S^{-1}M$.

Convsersely, suppose $\mathfrak{q} \in Ass_{S^{-1}E}(S^{-1}M)$. Then $\mathfrak{q} = S^{-1}\mathfrak{p}$ for some $\mathfrak{p} \in Spec(E)$ with $\zeta \not\leq \mathfrak{p} \forall \zeta \in S$ and $\mathfrak{q} = ann(x^S)$ for some $s \in M$. Write $\mathfrak{q} = \sum_{i=1}^{n} \zeta_i$ where ζ_i are principal.

Write $\mu_i : E \to M$ for the canonical corolla morphism sending 1 to ζ_i. Let $\mu_x : E \to M$ send 1 to x.

$$E \overset{\mu_i}{\to} E \to M \to S^{-1}M$$

□

4.29. **Dimension [59].**

4.30. **Associated Primes [62].** see stacks project

Definition 247. Let E be a calyx and M an E-corolla. A prime $\mathfrak{p} \in E$ is called an associated prime of M if there is a principal element $\zeta \in M$ such that $ann(\zeta) = \mathfrak{p}$. We denote the set of all such primes as $Ass_E(M)$ or just $Ass(M)$.

Lemma 248. *Let E be a calyx and M an E-corolla. $Ass(M) \subseteq Supp(M)$.*

Proof. Take $\mathfrak{p} \in Ass_E(M)$. If $x_\mathfrak{p} = 0_\mathfrak{p}$ in $M_\mathfrak{p}$ then $(x : \zeta) = 0$ for each principal element $\zeta \in M_\mathfrak{p}$. Thus $\zeta x \leq \zeta(x : \zeta) = 0$, so that $\zeta \in ann(x)$. Thus $x_\mathfrak{p} \neq 0$. □

Lemma 249 (not done)**.** *Let E be a calyx and take an exact sequence* $0 \to N \overset{\mathfrak{f}}{\to} M \overset{\mathfrak{g}}{\to} L \to 0$ *of E-corollas. $Ass(M) \subseteq Ass(N) \cup Ass(L)$.*

Proof. Take $\mathfrak{p} \in Ass(M)$ and an element $x \in M$ such that $ann(x) = \mathfrak{p}$. If $x \leq \mathfrak{f}_*(1)$ then $\mathfrak{p} \in Ass(N)$. Otherwise we show $\mathfrak{p} \in Ass(L)$. It suffices to show that $ann(x) = ann(\mathfrak{g}_*(x))$. □

4.31. **Completion [95].** c.f. the stacks project.

Definition 250. Let E be a calyx with element $\mathfrak{a}$. Form the poset category $\mathbb{Z}_{\leq 0}$, oredered in the usual way. Define a functor $\Phi : \mathbb{Z}_{\leq 0} \to \mathbf{Cal}$ where $\Phi(n) = [\mathfrak{a}^{-n}]\ \forall n \in \mathbb{Z}_{\leq 0}$. We define for a morphism $\alpha : n \to m$ set $\Phi(\alpha)$ to be the canonical quotient morphism $[\mathfrak{a}^{-n}] \to [\mathfrak{a}^{-m}]$. From this functor Φ we form the completion of E with respect to $\mathfrak{a}$: $\hat{E} = \varprojlim \Phi$. An element $\mathfrak{b}$ of $\hat{E}$ can be represented as a sequence $\{\mathfrak{b}_n\}_{n\geq 0}$ where $\mathfrak{b}_n \in [\mathfrak{a}^n]$ for each $n \in \mathbb{N}_{\geq 0}$ and $\mathfrak{b}_n = \mathfrak{b}_{n+1}$ in $[\mathfrak{a}^n]$. There is a canonical morphism $E \to \hat{E}$. If M is an E-corolla, we define the completion as follows: define a functor $\Phi : \mathbb{Z}_{\leq 0} \to E-$-**Cor** where $\Phi(n) = [\mathfrak{a}^{-n}M]\ \forall n \in \mathbb{Z}_{\leq 0}$. We define for a morphism $\alpha : n \to m$ set $\Phi(\alpha)$ to be the canonical quotient morphism $[\mathfrak{a}^{-n}M] \to [\mathfrak{a}^{-m}M]$. From this functor Φ we form the completion of E with respect to $\mathfrak{a}$: $\hat{E} = \varprojlim \Phi$. An element x of $\hat{M}$ can be represented as a sequence $\{x_n\}_{n\geq 0}$ where $x_n \in [\mathfrak{a}^n M]$ for each $n \in \mathbb{N}_{\geq 0}$ and $x_n = x_{n+1}$ in $[\mathfrak{a}^n M]$. We can view M as an $\hat{E}$ corolla. Thus there are canonical maps $M \to \hat{M}$ and therefore $M \otimes_E \hat{R} \to \hat{M}$. Moreover, completion forms a functor E-**Cor** $\to \hat{E}$-**Cor**.

