Reduced Words of (p,q)-Clans

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Summer 2017

1 Introduction

Over the course of my REU, I studied characterizations of reduced words of $(p, q)-$clans, which have analogous properties to reduced words of permutations and reduced words of involutions. I determined a characterization of equivalence classes of reduced words. I also found a characterization for reduced words in the $W-$sets of arbitrary matchless $(p, q)-$clans. I investigated finding the $(p, q)-$clan with the most reduced words and came up with numerical data and conjectures.

2 Preliminaries

Definition 1. Define a (p, q)-clan to an involution in S_n, where with each fixed point we associate a "+" or a "−", such that $n = p + q$ and the total charge is $p - q$.

Example 1. 1^+43^+2 is a $(3, 1)$-clan.

Clans may also be represented by their matchings and signed fixed points, e.g., the clan 1^+43^+2 is described by $(2, 4), (1^+), (3^+),$.

Definition 2. Define elementary switches s_i to be a unary operation on (p, q)-clans such that:

- If in clan C, i and $i + 1$ are opposite signed fixed points, s_iC is the same clan, except i and $i + 1$ are matched.
- If in clan C, i is a fixed point and $i+1$ is matched with $k, k > i$, s_iC is the same clan, except k and i are matched, $i + 1$ is a fixed point whose sign is the sign of i in the clan C,
- If in clan C, $i+1$ is a fixed point and i is matched with $k, k < i$, s_iC is the same clan, except k and $i + 1$ are matched, and i is a fixed point whose sign is the sign of $i + 1$ in the clan C,
- If in clan C, i is matched to j and $i + 1$ is matched to k, $k > i$, s_iC is the same clan, except i is matched with k and $i + 1$ is matched with j.

Taking the relation $s_iC > C$ and its transitive closure, we get arrive at a poset structure on (p, q)-clans, with unique maximal element $M =$

$$(1, n), (2, n - 1) \cdots , (p, q + 1), p + 1^-, p + 2^-, \cdots q^-$$

if $q \geq p$ and

$$(1, n), (2, n - 1) \cdots , (q, p + 1), q + 1^+, q + 2^+, \cdots p^+$$

otherwise.

Definition 3. Define a reduced word of a $(p, q) - clan\ C$ to be a sequence $s_{a_1}, s_{a_2}, \cdots, s_{a_n}$ of minimal length, such that $M = s_{a_i} \cdots s_{a_2} s_{a_1} C$.

Example 2. $s_1 s_3 s_2 s_3$ is a reduced word for the clan $+--+$

2.1 Permutations/W-sets

Then the elementary switches satisfy the coxeter relations: that is,

$$s_i s_j C = s_j s_i C \text{ if } |i - j| > 1 \text{ for any clan } C, \text{ and}$$

$$s_i s_j s_i C = s_j s_i s_j C \text{ if } |i - j| = 1 \text{ for any clan } C$$

.

Because of this, we may correspond reduced words with permutations: we let the reduced word $s_{a_1}, s_a 2, \cdots s_{a_i}$ to correspond the permutation $s_{a_1} s_{a_2} \cdots s_{a_i}$, where the product is evaluated right to left by convention. Thus the word $s_1 s_3 s_2 s_3$ corresponds to the permutation 2431. Then if any reduced word of a permutation w is a reduced word for a clan C, all reduced words of w are reduced words for C.

It's now natural to define the $W - set$ of a clan C.

Definition 4. Define the $W-$set of a clan C (to be the set of permutations w such that reduced words of w are reduced words of C.

Algorithm 2.20 in [1] allows us to generate W-sets. Throughout this paper I will use it repeatedly.

3 My Work

3.1 The equivilence relation and shapes with labellings

Given a reduced word of a clan C, all other reduced words corresponding to the same permutation must be reduced words of C. Then, we are motivated to ask: given a permutation w is in the W-set of C, what other permutations must also be in C's W-set?

