OBSTRUCTING EMBEDDINGS OF FINITE R-PARTITE RIGHT-ANGLED ARTIN GROUPS INTO MAPPING CLASS GROUPS

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The author would like to thank Wouter van Limbeek for his help in preparing this document.

Abstract. We introduce right-angled Artin groups and their embeddings into mapping class groups and detail significant research on obstructions to such embeddings. We strengthen an obstruction given in [11] which is independent of the standard obstructions coming from chromatic and clique numbers. We conclude with further research directions and unanswered questions from the literature.

Contents

University of Michigan REU Program 2017

This article is a product of my work with my mentor Wouter van Limbeek as a participant in the University of Michigan Research Experience for Undergraduates program in the summer of 2017. I'd like to extend my thanks to the University, for this opportunity and their support of my research, and to the other REU participants, for helpful and enjoyable conversations along the way.

The first author was supported in part by NSF Grant #000000.

1. Introduction

We survey the relevant areas of research and the results that support and motivate our work.

1.1. Embedding RAAGs into mapping class groups.

1.1. Embedding RAAGs into mapping class groups. Geometric group theory has for a long time concerned itself with the mapping class groups (see 2.1) of surfaces, both their algebraic characterization and their geometric actions on spaces. The subgroups of mapping class groups has proved to be an especially interesting and rich field of study. This paper engages the topic of subgroups of mapping class groups constructed from finite graphs called *right-angled Artin groups*, or RAAGs for short (see 2.2). For much insightful research into the field, one can turn to the works of Koberda and Kim, in [7], [8], [9], and in particular for an in-depth treatment, to [10]. It has surprisingly been shown that every right-angled Artin group embeds into the mapping class group of some finite type surface. For a construction, see [6]. It has also been shown that no mapping class group contains every RAAG (this turns out to be a graph-theoretic result, see [8]).

The characterization of right-angled Artin subgroups of mapping class groups is interesting not only for the wealth of examples of subgroups of mapping class groups that these embeddings exhibit, but also for the geometric information that RAAG subgroups contain. For instance, it has been shown that the action of RAAG subgroups on Teichmüller space is convex cocompact. The geometric data of RAAG subgroups is often easier to grasp than this, though, and indeed Koberda has shown in [10] that every subgroup of the mapping class group generated by finitely many Dehn twists about simple closed curves is (when suitable powers are taken) a RAAG.

We are interested here in particular with when RAAGs do not embed into mapping class groups. That is, with the questions about what constraints on graphs and surfaces imply that certain RAAGs do not embed into the mapping class group. A general motivating question is the following. For a positive integer n, let $g(n)$ denote the minimal genus of a surface such that there is an embedding $A(\Gamma) \hookrightarrow \mathrm{Mod}(S_{g(n)})$, where $S_{g(n)}$ denotes a surface of genus $g(n)$ and $|\Gamma| = n$. How does $g(n)$ grow with n?

Some facts about such obstructions have been obtained in the literature (see below), but there is still no general characterization. The answer to the question just stated is also unknown. The research surrounding and within this paper deals with, and is motivated by, rigidity phenomena like this in mapping class groups, and how to describe RAAG embeddings.

1.2. The curve graph and its clique graph. To fully understand how RAAGs embed into mapping class groups, it is extremely helpful to consider graphs associated to the surfaces in question. The prototypical example of such a graph is the *curve graph* $\mathcal{C}(S)$ associated to a surface S, see 2.1.1.

The curve graph (and the associated simplicial complex: the curve complex) have also been highly researched. A wide array of graph-theoretic properties of the curve graph are known (e.g. in non-trivial cases it is always connected, it has infinite diameter (convincing yourself that this is surprising is a fun exercise), et al.). In particular, Koberda and Kim have shown that given a (finite type) surface S and a finite graph G, induced subgraphs $G \leq \mathcal{C}(S)$ give rise to group embeddings $A(G) \hookrightarrow \mathrm{Mod}(S)$, and group embeddings $A(G) \hookrightarrow \mathrm{Mod}(S)$ induce embeddings as induced graphs $G \leq \mathcal{C}(S)_k$, where $\mathcal{C}(S)_k$ is the *clique graph of the curve graph*. The vertices of the clique graph are cliques of vertices of the curve graph (that is, sets of vertices that form a complete subgraph of the curve graph), where curves are thought of as 1-cliques. Vertices are adjacent when the corresponding sets of vertices form a clique. This suggests that there exist graph-theoretic obstructions to embedding RAAGs into mapping class groups. The automorphism group of $\mathcal{C}(S)$ is the mapping class group. (This adheres to an often-repeated metaconjecture of Ivanov that whenever a complex is associated to a surface, its automorphism group is the (extended) mapping class group. For a precise formulation, see a recent paper by Tara Brendle and Dan Margalit, [13].)

1.3. **Obstructions of embeddings.** There are two well-known obstructions to embeddings of RAAGs into mapping class groups, one graph-theoretic and the other group-theoretic. Of course these two sides of the RAAG coin correspond in interesting ways.

The group-theoretic obstruction comes from the abelian subgroups of the mapping class group. The abelian subgroups of the mapping class group have been widely studied; see in particular [3], where the authors Birman, Lubotzky, and McCarthy show that the maximal torsion-free rank of an abelian subgroup in the mapping class group is equal to the *complexity* of the surface, i.e. the number of (homotopy classes of) curves in a maximal multicurve on the surface. E.g., the complexity of the surface $S_{g,n}$ of genus g with n punctures can be shown to be $3g-3+n$. The abelian subgroups of right-Angled artin groups correspond one-to-one with cliques among the vertex generators, so that this fact may also be checked graph-theoretically. When one wishes to obstruct an embedding of a RAAG into a specified mapping class group, one can compare the clique number of the graph, defined to be the size of the maximal clique, to the abelian subgroups of the mapping class group. This should then be thought of as a local obstruction: it suffices to check the 1-balls of the the vertex generators in question to see whether or not they form a clique, i.e. whether or not they obstruct embeddings this way.