why can we view M as an $\hat{E}$ corolla?

Lemma 251.

4.32. **Regular Local Rings [105].**

Definition 252. We say a local Noetherian calyx E is a regular local calyx if $dim_I(\mathfrak{m}/\mathfrak{m}^2) = dim(E)$.

Definition 253 (The Discrete Valuation Calyx)**.** The discrete valuation calyx is the calyx whose underlying lattice $\mathbb{N}_{\geq 0}$ is ordered such that $n \geq m$ when $n \geq m$ as integers (for nonzero n, m) and $0 \leq n \ \forall n \in \mathbb{N}$. Multiplication nm of nonzero elements is defined as sum, and we set $0n = 0$ for all $n \in \mathbb{N}_{\geq 0}$. The prime elements of a discrete valuation calyx are 2 and 0.

Example 254. The

Lemma 255 (not done)**.** *Any regular local ring is a domain.*

Proof. By Krull's intersection theorem, lemma 4.22, $\bigcap_{n \in \mathbb{N}_{\geq 0}} = 0$. Let $\mathfrak{a}, \mathfrak{b} \in E$ be such that $\mathfrak{a}\mathfrak{b} = 0$. Take $n, m \in \mathbb{N}_{\geq 0}$ maximal such that $\mathfrak{a} \in \mathfrak{m}^n$, $\mathfrak{b} \in \mathfrak{m}^m$. □

Theorem. *A calyx E is a regular local calyx of dimension 1 if and only if it is isomorphic to the discrete valuation calyx $\mathbb{N}$*

Proof. One direction is obvious. So let $(E, \mathfrak{m})$ be a local calyx of Krull dimension one, with $I \cong [\mathfrak{m}^2, \mathfrak{m}]$. $\mathfrak{m} \neq \mathfrak{m}^2$, so $0 \leq \mathfrak{m}^2 < \mathfrak{m}$. In particular $0 \neq \mathfrak{m}$. Take a principal element $\zeta \leq \mathfrak{m}$ with $\zeta \not\leq \mathfrak{m}^2$. Then $\zeta + \mathfrak{m}^2 = \mathfrak{m}$. By corollary 215 $\zeta = \mathfrak{m}$. So $\mathfrak{m}$ is principal. By theorem 4.22 $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = 0$. Take $\mathfrak{a} \in E$ not equal to 1. If $\mathfrak{a} \leq \mathfrak{m}^n$ for each $n \in \mathbb{N}_{\geq 0}$ then $\mathfrak{a} = 0$. Otherwise, $\mathfrak{a} \not\leq \mathfrak{m}^n$ for some $n \in \mathbb{N}_{\geq 0}$. Take n minimal such that $\mathfrak{a} \not\leq \mathfrak{m}^{n+1}$. $(\mathfrak{a} : \zeta^n) \not\leq (\zeta^{n+1} : \zeta^n) = \zeta$ since ζ is principal. Thus $(\mathfrak{a} : \zeta^n) \not\leq \mathfrak{m}$, so that $(\mathfrak{a} : \zeta^n) = 1$. Then $\mathfrak{a} = \zeta^n$. □

Guess: a regular local calyx of dimension d is isomorphic to $I[[x_1, ..., x_n]]$ for some n. is a regular local calyx automatically complete?