Definition 5. Define an equivalence relation $\equiv$ on permutations to be

$$v \equiv w \text{ if and only if } \forall C\ v \in W(C) \iff w \in W(C)$$

Definition 6. Define the labelled shape of a permutation as follows: while implementing the algorithm in [1], keep track of which +'s and -'s are paired. On the ith step, if you pick an arc, label it with an i. If you pick a $+$ and a $-$, pair their locations and label the pair with i. (Note: the pairing does not keep track of which fixed point was a $+$ and which was a $-$)

We distinguish between arcs and paired fixed points by underlining the paired fixed points. Note labelled shapes are in one-to-one correspondence with algorithm implementations and hence permutations. Also note unpaired fixed points' sign.

Example 3. In the clan $+--+$, with the permutation 2431, we get the labelled shape $\underline{(1,2)}^1, \underline{(3,4)}^2$

From the algorithm, we must satisfy the following conditions:

1. Given paired fixed points $\underline{(a,b)}^i, \underline{(c,d)}^j$, we may not have $a < c < b < d$.

2. Additionally, for paired fixed points $\underline{(a,b)}^i, \underline{(c,d)}^j$, if $i < j$, we may not have $a < c < d < b$.

3. Furthermore, given paired fixed points (m, n), any fixed point p with $m < p < n$ must be paired.

4. Given two matchings, one nested in the other, the outer matching must be labelled with the lower number.

5. Finally, no unpaired fixed point p satisfies $a < p < b$ for fixed points $\underline{(a,b)}^i$ paired.

Definition 7. Define an unlabelled shape to be a labelled shape after removing the labels.

To determine which clans a permutation w is in the W-set of, we may take w's unlabelled shape, and then assign $+$'s and $-$'s to the paired fixed points, such that within each pair one fixed point receives a $+$ and one a $-$. Then each distinct assignment of signs corresponds to a clan which w is in the W-set of.

Theorem 1. *$w \equiv v$ if and only if w and v have the same unlabelled shape.*

First, we prove contrapositive of the only if direction. Suppose w and v have different unlabelled shapes.

Case 1: Let (a, b) be a matching in w's shape but not v's. Then they must be in the W-sets different clans, as every clan corresponding with w in its W-set will have (a, b) as a matching but no clan with v in its W-set will.

Case 2: Let $\underline{(a,b)}$ be paired fixed points in w's shape but not v's.

Case 2a: If a or b is not a fixed point for v, we're done, as the property of being a fixed point of a clan is preserved when taking shapes.

Case 2b: If at least one of a and b, WLOG a, is unpaired in v's shape. Also WLOG, suppose it is a^+. Then w is in the W-set of a clan with a^- and b^+ as fixed points, but v is not.

Case 2c: If both a and b are paired fixed points (say with c and d, respectively), v is in the W-set of a clan which has a and b as fixed points both with $+$'s associated, while w is not.

Now, we prove the the if direction. Suppose w and v have the same unlabelled shape. Then w and v have the same matchings and the same pairings of fixed points. Thus, any choice of $+$'s and $-$'s for w's paired fixed points may be copied for v's paired fixed points, and vice versa. This completes the proof.

Theorem 2. *$\equiv$ is the transitive closure of the Coxeter relations and the relation*

$$ws_i \sim ws_{n-i}, 1 \leq i \leq \min(p,q) - 1 \tag{1}$$

Note: In terms of labelled shapes, ws_i ws_{n-i} corresponds to being able to switch labels i and $i+1$, assuming that this leaves an allowed labelling for a shape. (Note that in terms of permutations, this relation can be written as $v \sim vs_i s_{n-i}$ as long as the reduced words are the same length which is true iff $vs_i s_{n-i}$ is in the W-set which is true iff switching i and $i+1$ gives an allowed labelling.)

Proof: define a function f of a labelled shape:

$$f(w) = |\{(\underline{(a,b)}^i, \underline{(c,d)}^j) : i < j, b > d\}| + |\{(a,b)^i, (c,d)^j) : i < j, a > c\}| + \sum_{(a,b)^k} k$$

.

Then, within all possible labelings of a shape, this function is minimized at the unique labelled shape where the m matchings have labels 1 through m and the two other sets are empty. Clearly this is a lower bound. To see why it is achievable and unique, note that $|\{(\underline{(a,b)}^i, \underline{(c,d)}^j) : i < j, a > c\}| = 0$ uniquely determines the labels of paired fixed points (when ordered by right fixed point, the labels must be increasing). Similarly, $|\{(a,b)^i, (c,d)^j) : i < j, a > c\}| = 0$ implies that when matchings are ordered by left endpoint, the labels must be increasing. Furthermore, this labelling satisfies all conditions for labelled shapes.