The graph-theoretic obstruction comes from chromatic number of the defining graph and the clique graph. This should be thought of as a global construction, since in order to compute the chromatic number of a graph, it is *necessary* to examine every vertex in question (for instance, cyclic graphs of odd and even parity have distinct chromatic numbers, yet differ in only one vertex; it does not suffice to check any proper subset of vertices to determine chromatic number of these graphs). Koberda has proved that chromatic number can obstruct embeddings of RAAGs into mapping class groups. To do this, it was shown that when the curve graph has finite chromatic number (this is always true), so does the clique graph of the curve graph. Then one can appeal to the fact (stated above) that group embeddings $A(G) \hookrightarrow \mathrm{Mod}(S)$ induce graph embeddings $G \leq \mathcal{C}(S)_k$ as induced subgraphs.

Additionally, the authors of [11], Bering, Conant, and Gaster, introduce a novel obstruction which they call the nested complexity length, NCL, of the defining graph. The authors show that NCL combined with the two previously described obstructions do not suffice to totally obstruct RAAG embeddings into mapping class groups. They also introduce the graph which they call $K_r(2)$, the complete r-partite graph on sets of 2 vertices. The authors employ $K_r(2)$ in order to obstruct NCL and determine that it is insufficient to obstruct RAAG embeddings as a whole (they consider $K_r(t)$ for all t, but the case of $t=2$ is the most interesting). In particular they show that for large t, $K_r(t)$ embeds into $\mathrm{Mod}(S)$ when r is not too large, where r is related to the surface S by $r = g + \lfloor \frac{g+p}{2} \rfloor$, where $S = S_{g,p}$. We prove an analogous strengthening which is that $K_r(2)$ is obstructed for all r considered in [11].

1.4. **Statement of the main theorem.** In particular, we show:

Theorem 1.1. *For a finite-type surface $S = S_{g,p}$ of genus g with p punctures let $r = r(g,p) = g + \lfloor \frac{g+p}{2} \rfloor$. The RAAG $A(K_r(2)) \cong F_2^r$ does not embed into* $\mathrm{Mod}(S)$.

There are assumptions on how large g and p need to be in order for this to be true; they are handled in a later section. In [11], the authors show that $K_r(t)$ embeds as an induced subgraph of the curve graph if and only if $r \leq r(g,p)$. Thus this result strengthens theirs, in that it strongly obstructs embeddings of the $K_r(t)$ as a function of t as well.

The RAAG $A(K_r(2))$ is equal to a direct product of copies of the free group F_2. Koberda has noted briefly in [10, Prop. 6.15] that embeddings of products of F_n into mapping class groups of non-punctured, finite-type surfaces are obstructed. This paper expands on this result, additionally contributing the punctured case, which, we will see, poses many of its own challenges.

The paper will be organized as follows. We will begin with a section on preliminary results and definitions concerning mapping class groups and right-angled Artin groups, and the embeddings of the second into the first, and their obstructions. We will record, for the general record and for our own purposes, the computation of the centralizer of a reducible mapping class in $\mathrm{Mod}(S)$, which is easily obtainable by cutting along a canonical reduction system (to be defined). Then we will prove the main theorem, with some consequences listed afterwards.

We sketch the proof of the main theorem. First, we will observe that in some embedding $A(K_r(2)) \cong F_2^r \cong H_1 \times \cdots \times H_r \hookrightarrow \mathrm{Mod}(S_{g,p})$, there will be generators for some H_i which are reducible. We will assume that g, p are minimal so that this embedding is realized. Upon cutting along canonical reduction systems and appealing to the result about centralizers, we will reduce to looking at a connected subsurface, and the restriction of the generator to that surface. A contradiction will be proved combinatorially, by showing that this component of the cut surface will support an embedding with strictly smaller (in some sense to be defined) g, p than the original.

2. Preliminaries

This section gives some of the basic results and definitions necessary for the proof of the main theorem, with references to the proofs and some standard documents.

2.1. **Mapping class groups.** We introduce one of the two principal objects of our study (and of geometric group theory in general), the *mapping class group* of a surface.

Given a (finite type) surface $S = S_{g,n}$ of genus g with n punctures, we define the *mapping class group* of S, denoted by by $\mathrm{Mod}(S)$, to be the quotient $\mathrm{Homeo}^+(S)/\sim$ where two orientation-preserving homeomorphisms $f, g : S \to S$ are related whenever there exists an isotopy between them, i.e. a continous map $H : S \times [0,1] \to S$, where $H(x,0) = f(x)$ and $H(x,1) = g(x)$ with $H(x,t) \in \mathrm{Homeo}^+(S)$ for all $t \in [0,1]$. (A similar definition using diffeomorphisms is equivalent.) When the boundary of S, denoted ∂S, is nonempty, the common assumption is that ∂S is fixed point-wise by all mapping classes; the mapping class group is often denoted $\mathrm{Mod}(S, \partial S)$ in this case. In some settings it is more convenient however to allow $\mathrm{Mod}(S)$ to permute boundary components or leave ∂S invariant set-wise. A classical reference is the text by Farb-Margalit, *Primer on Mapping Class Groups*, [1].

The mapping class group of a surface is often a highly complex and rich object of study, but there are various elements of mapping class group theory that are easily tangible at the outset. One such example is a particularly important mapping class (where a *mapping class* is an element of the mapping class group, i.e. an isotopy class of homeomorphisms of S) called a *Dehn twist*. Consider a simple closed curve γ on S (i.e. an embedded circle); remove a small annular neighborhood of γ from S. On this neighborhood, apply a "twist map", where points near the center of the annulus, i.e. away from the two boundary components, are rotated 360 degrees, while the boundary circles are fixed. We have twisted around the core of the annulus. Pasting this newly twisted annulus back into S with the same boundary configurations gives a homeomorphism of S, the class of which is called the Dehn twist about γ, which is denoted T_γ.