5. Schemes

5.1. **Intrinsic Definitions.** credit to hartshorne.

Definition 256. Let E be a cupulate calyx. Recall that for a prime ideal $\mathfrak{p}$ in E, $E_\mathfrak{p}$ is the localization of E at $\mathfrak{p}$, and that, for an element $\mathfrak{s} \in E$, $E_\mathfrak{s}$ is the localization at the multiplicative set $\{\mathfrak{s}^n : n \in \mathbb{N}_{\geq 0}\}$. For an open set $U \subseteq Spec(E)$, define E_U to be the set of functions $\phi : U \to \amalg_{\mathfrak{p} \in U} E_\mathfrak{p}$ such that $\phi(\mathfrak{p}) \in E_\mathfrak{p}$ and such that ϕ is locally an element of E. More precisely, we require that for each $\mathfrak{p} \in U$ there is a neighborhood $V \subseteq U$ of $\mathfrak{p}$ and an element $\mathfrak{a} \in E$ such that $\phi(\mathfrak{q}) = \phi_\mathfrak{q}(\mathfrak{a})$ for each $\mathfrak{q} \in V$, where $\phi_\mathfrak{q} : A \to A_\mathfrak{p}$ is the canonical map. This construction determines a calyx E_U.

Definition 257. Let E be a calyx. The spectrum of E is a pair consisting of of the topological space $Spec(E)$ together with the sheaf of rings Ω where $\Omega(U) = E_U$. It should be clear what the restriction maps are.

Lemma 258. *Let E be a calyx with elements $\zeta_1, ..., \zeta_n \in E$ such that $\sum_{i=1}^n \zeta_i = E$. Form the poset category $\Lambda = \{0,1\}^n - \{(0,0,...,0)\}$ ordered where $(a_1,...,a_n) \leq (b_1,...,b_n)$ when $a_i \leq b_i$ $\forall 1 \leq i \leq n$ for $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ in $\{0,1\}$. Form the functor $\Phi : \Lambda \to$ **Cal** where $(a_1,...,a_n) \mapsto E_{\prod_{1\leq i\leq n, a_i=1}\zeta_i}$ and $0 \mapsto E_{\zeta_1\cdots\zeta_n}$. For each morphism $\lambda : (a_1,...,a_n) \leq (b_1,...,b_n)$ in Λ, let $\Phi(\lambda) : E_{\prod_{1\leq i\leq n, a_i=1}\zeta_i} \to E_{\prod_{1\leq i\leq n, b_i=1}\zeta_i}$ be the canonical localization map. We show that $\varprojlim \Phi \cong E$.*

Proof. The canonical localization maps $E \to \Phi(a)$ for $a \in Obj(\Lambda)$ form a cone for Φ, and thus a map $\phi : E \to \varprojlim \Phi$ such that $E \to \varprojlim \Phi \to \Phi(a) = E \to \Phi(a)$ for each $a \in Obj(\Lambda)$. We construct an inverse map $\psi : \varprojlim \Phi \to E$ for ϕ. Write $\Phi_{a\leq b}$ for $\Phi(\lambda)$ where $\lambda : a \to b$ is a morphism in Λ. Take a collection of objects $\mathfrak{a}_a \in \Phi(a)$ such that $\Phi_{a\leq b}(\mathfrak{a}_a) = \mathfrak{a}_b$ for each pair $a \leq b$ in Λ. Let $\Phi([\{\mathfrak{a}_a\}_{a\in\Lambda}]) = \bigcap_{a\in A} \mathfrak{a}_a$.

Clearly $\psi \circ \phi = id$.