Now suppose we have a labelling which does not minimize f. I show we may switch the labels i and $i+1$ for some i to decrease f.

Case 1: the labels of matchings are not all less than the labels of paired fixed points. Then let i be such that i is the label of a pairing, and $i+1$ is the label of a matching. Then we may switch the labels i and $i+1$ without violating any conditions on labelings. Then f is decreased by 1.

Case 2: There exist pairings $\underline{(a,b)}^k, \underline{(c,d)}^j : k < j, b > d$. Then there exist pairings $\underline{(m,n)}^i, \underline{(x,y)}^{i+1} : n > y$. Then we may swap the labels i and $i+1$, decreasing f by 1.

Case 3: There exist matchings $(a,b)^k, (c,d)^j) : k < j, a > c$. Then there exist matchings $(m,n)^{i+1}, (x,y)^i) : m > x$. Then we may swap the labels i and $i+1$ to get another allowed labelling, decreasing f by 1.

By applying applying process repeatedly, we must eventually reach the minimal clan, which proves the $\sim$ connects permutations in the equivalence classes of $\equiv$, and hence $\equiv$ is the transitive closure of $\sim$ and the Coxeter relations.

3.2 Codes

Now, one might ask: given a clan C, which equivalence classes make up $W(C)$?

Definition 8. Define the code of a labelled shape of a matchless (p,q)-clan C to be a list of length $\max(p,q)$ in the following way:

For $i < \min(p,q)$, let a_i be the left endpoint of the pairing with label i. Let b_i be the number of paired fixed points less than a_i whose label is less than i. Let s be the sign of a_i. Then the ith entry of the code is $(a_i - b_i)^s$. The last $|p-q|$ entries of the code are all signs: +'s if $p > q$ and −'s otherwise.

Codes are in 1-to-1 correspondence with labelled shapes of clans (and hence pairs of permutations and clans (w,C) with $w \in W(C)$: to recover the labelled shape and clan from its code, the following algorithm may be used.

Let $k(i)^{s(i)}$ be the ith entry of a code K. On the ith step, pair the $k(i)$th and $k(i)+1$st fixed points (counting from the left) not paired in previous steps. Let the $s(i)$ be the sign of the left just-paired fixed point (and $-s(i)$ be the sign of the right just-paired fixed point). The sign of the $|p-q|$ unpaired fixed points is determined by the sign of the final entries of the code.

Note the ith entry in a code of a (p,q)-clan C is at most $p+q+1-2i$. Furthermore, this condition is sufficient for a list of length $p+q$ with $|p-q|$ signs of the appropriate type at the end to be the code of a labelled shape.

Definition 9. Define the following equivalence relation on codes: $c \simeq d$ if and only if c and d are codes for the same clan.

Note the following relations:

1.
$$(\cdots, i^{s_i}, j^{s_j}, \cdots) \simeq (\cdots, j+2^{s_j}, i^{s_i}, \cdots) \text{ if } i-j \geq 2.$$
In terms of labelled shapes, this says swapping the labels k and $k+1$ (where k is the location of i in the code) maintains equivalence (assuming after swapping the labels we still have an allowed labelled shape). This is the same as equation (1) in 2.1.

2.
$$(\cdots, i^{s}, i^{s}, \cdots) \simeq (\cdots, (i+1)^{-s}, i^{s}, \cdots).$$
In terms of labelled shapes, this corresponds to taking the labelled pairs $(a,b)^k, (c,d)^{k+1}$ and changing them to $(b,c)^k, (a,d)^{k+1}$ (with $a<b<c<d$) being equivalent, as long as the signs of the fixed points permit such pairings (again where k is the left shown location in the code).

3.
$$(\cdots, i+1^{-s}, s, \cdots) \simeq (\cdots, i^{s}, s, \cdots).$$
In terms of labelled shapes, this relation corresponds to taking the paired fixed points with the smallest label, and changing which fixed points are paired. If, for example, the unpaired fixed points have positive sign, this relation corresponds to pairing $-$ with the $+$ on its right being equivalent pairing it with the $+$ on its left.