Dehn twists are intuitively clear and easy to draw pictures of on simple surfaces (think about what a Dehn twist around a simple closed curve on a torus does to other simple closed curves, for instance), and have remarkable algebraic properties in $\mathrm{Mod}(S)$. In particular, it has been shown that the mapping class group of the genus g surface S_g can be generated by Dehn twists. We will see in section 2.1.3 that Dehn twists can be used to define much more complicated mapping classes which are called pseudo-Anosov, whose principal exponent was Bill Thurston (see [5]).

2.1.1. *The Nielsen-Thurston classification.* The mapping class group is a group of some amount of mystery when it is first encountered. However since its outset, the study of mapping class groups has greatly formalized the subject, to the point where there is a three-fold classification of mapping classes, known as the *Nielsen-Thurston trichotomy.*

The Nielsen-Thurston classification of mapping classes dictates that any $[f] \in \mathrm{Mod}(S)$ is one of the following:

(1) a periodic (or finite-order) element;
(2) a pseudo-Anosov element;
(3) a reducible element (i.e. it fixes a nonempty set of curves up to isotopy).

The third set of mapping classes may be reduced to combinations of the first and second. Psued-Anosovs have significantly more structure than periodic elements; we will define them in section 2.1.3. This result is now a fundamental tool in mapping class group theory, and allows otherwise complicated questions about mapping class groups to be reduced to simpler, finite checks.

2.1.2. *Pseudo-Anosov mapping classes.* To fully grasp the power of the Nielsen-Thurston classification and the richness of the mapping class group, it is necessary to study the most complicated of the mapping class group types, *pseudo-Anosov* mapping classes. We require the notion of a foliation of a topological surface; the reader can find a detailed study in [12, Chapter 14].

Constructions of pseudo-Anosov maps are nontrivial, and there are various well-known ways of going about them. See in particular [1, Section 14.1]. We explore the construction that uses branched covers of the torus. Pseudo-Anosov maps are so-named because of their derivation from the more well-behaved *Anosov maps* on the torus, from which we build pseudo-Anosovs in the following manner.

From hyperbolic geometry, we know that the isometries of the upper half plane (isomorphic to PSL(2, $\mathbb{R}$)) are classified by their fixed points. Hyperbolic elements are those that fix two distinct points in $\partial\mathbb{H}$, and acts by translation across the unique line between them, one endpoint acting as a sink and the other as source. This is equivalent to saying that the corresponding element of SL(2,$\mathbb{Z}$) has two distinct real eigenvectors, and since its determinant is 1, they are inverses. Call these two $\lambda > 1$ and $\lambda^{-1} < 1$. Thus any hyperbolic element τ gives information about a mapping class $f \in \mathrm{Mod}(T^2) \cong \mathrm{SL}(2,\mathbb{Z})$; this information is called an *Anosov package*. Specifically we have two foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ on T^2 satisfying

(1) each leaf of $\mathcal{F}^s$ and $\mathcal{F}^u$ is the image of an injective map $\mathbb{R} \to T^2$,
(2) $\mathcal{F}^s$ and $\mathcal{F}^u$ are transverse at all points,
(3) There is a natural transverse measure μ_s (respectively μ_u) assigning a measure to each arc transverse to $\mathcal{F}_s$ (respectively $\mathcal{F}_u$) obtained by realizing the foliations by straight lines in some flat metric on T^2, and declaring the measure of a transverse arc to be the total variation in the direction perpendicular to the foliation,
(4) there is an affine representative $\phi \in \mathrm{Mod}(T^2)$ of f satisfying

$$\phi(\mathcal{F}_u, \mu_u) = (\mathcal{F}_u, \lambda\mu_u), \quad \phi(\mathcal{F}_s, \mu_s) = (\mathcal{F}_s, \lambda^{-1}\mu_s).$$

Then to define a pseudo-Anosov mapping class on a higher-genus surface S_g, one fixes a branched cover $p : S_g \to T^2$, and an Anosov mapping class $\phi \in \mathrm{Mod}(T^2)$, i.e. a pair of transverse measured foliations descending from $\mathbb{R}^2$ and an affine representative ϕ as in (4). After passing to a power, we may assume that ϕ fixes the branch points, so that we may lift ϕ to obtain a homeomorphism of S that is locally given by ϕ: the isotopy class of such a map is a pseudo-Anosov mapping class in $\mathrm{Mod}(S_g)$.

There are many ways of thinking about pseudo-Anosov maps of which the above construction is simply an easily stated one. The reader can and should refer to [1] for further reading.

2.1.3. *Canonical reduction systems.* For the purposes of this paper, we also need to deal with reducible mapping classes in some detail.

Reducible mapping classes are those elements f for which there exists a nonempty set of pairwise disjoint non-isotopic essential simple closed curves $C = \{c_1, ..., c_n\}$ such that the c_i are permuted by f, i.e. $f(C) = C$. Such a set C is a *reduction system* for f, and a reduction system is called *maximal* when it is maximal with respect to inclusion of reduction systems. In [3] the authors define a *canonical reduction system* for the reducible mapping class f, defined as the intersection of all maximal reduction systems for f, which we denote A. A is also a reduction system for f, as for any $c \in A$ we must have $f([c])$ lying in each maximal reduction system for f, since f permutes the curves in each maximal reduction system. The other qualities needed to be checked are also inherited. A is canonical in the sense that its definition requires no choice of reduction system, but also in the following sense: let the set of maximal reduction systems for f be denoted $\mathrm{MRS}(f)$. Then for all $g \in C_{\mathrm{Mod}(S)}(f)$ we have that $gA = A$ and

$$\begin{aligned} gA &= \bigcap_{C \in \mathrm{MRS}(\tau)} gC \\ &= \bigcap_{C \in g\mathrm{MRS}(\tau)} C \\ &= \bigcap_{C \in \mathrm{MRS}(\tau)} C \\ &= A. \end{aligned}$$

Dehn twists are examples of reducible mapping classes, whose canonical reduction system is the curve around which the surface is twisted. We make use of canonical reduction systems in the proof of the main theorem.