To show $\phi \circ \psi = id$, take $[\{\mathfrak{a}_a\}_{a\in\Lambda}]$ in Λ and let $x = \Phi([\{\mathfrak{a}_a\}_{a\in\Lambda}])$. Write a_i for the element of Λ which is 1 in the ith slot and 0 elsewhere, and write $\mathfrak{a}_i$ for $\mathfrak{a}_{a_i}$. Observe that $(\mathfrak{a}_j : \zeta_i) = (\mathfrak{a}_i : \zeta_j)$, so that

$$\begin{aligned}
&(x : \zeta_j)\\
=&\left(\bigcap_{a\in\Lambda} \mathfrak{a}_a : \zeta_j\right)\\
=&\left(\bigcap_{i=1}^n \mathfrak{a}_i : \zeta_j\right)\\
=&\bigcap_{i=1}^n (\mathfrak{a}_i : \zeta_j)\\
=&\bigcap_{i=1}^n (\mathfrak{a}_j : \zeta_i)\\
=&\left(\mathfrak{a}_j : \sum_{i=1}^n \zeta_i\right)\\
=&(\mathfrak{a}_j : 1) = \mathfrak{a}_j
\end{aligned}$$

□

Theorem. *Let E be a cupulate calyx (recall definition 143) and let $(Spec(E), \Omega)$ its spectrum. For any principal element $\zeta \in E$, the calyx $\Omega(D(\zeta))$ is isomorphic to E_ζ. In particular, taking $\zeta = 1$, $\Omega(Spec(E)) = E_\zeta$.*

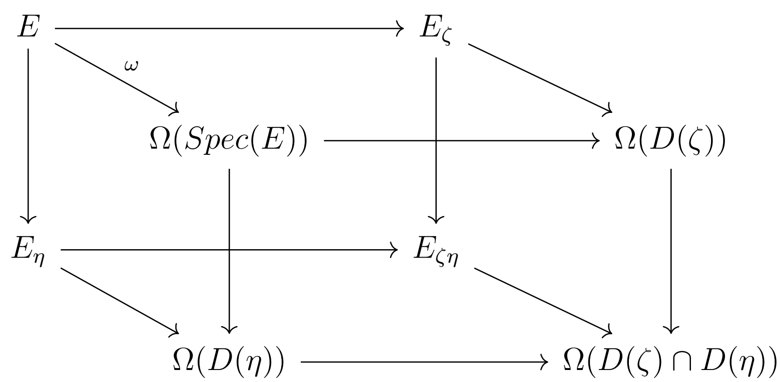
Proof. Passing to $(D(\zeta), \Omega|_{D(\zeta)})$, which is also cupulate, it suffices to show that $\Omega(Spec(E)) \cong E$. Define $\phi : E \to \Omega(Spec(E))$ by sending $\mathfrak{a}$ to the section α on E assigning to each prime $\mathfrak{p}$ the element $\mathfrak{a}^{\mathfrak{p}} \in E_{\mathfrak{p}}$.

To show ϕ is injective, take $\mathfrak{a}, \mathfrak{b} \in E$ such that $\mathfrak{a}^{\mathfrak{p}} = \mathfrak{b}^{\mathfrak{p}}$ for each $\mathfrak{p}$ in $Spec(E)$. $\mathfrak{a} = \mathfrak{b}$ by lemma 167.

Take $\alpha \in \Omega(Spec(E))$. Cover $Spec(E)$ with open sets $\{U_i\}_{i\in I}$ such that there are $\{\mathfrak{a}_i\}_{i\in I}$ in E with $\alpha(\mathfrak{p}) = (\mathfrak{a}_i)_{\mathfrak{p}}$ for $\mathfrak{p} \in U_i$. Without loss of generality, $\{U_i\}_{i\in I}$ can be taken to be principal open sets $D(\zeta_i)$, since such sets form a basis for $Spec(E)$. Since $1 \in E$ is compact, $Spec(E)$ is compact as a topological space, so that I can be taken to be a finite set.