Theorem 3. $\simeq$ *is the transitive closure of these relations.*

First, we prove that, given labelled shapes C and D with the same unlabelled shapes, relation 1 is enough to show $C \simeq D$.

Suppose labelled shape C has the unlabeled shape S with k pairings. We may associate a permutation $w \in S_k$ with C, where $w(i)$ is defined to be the position of the pair with label i, when pairs are ordered by their right endpoint. Then distinct labellings of the same shape have distinct associated permutations. Out of all possible permutations, one is first lexicographically. Furthermore, it's not hard to see that, for every shape, there exists a labelling of that shape with the identity permutation associated to it.

Second, we prove that for every labelled shape C with unlabeled shape S and associated permutation w not the identity, there is another labelling D of S with associated permutation v, such that v is before w in lexicographic ordering, and $C \simeq D$ by relation 1.

Since w is not the identity, it contains at least one inversion. That is, there exists i such that $w[i] > w[i+1]$. From the definition of w, it must be the case that in labelled shape C, the right endpoint of the pair with label $i+1$ is left of the right endpoint of the pair with label i. Then if we switch labels i and $i+1$, we still get an allowed labelling. Letting D be the labelling attained by applying this switch (and v its associated permutation), we have $C \simeq D$ by relation 1, and v is before w lexicographically. Then we're done, since lexicographic ordering is a finite ordering.

Third, we prove that, given two labelled shapes C and D with the same unpaired fixed points, relations 1 and 2 are enough to show $C \simeq D$. We will show this by a similar method, picking a finite ordering and showing for every labelled shape C with some fixed points not first in the ordering, C is related to another labelled shape D by relations 1 and 2, where D is before C in the ordering.

Fix a (p, q)-clan, and set of unpaired fixed points F (with $|F| = |p - q|$). It suffices to show the statement is true for (p,q)-clans where $p = q$, since no paired fixed points have an unpaired fixed point between them. (A corollary: between unpaired fixed points are equal numbers of positive and negative paired fixed points.)

Order the paired fixed points of an unlabelled shape for a (p, p)-clan by their left endpoints, and with each unlabelled shape S associate a word z_S, where $z_S(i)$ is $|a - b|$, where (a^+, b^-) is the ith pair. Then consider a total ordering on shapes of a fixed (p, q)-clan to be the lexicographic ordering on words.

Now, note that the shape corresponding to $(\cdots, (i+1)^{-s}, i^s, \cdots)$ is later than the shape corresponding to $(\cdots, i^s, i^s, \cdots)$ in this ordering. Thus, it suffices to show that, for every shape not minimal in lexicographic ordering, the shape has a labelling whose code of the form

$$(\cdots, (k+1)^{-t}, k^t, \cdots),$$

as then it is related to a code whose shape is earlier in the ordering.

Now, let the minimal shape of a (p, p)-clan be M. Fix a (p, p)-clan C whose shape is not M. Let a be minimal such that (a^l, b^{-l}) is a fixed point pair in M, and (a^l, b'^{-l}) is a fixed point pair in C, with $b' > b > a$. Then, notice in the set of fixed points $\{a+1, \cdots, b\}$ exactly one more fixed point has sign $-l$ than sign l. Thus, in clan C, at least one fixed point $a < d \le b$ with sign $-l$ is paired with e^l for some $e > d$.

Let d' be the minimum such fixed point, and let its pair (in C) be (d'^{-l}, e'^l). Now let j' be the maximum $j < d$ such that j is paired with k for some $k > e'$. (let k' be the fixed point j is paired with.) Note j' exists, as $j = a$ works. Thus by the minimality of d', the sign of j' is l, and hence the sign of k' is $-l$.

Finally, note that we may, when applying the algorithm in https://arxiv.org/pdf/1409.4227.pdf, select (d'^{-l}, e'^l) nth, and (j'^l, k'^{-l}) $n+1$th, where n is the number of paired fixed points (x^t, y^{-t}) : $j' < x < y < k'$ Then, by applying the algorithm in this way, we end up with a code containing

$$(\cdots, j' + 1^{-l}, j'^l, \cdots)$$

(nth and $n+1$th locations in the code shown), which is of the desired form.