2.1.4. *The capping and inclusion homomorphisms.* Some obvious questions about the mapping class group arise when we consider surfaces that are made from other surfaces, either by "capping off" boundary components with (potentially punctured) disks, or by including one surface as a subsurface of another. Each of these induces a homomorphism of mapping class groups, and we call these the *capping* and *including* homomorphisms. For precise statements of these homomorphisms and proofs, see [1, 3.6].

Given some surface S with boundary $\partial S = \{\gamma_1, ..., \gamma_m\}$, we may want to construct a surface without boundary by capping the boundary components with a once-punctured disk (replacing the boundary with punctures) or an unpunctured disk (removing the boundary altogether). In either case, we write $\mathcal{C}ap(S)$ for such a capped surface (with some abiguitiy, however in later portions of this paper the kind of capping being used will be stated clearly). We will denote the homomorphism induced by the capping map by $\mathcal{C}ap$. In the former case, for S a punctured surface with punctures $p_1, ..., p_k$ with boundary curve γ capped by a disk with puncture p_0, we have that the following sequence is exact:

$$1 \to \langle T_\gamma \rangle \to \mathrm{Mod}(S, \{p_1, ..., p_k\}) \xrightarrow{\mathcal{C}ap} \mathrm{Mod}(\mathcal{C}ap(S), \{p_0, p_1, ..., p_k\}) \to 1.$$

In the latter case, the kernel is slightly more badly behaved, and is isomorphic to the fundamental group of the unit tangent bundle of $\mathcal{C}ap(S)$. That is, the following sequence is exact:

$$1 \to \pi_1(T^1(\mathcal{C}ap(S))) \to \mathrm{Mod}(S, \{p_1, ..., p_k\}) \xrightarrow{\mathcal{C}ap} \mathrm{Mod}(\mathcal{C}ap(S), \{p_1, ..., p_k\}) \to 1.$$

This case occurs in the proof of the main theorem.

The analogous homomorphism induced by the inclusion of a surface Σ_1 into another Σ_2 is also phrased in terms of the boundary of Σ_1. Specifically, whenever two boundary components β_1, β_2 of Σ_1 in Σ_2 are homotopic, i.e. bound an annulus, the product $T_{\beta_1} T_{\beta_2}^{-1}$ is trivial in $\mathrm{Mod}(\Sigma_2)$, and whenever a boundary component of Σ_1 is null-homotopic in Σ_2, i.e. bounds a disk, the Dehn twist around it is trivial. It turns out that these entirely compose the kernel, so that we have $\mathrm{Mod}(\Sigma_1, \partial\Sigma_1)/\langle T_{\gamma_i}, T_{\beta_i} T_{\beta_i}^{-1} \rangle \cong \mathrm{Mod}(\Sigma_2, \partial\Sigma_2)$.

2.2. **Right-angled Artin groups.** *Right-angled Artin groups* (or RAAGs) are the second principal object of our focus.

Right-angled Artin groups, and in particular right-angled Artin subgroups of mapping class groups, are widely researched groups in geometric group theory. To a finite simplicial graph G with vertex set $V(G)$ and edge set $E(G)$, we associate the group $A(G)$, known as the right-angled Artin group (or RAAG) associated to G, defined as

$$A(G) = \langle V(G) : [v_i, v_j] = 1 \text{ if and only if } \{v_i, v_j\} \in E(G)\rangle.$$

That is, the RAAG associated to G has generator set the vertices of G, which we call the *vertex generators* for $A(G)$, where two generators commute if and only if the corresponding vertices are adjacent, and these are the only relations. As a general reference, see [4].

Right-angled Artin subgroups of mapping class groups have been widely studied (see, e.g. [6], [7], [8], [9], [10]). In particular, Koberda-Kim have developed a system of analogies between graph-theoretic properties of RAAGs and group-theoretic properties of mapping class groups.

We will be primarily focused with the RAAG associated to the finite r-partite graph on 2 vertices, defined in [11] as $K_r(2)$. As was detailed above, $K_r(2)$ has interesting properties related to obstructing embeddings of right-angled Artin groups into mapping class groups. This follows from the group structure of $A(K_r(2))$: if we enumerate the vertex generators of $A(K_r(2))$ by $f_1, g_1, ..., f_r, g_r$, we see that $\langle f_i, g_i\rangle$ forms a free group, so that $A(K_r(2)) \cong F_2 \times \cdots \times F_2$, a direct product of free groups of order 2.

2.2.1. *The curve complex.* The curve graph $\mathcal{C}(S)$ associated to a (for our purposes, finite-type) surface S is a simplicial graph whose vertices are isotopy classes of essential simple closed curves in S, where vertices are adjacent if and only if the isotopy classes admit disjoint representatives. A curve in S is *essential* when it is not null-homotopic or homotopic to a boundary component or puncture. Throughout this paper, a *curve* on a surface will mean a vertex of the curve graph, i.e. the isotopy class of an essential simple closed curve in S.

The curve graph gives us one of the most powerful tools for embedding RAAGs into mapping class groups, since topological and geometric properties of the surface can often be greatly simplified by reducing them to graph-theoretical problems on $\mathcal{C}(S)$ (however that should note be thought of as making them any easier!). Our main references for curve graph theory will be Kim-Koberda, some of whose work (namely [7], [8], [9], [10]) was described above.

3. Centralizers of reducible mapping classes

We record in this section the general study of the centralizers of reducible elements, the proof of which is used in the proof of Theorem 1.1.