We aim to show there is $\mathfrak{a} \in E$ with $\mathfrak{a}_{\mathfrak{p}} = \alpha(\mathfrak{p})$ for each prime $\mathfrak{p} \in Spec(E)$.

First we show by induction on $|F|$ that for each finite $F \subseteq I$ there is $\mathfrak{a} \in E_{\sum_{i\in F}\zeta_i}$ such that $\mathfrak{a}_{\zeta_i} = \alpha(\mathfrak{p})$. The case for $|F| = 1$ is automatic. Suppose the claim is true for some finite set $F \subseteq I$ and take $i \in I$. Write $\zeta = \zeta_i$ and $\eta = \sum_{j\in F}\zeta_j$. Passing to $E_{\zeta+\eta}$ we have $\zeta + \eta = 1$ and $\zeta\eta = \zeta \cap \eta$. Let $\mathfrak{a} \in E_\zeta$ be such that $\mathfrak{a}_{\mathfrak{p}} = \alpha(\mathfrak{p})$ for $\mathfrak{p} \in D(\zeta)$. Let $\mathfrak{b} \in E_\eta$ be such that $\mathfrak{b}_{\mathfrak{p}} = \alpha(\mathfrak{p})$ for each $\mathfrak{p} \in D(\eta)$. Then $\mathfrak{b}_{\mathfrak{p}} = \mathfrak{a}_{\mathfrak{p}}$ for $\mathfrak{p} \in D(\zeta) \cap D(\eta)$. The following diagram of canonical maps commutes:



Take $x = \mathfrak{b} \cap \mathfrak{a}$. Since $(\mathfrak{a} : \eta) = (\mathfrak{b} : \zeta)$, we have

$$(x : \zeta) = (\mathfrak{a} \cap \mathfrak{b} : \zeta) = (\mathfrak{a} : \zeta) \cap (\mathfrak{b} : \zeta) = (\mathfrak{a} : \zeta) \cap (\mathfrak{a} : \eta) = (\mathfrak{a} : \zeta + \eta) = \mathfrak{a}$$

and

$$(x : \eta) = (\mathfrak{a} \cap \mathfrak{b} : \eta) = (\mathfrak{a} : \eta) \cap (\mathfrak{b} : \eta) = (\mathfrak{b} : \zeta) \cap (\mathfrak{b} : \eta) = (\mathfrak{b} : \zeta + \eta) = \mathfrak{b}$$

With ω as in the diagram above, $\omega(x) = \alpha|_{Spec(E)}$, as claimed.

Now the claim follows from lemma 172. □

Lemma 259. *Take a calyx E and let U be an open set of $Spec(E)$. Let Ω be the structure sheaf of $Spec(E)$. $\Omega(U) \cong E_{\sum_{\zeta\in E^{princ.}, \zeta\not\leq\mathfrak{p}\ \forall\mathfrak{p}\in U}\zeta}$.*

Proof. By lemma 172

$$\Omega(U) \cong \varprojlim_{\zeta\not\leq\mathfrak{p}\ \forall\mathfrak{p}\in U} \Omega(D(\zeta)) \cong \varprojlim_{\zeta\not\leq\mathfrak{p}\ \forall\mathfrak{p}\in U} E_\zeta \cong E_{\sum_{\zeta\not\leq\mathfrak{p}\ \forall\mathfrak{p}\in U}}$$

□

Lemma 260. *Let E be a calyx and $(Spec(E), \Omega)$ its spectrum. Then the stalk $\Omega_{\mathfrak{p}}$ is isomorphic to $E_{\mathfrak{p}}$.*

Proof. For each $\mathfrak{p} \in \mathit{Spec}(E)$ let $P_{\mathfrak{p}}$ be the set of principal elements not contained in $\mathfrak{p}$. $\Omega_{\mathfrak{p}} \cong \varinjlim_{\zeta \in P_{\mathfrak{p}}} \Omega(D(\zeta)) \cong \varinjlim_{\zeta \in P_{\mathfrak{p}}} E_\zeta \cong E_{\mathfrak{p}}$ by lemma 173. □