We have now shown that if two (p,q)-clans have the same unpaired fixed points, they are related by the closure of relations 1 and 2.

Suppose we have two shapes of a (p,q)-clan C, X and Y. Without loss of generality, assume, $p > q$. Let the unpaired fixed points of X be $(x_1, x_2, \cdots x_{p-q})$, and Y's be $(y_1, \cdots y_{p-q})$, both in increasing order. Let $m \leq q+1$ be maximal such that $x_k = y_k$ for all $k \leq m-1$. Also without loss of generality, assume $y_m < x_m$. We prove X and Y are related by induction on $p-q-m$. The base case, where $q-m=-1$, is clear: X and Y then have the same unpaired fixed points.

Assume by strong induction the statement is true for $q-m < q-n$ (i.e., $m > n$). We must prove it when $q-m = q-n$, i.e. $n=m$.

I claim the exists a labelled shape A with unpaired fixed points $(x_1, \cdots, x_{p-q})$ which is related to a labelled shape B with unpaired fixed points $(x_1, x_2, \cdots, x_{n-1}, y_n, x_{n+1}, \cdots, x_{p-q})$ by relation 3.

We apply the algorithm in [1] in order to find labelled shapes A and B, first pairing fixed points between x_1 and x_{n-1}, and then fixed points between x_{n+1} and x_{p-q}, making the same choices at every step for the labelings of A and B. Since X and Y are shapes for C, there must also be an equal number of fixed points of each sign in C between x_{n-1} and y_n, and between x_n and x_{n+1}. Also, between y_n and x_n inclusive, we must have one more positive fixed point than negative. We then continuing applying the algorithm to fixed points between x_{n-1} and y_n, and then between x_n and x_{n+1}, pairing points between x_{n-1} and y_n only with each other (and similar between x_n and x_{n+1}) until all such points have been paired. Note we may again make the same choices at each stage for A and B.

Then we continue applying the algorithm to get the labelled shape A, following the condition that we pair y_n last. Note this is possible; there's at least one positive and one negative not-yet-paired fixed point among $(y_n+1, \cdots x_{n+1}-1$ unless $x_{n+1}-1$ was the only positive not-yet-paired fixed point, and among any list containing both pluses and minuses, there's a plus and minus adjacent to each other. Until the last step, we apply the algorithm in the same way to get the labelled shape B. We have one fixed point left: k^-. To finish the algorithm for labelled shape A, we paired k with y_n. To finish the algorithm for labelled shape B, pair k with x_n.

Then the codes for the chosen labelings A and B are related by relation 3.(A's is $(\cdots, n^+, +, \cdots)$ while B's is $(\cdots, n+1^-, +, \cdots)$) Since B and Y share the same first n unpaired fixed points, by induction B and Y are related. Thus $X \simeq A \simeq B \simeq Y$, so the induction is complete.

4 A Conjecture

I also considered the following question: Fixing p and q, which (p,q)-clan has the most reduced words? The answer must be a matchless clan, and the number of reduced words $n(C)$ of a matchless clan C is be given nicely (this is part known, not a conjecture) by the following:

Define $+_C$ be the set of positive fixed points of C, and defining $-_C$ similarly, Then

$$n(C) = \frac{(pq)!}{\prod_{a \in +_C, b \in -_C} |a - b|}$$

Conjecture 1. Let $d_1 < \cdots < d_p$ be the locations of the positive fixed points in the (p,q)-clan with the most reduced words. Let $e_1 < \cdots < e_p$ be the locations of the positive fixed points in the (p,q+1)-clan with the most reduced words. Then

$$d_i \leq e_i \leq d_i + 1 \text{ for all } i.$$

I attempted to find a proof and wrote a program to gather numerical evidence. Though I found interesting lemmata, ultimately the problem remains unsolved.

References

[1] Can, Mahir Bilen, et al. Chains in Weak Order Posets Associated to Involutions, 15 Sept. 2014, arxiv.org/abs/1409.4227.