Given a surface $S = S_{g,p}$ of genus g and number of punctures p and a reducible mapping class $f \in \mathrm{Mod}(S)$, let A be the canonical reduction system associated to f. Then cutting along A, we obtain a (possibly disconnected) surface S', whose components we denote $S'_1, ..., S'_n$. Then the permutations of homeomorphic components represent periodic mapping classes in $\mathrm{Mod}(S', \partial S')$; call the group of such permutations G. Then G is the product of symmetric groups, as is detailed explicitly below. Given a mapping class in the centralizer of f in $\mathrm{Mod}(S)$, it induces a set of permutations of homeomorphic components, i.e. an element of G. This is clearly a surjective group homomorphism; we denote by $C^0_{\mathrm{Mod}(S)}(f)$ its kernel. Then (by definition) the following sequence is exact:

$$1 \longrightarrow C^0_{\mathrm{Mod}(S)}(f) \longrightarrow C_{\mathrm{Mod}(S)}(f) \longrightarrow G \longrightarrow 1.$$

Describing $C^0_{\mathrm{Mod}(S)}(f)$ is now the important step. For a mapping class $\varphi \in C^0_{\mathrm{Mod}(S)}$, we know that $\varphi(S'_i) = S'_i$ for every i. Then we may restrict φ to a subsurface to obtain an element $\varphi|_{S'_i} =: \varphi_i \in \mathrm{Mod}(S'_i, \partial S'_i)$. Each φ_i lifts to an element of $\mathrm{Mod}(S, \partial S)$ with support contained in $S'_i \subset S$. Since we assume that φ centralizes f, no two φ_i have overlapping supports. By disjointness, it is clear that in $\mathrm{Mod}(S, \partial S)$ we have $\varphi = \prod_i \varphi_i$.

In other words, to compute the meaningful subgroup of the centralizer of f, it suffices to look at the mapping class groups of the components of the cut surface S', where we only consider the elements that leave every component invariant, so that they lift to elements of $\mathrm{Mod}(S)$ that centralize f. We can decompose each lift into products of mapping classes on subsurfaces of the cut surface obtained by removing the canonical reduction system for f.

4. The proof of the main theorem

4.1. **Proof of Theorem 1.1.** Suppose there is as surface $S = S_{g,p}$ so that there is an embedding $i : F_2^{r(g,p)} \hookrightarrow \mathrm{Mod}(S)$, where $r(g,p) = g + \lfloor (g+p)/2 \rfloor$. Choose S so that $r = r(g,p)$ is minimal such that $F_2^r \hookrightarrow \mathrm{Mod}(S)$. We build an embedding $F_2^{r'} \hookrightarrow \mathrm{Mod}(S)$ for $r' < r$ as follows: write $\langle f_1, g_1, ..., f_r, g_r \rangle = F_2^r$. By the Nielsen-Thurston classification, each f_i, g_j is finite order in the mapping class group, reducible, or pseudo-Anosov. Since F_2^n has no torsion elements, no f_i, g_j is finite order in $\mathrm{Mod}(S)$. Any (fully-supported) pseudo-Anosov mapping class $[f] \in \mathrm{Mod}(S)$ with pseudo-Anosov representative f has virtually cyclic centralizer in $\mathrm{Mod}(S)$ (see [2]), however if some $f_i \in i(F_2^r)$ has virtually cyclic centralizer, then $\langle f_2, g_2, ..., f_r, g_r \rangle \cong F_2^{r-1} \leq C_{\mathrm{Mod}(S)}(f)$ has rank 0, i.e. $r = 1 = g + \lfloor (g+p)/2 \rfloor$, so $g = 1$ with $p = 0$ or $g = 0$ with $p \in \{2, 3\}$, in which cases see the low-complexity cases below. We conclude then that no finite-order or pseudo-Anosov elements are present in $i(F_2^r) \leq \mathrm{Mod}(S)$.

Additionally, we can observe that the image of i does not lie entirely in (powers of) Dehn twists. This follows from [11, Lemma 30], since in this case we have a graph embedding $K_r(2) \to \mathcal{C}(S)$, which directly contradicts the cited result for r as given.

Thus, let f_1 be reducible and not a product of powers of Dehn twists, and let $A_1 = \{\alpha_1, ..., \alpha_n\}$ be the canonical reduction system associated to f_1. Write $\Gamma := F_2^{r-1} = \langle f_2, g_2, ..., f_r, g_r \rangle$. We have $\Gamma \leq C_{\mathrm{Mod}(S)}(f_1)$. We aim to examine the (potentially disconnected) surface with boundary $(S', \partial S')$ obtained by removing an open regular neighborhood of A from S, whose connected components we denote $S'_1, ..., S'_k$. We have the following exact sequence describing the relationship between the respective mapping class groups

$$1 \longrightarrow C^0_{\mathrm{Mod}(S)}(f_1) \longrightarrow C_{\mathrm{Mod}(S)}(f_1) \longrightarrow \prod_{1 \leq j \leq \ell} \mathrm{Perm}(n_j) \longrightarrow 1$$

where ℓ is the number of homeomorphism types among the S'_i, and n_j is the number of components in the jth homeomorphism class. $C^0_{\mathrm{Mod}(S)}(f_1)$ is defined to be the kernel of the homomorphism $C_{\mathrm{Mod}(S)}(f_1) \to \prod_{1 \leq j \leq \ell} \mathrm{Perm}(n_j)$, which is clearly surjective. We record here that the finiteness of the product group $\prod_{1\leq j\leq \ell}\mathrm{Perm}(n_j)$ shows that $C^0_{\mathrm{Mod}(S)}(f)$ sits in $C_{\mathrm{Mod}(S)}(f_1)$ with finite index. Denote the composition $F:\Gamma\to\prod_{1\leq j\leq\ell}\mathrm{Perm}(n_j)$ and let $\Gamma_0:=\ker(F)$ be the subgroup of mapping classes that do not permute homeomorphic components. We show in Lemma 4.1 that, after passing to a subgroup, we may assume that $\Gamma_0\cong\Gamma\cong F_2^{r-1}$ is a direct product of free groups.