Definition 261. A sheaf of calyxes is called a calical sheaf. A morphism $(X, \Omega_X) \to (Y, \Omega_Y)$ is a pair $(\mathfrak{f}, \mathfrak{f}^\#)$ where $\mathfrak{f} : X \to Y$ is a continuous map of topological spaces and $\mathfrak{f}^\# : \Omega_Y \to \Omega_X \circ \mathfrak{f}^{-1}$ is a morphism of sheaves, where $\mathfrak{f}^{-1}$ is viewed as a poset functor from the topology of Y to the topology of X. Thus the class of calical sheaves with calical sheaf morphisms forms a category. Write $\mathfrak{f}$ for a morphism $(\mathfrak{f}, \mathfrak{f}^\#)$ of calical sheaves.

Definition 262. Let (X, Ω) be a calical sheaf. If furthermore $\Omega_{\mathfrak{p}}$ is a local calyx for each $\mathfrak{p} \in X$ then we say (X, Ω) is a locally calical sheaf or a locally calical space. The morphisms of locally calical spaces are not arbitrary morphisms of calical sheaves, but instead morphisms $(\mathfrak{f} : (X, \Omega_X) \to (Y, \Omega_Y)$ such that for each $\mathfrak{p} \in X$, the canonical map $\mathfrak{k} : \Omega_{X\mathfrak{p}} \to \Omega_{Y\mathfrak{f}(\mathfrak{p})}$ has $\mathfrak{k}^*(\mathfrak{m}_Y) = \mathfrak{m}_X$ where $\mathfrak{m}_X$ is the maximal element in $\Omega_{X\mathfrak{p}}$ and $\mathfrak{m}_Y$ is the maximal element in $\Omega_{Y\mathfrak{p}}$. This construction forms a category.

Lemma 263. *The forgetful functor Φ from locally calical spaces to topological spaces distributes over limits and colimits.*

Lemma 264. *The forgetful functor Φ from locally ringed spaces to topological spaces distributes over limits and colimits.*

Lemma 265. *There is a covariant functor Φ from calyxes to locally calical spaces.*

Proof. It should be clear how Φ acts on objects. We construct $\Phi(\mathfrak{f})$ for a morphism of calyxes $\mathfrak{f} : E \to F$ as follows: define $\alpha : \mathit{Spec}(F) \to \mathit{Spec}(E)$ where $\alpha(\mathfrak{p}) = \mathfrak{f}^*(\mathfrak{p})$. Then

$$\begin{aligned}
&\alpha^{-1}(V(\mathfrak{a})) \\
=&\{\mathfrak{p} \in F : \alpha(\mathfrak{p}) \in V(\mathfrak{a})\} \\
=&\{\mathfrak{p} \in F : \mathfrak{a} \le \mathfrak{f}^*(\mathfrak{p})\} \\
=&\{\mathfrak{p} \in F : \mathfrak{f}^*(\mathfrak{a}) \le \mathfrak{p}\} \\
=&V(\mathfrak{f}^*(\mathfrak{a}))
\end{aligned}$$

so that α is continuous. For each $\mathfrak{p} \in \mathit{Spec}(F)$, $\mathfrak{f}$ induces a canonical map $\mathfrak{f}_{\mathfrak{p}} : A_{\mathfrak{f}^*(\mathfrak{p})} \to B_{\mathfrak{p}}$. Now for any open set $U \subseteq \mathit{Spec}(E)$, there is a calyx morphism $\alpha^\# : \mathit{Spec}(E)(U) \to \alpha_* \mathit{Spec}(F)(U)$ where $\alpha^\#(\mathfrak{s})_*(\mathfrak{p}) = \mathfrak{f}_{\mathfrak{p}*}(\mathfrak{s}(\alpha(\mathfrak{p})))$. This makes $(\alpha, \alpha^\#)$ a map of locally calical spaces. □

Lemma 266 (not done yet)**.** *The functor Φ as in* **??** *is full and faithful.*

Proof. It should be clear that Φ is faithful. Take a morphism of locally calical spaces $(\alpha, \alpha^\#)$. $\alpha^\#$ induces a map $\alpha : \Omega_{\mathit{Spec}(E)}(\mathit{Spec}(E)) \to \Omega_{\mathit{Spec}(F)}(\mathit{Spec}(F))$. By lemma 5.1 $\Omega_{\mathit{Spec}(E)}(\mathit{Spec}(E)) \cong E$ and $\Omega_{\mathit{Spec}(F)}(\mathit{Spec}(F)) \cong F$. Thus $\alpha : E \to F$. Similarly, by lemma 260, the map of stalks $\alpha_p : (\Omega_{\mathit{Spec}(E)})_{\alpha(\mathfrak{p})} \to \Omega_{\mathit{Spec}(F)_{\mathfrak{p}}}$ is really a map $\alpha_p : E_{\mathfrak{p}} \to F_{\mathfrak{p}}$. □

Remark. Let (X, Ω) be a scheme. Since the underlying topological space of $\Phi(X)$ is the same as the underlying topological space of X, $\Phi(X)$ is determined by $\Phi(X)_{\mathfrak{p}}$ for primes $\mathfrak{p}$.

Definition 267. An affine carpel is a locally calical space (X, Ω_X) which is isomorphic to the spectrum of some calyx. A carpel is a locally calical space (X, Ω_X) such that every point $\mathfrak{p}$ has an open neighborhood $U_{\mathfrak{p}}$ such that $(U, \Omega_X|_U)$ is an affine calical space. X is the underlying topological space of the carpel (X, Ω_X) and Ω_X is its structure sheaf. The class of carpels forms the class of objects of a category whose morphisms are morphisms of locally calical spaces. Thus carpels form a full subcategory of the category of calical spaces.

Example 268. Take the calyx $I = \{0, 1\}$, the unique domain calyx of dimension 0. $Spec(I)$ has an underlying space consisting of a single point, $\{*\}$. The structure sheaf Ω of $Spec(I)$ has $\Omega(*) = I$ and $\Omega(\varnothing) = 0$.

Lemma 269 (Coproducts in the category of locally calical spaces)**.** . *Let* $\{\mathcal{S}_i\} = \{(U_i, \Omega_i)\}_{i\in I}$ *be locally calical spaces. Form the sheaf* $\mathcal{S} = \amalg_{i\in I} S_i = (\amalg_{i\in I} U_i,)_{i\in I}, \Omega)$ *where* $\Omega(\amalg_{i\in I} V_i) = \{(\sigma_i)_{i\in I} : \sigma_i \in \Omega_i(V_i)\}$*. There are inclusion maps* $(\alpha_i, \alpha_i^{\#}) : \mathcal{S}_i \to \mathcal{S}$ *where* $\alpha_i : U_i \to \amalg_{i\in I} U_i$ *is the inclusion map and* $\alpha_i^{\#} : \Omega \to \alpha_* \Omega_i$ *sends* $(\mathfrak{s}_j)_{j\in I}$ *to* $\mathfrak{s}_i$*.*

Lemma 270 (Pushouts in the category of locally calical spaces)**.** .

Lemma 271.

5.2. **Some Examples.**

Example 272. Consider the calyx I. $Spec(I)$ consists of a single point $\{0\}$, and the localization of I at 0 is I, so that $\Omega_0 \cong I$ where Ω is the structure sheaf of $Spec(I)$.

Example 273. Let A be a discrete valuation ring. $A^{cal} \cong F$ where $F = \mathbb{N}_{\geq 0} \cup \{\infty\}$ has the usual order of $\mathbb{N}$, where sum in F is maximum of integers, intersection in F is minimum of integers, and product in F is sum of integers. F has two prime ideals, 0 and $\mathfrak{m}$ where $\mathfrak{m}$ is maximal in A.