For $1\leq i\leq k$ we define the projection map $p_i':C^0_{\mathrm{Mod}(S)}(f_1)\to\mathrm{Mod}(S_i',\partial S_i')$ to be projection onto the ith component. We also define the homomorphism $\mathcal{C}ap_i:\mathrm{Mod}(S_i',\partial S_i')\to\mathrm{Mod}(\mathcal{C}ap(S_i'))$ induced by the inclusion $S_i'\to\mathcal{C}ap(S_i')$ where boundary components are capped off by disks (see [1, Section 3.6]). Note that

$$g(S_i')+\lfloor g(S_i')+p(S_i')/2\rfloor=g(\mathcal{C}ap(S_i'))+\lfloor(g(\mathcal{C}ap(S_i'))+p(\mathcal{C}ap(S_i')))/2\rfloor.$$

Denote the composition $p_i=\mathcal{C}ap_i\circ p_i'$. We now have a well-defined homomorphism $\Phi:\Gamma_0\to\prod_{i=1}^k\mathrm{Mod}(\mathcal{C}ap(S_i'))$ that takes $\phi\mapsto(p_1(\phi),...,p_k(\phi))$. When the cut surface obtained from S by removing the canonical reduction system for f_1 is connected, the kernel of Φ is isomorphic to the fundamental group of the unit tangent bundle of the capped surface (see [1, Section 4.2]). In this case there is a well-known exact sequence of groups

$$1\longrightarrow\mathbb{Z}\longrightarrow\pi_1(T^1\Sigma)\longrightarrow\pi_1(\Sigma)\longrightarrow 1$$

where Σ is the surface whose components are the $\mathcal{C}ap(S_i')$. This is not in general the case, but in Lemma (???) we show that we may pass to appropriate subgroups in order to ...

Lemma 2 below shows that Γ_0 does not meet any copy of $\mathbb{Z}$, so that the Γ_0 is a subgroup of a surface group, i.e. it is free or a surface group. Lemma 4 below shows that the intersection $\Gamma_0\cap\ker(f)$ is free and contained in some free factor of Γ_0. Without loss of generality, assume $\ker(f)\leq H_{r-1}$, where $\Gamma_0=H_1\times\cdots\times H_{r-1}$ with each $H_i\cong F_2$. Now we have an inclusion $F_2^{r-2}\hookrightarrow\prod_i\mathrm{Mod}(\mathcal{C}ap(S_i'))$.

The most involved of the group-theoretic lemmas below shows that, perhaps after passing to free subgroups, in this arrangement we in fact have embeddings $F_2^{r_k}\hookrightarrow\mathrm{Mod}(\mathcal{C}ap(S_i'))$, where the exponents obey $\sum_k r_k=r$. We devote a subsequent section of the document to the proof. That given, consider the restriction maps $\mathrm{res}_i:F_2^{r-2}\hookrightarrow\mathrm{Mod}(\mathcal{C}ap(S_i'))$. If we show that for some index k that $r_k\geq g(\mathcal{C}ap(S_i'))+\lfloor\frac{g(\mathcal{C}ap(S_i'))}{2}\rfloor$, then the proof is finished, since this contradicts the minimality of r. No r_k is greater than or equal to r, since the capped surfaces $\mathcal{C}ap(S_i')$ have strictly fewer punctures than S (namely none) and smaller or the same genus.

So assume no such k exists, so that $r_k<g(\mathcal{C}ap(S_k'))+\lfloor\frac{g(\mathcal{C}ap(S_i'))}{2}\rfloor$ for each k. Recall that the genus of S is g and the number of punctures is p. We have

$$\begin{aligned}\sum_i r_i&<\sum_i g(\mathcal{C}ap(S_i'))+\left\lfloor\frac{g(\mathcal{C}ap(S_i'))}{2}\right\rfloor\\&=\sum_i g(\mathcal{C}ap(S_i'))+\sum_i\left\lfloor\frac{g(\mathcal{C}ap(S_i'))}{2}\right\rfloor\\&\leq g+\sum_i\left\lfloor\frac{g(\mathcal{C}ap(S_i'))+p}{2}\right\rfloor\end{aligned}$$

$$\begin{aligned}
&\leq g + \left\lfloor \sum_i \frac{g(\mathcal{C}ap(S_i') + p}{2} \right\rfloor \\
&< g + \frac{(\sum_i g_i) + p}{2} \\
&\leq g + \left\lfloor \frac{g+p}{2} \right\rfloor \\
&= r,
\end{aligned}$$

contradicting the lemma. So let k be such that $r_k \geq r' := g(\mathcal{C}ap(S_k')) + \lfloor \frac{g(\mathcal{C}ap(S_i'))}{2} \rfloor$. Then we have $F_2^{r'} \hookrightarrow \mathrm{Mod}(\mathcal{C}ap(S_i'))$ with $r' < r$, contradicting the minimality of r and completing the proof.

4.1.1. *Lemmas.*

Lemma 4.1. $\Gamma_0 = \ker(f)$ *contains a copy of* $\Gamma = F_2^{r-1}$.

Proof. We write $\Gamma = H_1 \times \cdots \times H_{r-1}$ where each $H_i \cong F_2$. Then $\Gamma_0 \trianglelefteq H_1 \times \cdots \times H_{r-1}$. Observe that each $\Gamma_0 \cap H_i \leq H_i$ is finite index, since if not the overgroup Γ_0 has infinite index in Γ. Then since subgroups of free groups are free, each $\Gamma_0 \cap H_i$ is free, so each contains a copy of F_2, so Γ_0 contains $(\Gamma_0 \cap H_1) \times \cdots \times (\Gamma_0 \cap H_{r-1}) \cong F_2^{r-1}$. ∎

Lemma 4.2. F_2 *contains no nontrivial normal cyclic subgroups.*

Proof. Let $X = \langle t \rangle \leq F_2$ be normal and fix $x \in F_2$. Then $xtx^{-1} = t^{\pm 1}$; in the first case we have that $\langle x, t \rangle \leq F_2$ is abelian, but all subgroups of F_2 are free. Then one of x, t is 1, but x was generic. In the second case we have that x^n centralizes t for all n, so that $\langle x, t \rangle \leq F_2$ is again not free. We conclude in both cases that $X = \{1\}$. ∎

Corollary. For $r > 0$, F_2^r contains no normal copy of $\mathbb{Z}^k$ for any $k > 0$.