Example 274. Let $E \cong \mathbb{Z}^{cal}$ be the calyx corresponding to the integers. The prime elements of E are the prime ideals $\{(0), (2), (3), (5), (7), ...\}$ in $\mathbb{Z}$. Of these, the nonzero prime ideals are all maximal and therefore closed. The ideal (0) is a generic point for $Spec(E)$. The localization of E at each element is a discrete valuation ring and therefore isomorphic to the calyx $\mathbb{N}_{\geq 0} \cup \{\infty\}$.

Example 275. Similarly, $k[x]^{cal}$ (with k algebraically closed) is a dimension 1 calyx with prime ideals $(x - a)$ for $a \in k$, and 0. Closed sets in $k[x]$ correspond to finite sets of nonzero prime ideals and the set of all elements. 0 is a generic point for $k[x]^{cal}$.

Example 276. More generally, take any dedekind domain A. $A_{\mathfrak{p}}^{cal}$ is a discrete valuation ring for each prime ideal $\mathfrak{p} \subseteq A$. The topology on $Spec(A)$ has as closed sets finite sets of points and the whole set.

Example 277 (not done)**.** Let S be a smooth algebraic curve over an algebraically closed field.

5.3. **Projective Calical Schemes.**

5.4. **First Properties of Schemes.** (off of hartshorne) c.f. chapter 3.

Definition 278. A calical scheme is connected if its topological space is connected. A calical scheme is irreducible if its underlying topological space is irreducible.

Lemma 279. *Let* $X = Spec(E)$*, the spectrum of a calyx.* X *is irreducible if and only if* $nil(E)$ *is prime.*

Proof.

$$\begin{aligned}
&nil(E) \text{ is irreducible}\\
&V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(0) \Rightarrow V(\mathfrak{a}) = 0 \text{ or } D(\mathfrak{b}) = 0\\
&V(\mathfrak{a}\mathfrak{b}) = V(0) \Rightarrow V(\mathfrak{a}) = 0 \text{ or } V(\mathfrak{b}) = 0\\
&\mathfrak{a}\mathfrak{b} \leq nil(E) \Rightarrow \mathfrak{a} \leq nil(E) \text{ or } \mathfrak{b} \leq nil(E)
\end{aligned}$$

□

Definition 280. A calical scheme X is reduced if for every open set U, the intersection of primes in the calyx $\Omega_X(U)$ is 0.

Lemma 281 (not done)**.** *Let $X = Spec(E)$, the spectrum of a calyx. X is reduced if and only if $nil(E) = 0$.*

Proof. One direction is obvious. Conversely, suppose $nil(E) = 0$ and take an open set $D(\mathfrak{a}) \subseteq X$. □

Definition 282. A scheme X is integral if for every open set $U \subseteq X$, the calyx $\Omega_X(U)$ is a domain.

Lemma 283. *A calical scheme X is integral if it is reduced and irreducible.*

Proof. An integral calical scheme is clearly reduced when it is integral.

Suppose X is reducible. Then X can be expressed as the disjoint union of open sets U and V. $\Omega_X(X) \cong \Omega_X(U) \, \Pi \, \Omega_X(V)$.

Suppose X is reduced and irreducible. Let $U \subseteq X$ be an open subset and suppose $\mathfrak{a}, \mathfrak{b} \in \Omega(U)$ have $\mathfrak{a}\mathfrak{b} = 0$. Let $Y = \{\mathfrak{p} \in U : \mathfrak{a}_{\mathfrak{p}}\}$ and let $Z = \{\mathfrak{p} \in U :\}$. □

5.5. **Smooth Schemes.**

5.6. **Separated and Proper Morphisms.**

5.7. **Sheaves of Corollas.** (taken from hartshorne)

Let (X, Ω_X) be a calical space. A sheaf of Ω_X-corollas.

5.8. **Divisors.**

5.9. **Projective Morphisms.**

5.10. **Differentials.**

5.11. **Formal Calical Schemes.**

6. Cohomology of Calical Sheaves

6.1.

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