Proof. Suppose we have $\mathbb{Z}^k \trianglelefteq F_2^r = \langle f_1, g_1, ..., f_r, g_r \rangle$. Then the subgroups $\mathbb{Z}^k \cap \langle f_i, g_i \rangle \trianglelefteq F_2$ give nontrivial normal cyclic subgroups of F_2 whenever $k > 0$, contradicting Lemma 2. ∎

Lemma 4.3. *If* $G \trianglelefteq F_2^d$ *is nontrivial and isomorphic to a subgroup of a surface group, then* G *is free and contained in one factor of* F_2^d.

In fact, all we will use is the fact that G does not contain any $\mathbb{Z}^2$, which follows from the corollary.

Proof. Clearly, if G is contained in some factor of F_2^d, then G is free. So suppose that there are distinct i and j so that $G \cap H_i, G \cap H_j \neq \emptyset$, and choose $h_1 \in G \cap H_i$ and $h_2 \in G \cap H_j$. Then h_1 and h_2 commute since they lie in different factors, so $\langle h_1, h_2 \rangle \cong \mathbb{Z}^2 \leq G$, a contradiction. Thus G must lie in only one factor, which implies that it is free. ∎

4.1.2. *A group-theoretic lemma of some significance.* In the statement of this lemma, we remove all reference to mapping class groups; the lemma is true for arbitrary groups G_i. The crux of the problem is that as nicely behaved as free groups are, their direct products behave quite badly. The subgroup structure of a direct product of free groups, for example, is very hard to understand. We lack the powerful tool, employed much above, that subgroups of the direct product share its structure (to say that, e.g., every subgroup of a direct product of free groups is itself a direct product of free groups is not the case), which is true for free groups.

Lemma 4.4. *For some $d, n > 0$, suppose that $F_2^d = H_1 \times \cdots \times H_d \hookrightarrow \prod_{i=1}^n G_i$, where each $H_i \cong F_2$. Then there exist subgroups $H_i' \cong F_2 \leq H_i$ such that $H_i' \hookrightarrow G_j$ for some j. Moreover, for each $1 \leq k \leq n$ there is r_k such that $F_2^{r_k} \hookrightarrow G_k$, with $\sum_k r_k = r$.*

Proof. We establish the first claim first. To show this, consider the composition of the restriction to a factor and projection onto a factor

$$H_i \hookrightarrow \prod_{i=1}^n G_i \twoheadrightarrow G_1.$$

If this is not injective, let its kernel be $K_1 \leq H_i$. Then since K_1 projects trivially onto the first factor G_1 but embeds into the product, we have an embedding $K_1 \hookrightarrow \prod_{i>1} G_i$ and may post-compose with the projection to obtain the same kind of map

$$K_1 \hookrightarrow \prod_{i>1} G_i \twoheadrightarrow G_2.$$

Now, inductively, assume that the homomorphisms $K_j \hookrightarrow \prod_{i>j} G_i \to G_{j+1}$ are as in the base case, i.e. with K_{m+1} the kernel of the homomorphism $K_m \hookrightarrow \prod_{i>m} G_i \to G_{m+1}$. If this process reaches G_n without terminating, the homomorphism $K_{n-1} \to G_n$ must be an embedding, otherwise we would obstruct the embedding $F_2^r \hookrightarrow \prod_{i=1}^n G_i$.

Then, we may assume that some K_r embeds into some factor G_s. K_r is a subgroup of $H_i \cong F_2$, so it is itself free. Passing to some subgroup, we may assume that $K_r \cong F_2$ as well. Replacing H_i with K_r and doing this for each i, we have embeddings of each factor H_i on the left into some factor G_j on the right.

For the second claim, we also proceed inductively. First, define the sets $T_j := \{j : H_i \hookrightarrow G_j\}$. Then consider any two-fold product $H_i \times H_j$ for $i, j \in T_1$. Then we have a map $H_i \times H_j \to G_1$. If this is not injective, denote its kernel $K_{i,j}^1$. Observe that $K_{i,j}^1$ is also free, by projecting onto a factor. Then since the subgroup generated by $(\prod_{i\neq j} H_i) \times K$ is their inner direct sum, we may replace H_j with a subgroup K' of K such that $K' \cong F_2$, with $\prod_{i\neq j} H_i \times K' \cong F_2^r$. Replacing H_j with K, we have $H_i \times H_j \hookrightarrow G_1$.

Now, inductively, assume that for some index set $i_1, ..., i_n \in T_1$, we have $H_{i_n} \times \cdots \times H_{i_n} \hookrightarrow G_1$. Then proceeding as in the base case, we consider $(H_{i_1} \times \cdots H_{i_n}) \times H_{i_{n+1}} \to G_1$. If this is not injective, let K be its kernel. Since we know that $H_{i_1} \times \cdots \times H_{i_n}$ embeds into G_1, we have $K \cap H_{i_1} \times \cdots \times H_{i_n}$ is trivial. Then $K \leq H_{i_{n+1}}$, so K is free. Passing to a suitable subgroup of K and replacing $H_{i_{n+1}}$ by K, we can assume that $H_{i_1} \times \cdots \times H_{i_n} \times H_{i_{n+1}}$ embeds into G_1. Proceeding this way for all G_j, T_j, we see that whenever the elements of a set of factors of F_2^r embed into G_j, so does their direct product. That is, we have $\prod_{i \in T_j} H_i = F_2^{|T_j|} \leq G_j$ for all j. Let $r_k := |T_k|$. Then $\sum_j r_k = \sum_j |T_j| = r$, as claimed. $\square$

Lemma 4.5 (Faux-Lemma)**.** *The embedding $F_2^{r-1} \hookrightarrow \prod \mathrm{Mod}(\mathcal{C}ap(\Sigma_i), *)$ where the capping is done by punctured disks, induces an embedding $F_2^{r-1} \hookrightarrow \prod \mathrm{Mod}(\mathcal{C}ap(\Sigma_i))$ where the capping is done by unpunctured disks. This amounts to avoiding point-pushing on subsurfaces.*

When the capping of the boundary components of the S_i is done by punctured disks, the kernel of the induced homomorphism $\prod_i \mathrm{Mod}(S_i', \partial S_i') \to \prod_i \mathrm{Mod}(\mathcal{C}ap(S_i'), *)$ is a direct product of copies of $\mathbb{Z}$, each generated by a Dehn twist around a boundary component. However, when the additional capping by unpunctured disks is applied, the kernel is the direct product of the $\pi_1(T^1 S_i')$, the fundamental groups of the unit tangent bundles of the capped connected subsurfaces. This is due to the Birman exact sequence: if $x_i \in \mathcal{C}ap(S_i')$, after capping by punctured disks, is a marked point signifying the puncture, then

$$1 \longrightarrow \pi_1(S_i', x_i) \longrightarrow \mathrm{Mod}(\mathcal{C}ap(S_i'), x_i) \longrightarrow \mathrm{Mod}(\mathcal{C}ap(S_i')) \longrightarrow 1$$

where the second capping is done by unmarked disks, is exact. The loops in the kernel $\pi_1(S_i', x_i)$ are obtained by "point-pushing" the x_i along non-trivial loops in S_i' based at x_i. See Farb-Margalit [1, Section 4.1].

When the cut surface obtained by removing (an open regular neighborhood of) the canonical reduction system for f_1 is disconnected, the kernel of Φ (as above) is a product of based fundamental groups of the cut surfaces, corresponding to point-pushing mapping classes.

Proof. Consider a subsurface $\mathcal{C}ap(\Sigma_i)$ and the induced embedding $F_2^{k_j} \hookrightarrow \mathrm{Mod}(\mathcal{C}ap(\Sigma_j), *)$. All cappings are assumed to be by punctured disks for the time being. If $k_j = 1$, don't do anything. Else, as above we may deduce that there are reducible elements among the generators for the embedded product of free groups. Cutting along the canonical reduction systems for these elements (which consist of curves in $\pi_1(\mathcal{C}ap(\Sigma_i))$), we may reduce to the $k_j = 1$ case for all subsurfaces. Now we may assume that $k_j = 1$ for all j.

For each subsurface, we apply the following operations. Note that if upon filling in the puncture (capping again, by unpunctured disks) and thus applying the $\mathcal{F}orget$ map on mapping class groups, a generator for the embedded copy of F_2 becomes trivial, then it was a point-pushing map in the sense of the Birman exact sequence. When this is the case, notate the point-pushing mapping class f. Then we replace f by a psuedo-Anosov whose support is the entire subsurface $\mathcal{C}ap(\Sigma_j)$ (where this capping is done by an unpunctured disk). Without loss of generality, we may assume that this mapping class does not commute with the twin generator for the embedded copy of F_2.

Now we may estimate as done above, i.e. we may assume that for each (newly reduced and replaced) $\mathcal{C}ap(\Sigma_k)$ that no corresponding r_k is such that $r_k < g + \lfloor \frac{g+p}{2} \rfloor$ and derive a contradiction with the same computation as above. We see that in the exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(T^1\Sigma) \longrightarrow \pi_1(\Sigma) \longrightarrow 1$$

used in the proof of the main theorem, that we are safe in the false assumption that Σ is a connected surface, since otherwise we may apply the above operations to arrive at a similar contradiction and close the proof. $\square$

4.1.3. *Low-complexity cases.* In certain low-complexity cases, for the purposes of this proof when g, p are relatively small, we need to rule out embeddings $F_2^r \to \mathrm{Mod}(S)$ by hand. In the case that occurs in the proof of the main theorem of $g + \lfloor (g+p)/2 \rfloor = 1$, we have either $g = 1$ with $p = 0$ or $g = 0$ with $p \in \{2, 3\}$. These correspond to the surfaces T^2 and the twice- and thrice-punctured spheres. Since $K_1(2)$ is a graph on two non-adjacent vertices, for these surfaces it is enough to give two essentially intersecting curves, so that their corresponding Dehn twists do not commute. These two mapping class will realize $K_1(2)$ in $\mathrm{Mod}(S)$. On the torus the standard generators of the fundamental group are two such curves, however the curve graphs of the genus 0 surfaces given do not have vertices (all curves are homotopic to punctures).

In fact, for a surface $S_{g,p}$ to even admit a finite-area complete hyperbolic metric, it must satisfy $2g + p > 2$ (the authors of [11] call such a surface as *hyperbolizable*). When this is the case, the above proof applies.

5. Conclusions and further research directions

The research surrounding this paper centered primarily on rigidity of the curve graph and the clique graph of the curve graph, following the results of Koberda and Kim on graph embeddings in $\mathcal{C}(S)$ vis a vis group embeddings in $\mathrm{Mod}(S)$. Some questions that we posed that went unanswered or did not appear in the research leading up to this document are stated here.

1) The work of Aramayona and Leininger (and additionally Hernandez) has shown that the curve graph is exhausted by rigid sets. A subgraph $\hat{G}$ of the curve graph is called *rigid* when for any embedded copy $G \leq \mathcal{C}(S)$ there is an automorphism ρ, that is, mapping class of S, such that $\rho(G) = \hat{G}$. To say that the curve graph is exhausted by rigid sets means that there is a nested sequence $G_0 \leq G_1 \leq \cdots$ of subgraphs of the curve graph such that $\bigcup_i G_i = \mathcal{C}(S)$ with each G_j rigid.

   It is a fact that the curve graph is quasi-isometric to the clique graph of the curve graph. Using these two facts together, is it possible to obstruct embeddings of RAAGs into $\mathrm{Mod}(S)$ by exploiting rigidity in $\mathcal{C}(S)$? That is:
   i) The curve graph embeds into its clique graph. Are rigid sets in the curve graph also rigid in the clique graph? Here you would need to show that the automorphism group of the clique graph is also the mapping class group.
   ii) Which sets are rigid in the clique graph? Is there an exhaustion as there is in the curve graph by rigid sets?
   iii) Can rigidity be exploited to obstruct a family of graph embeddings into $\mathcal{C}(S)_k$, and therefore RAAG embeddings into $\mathrm{Mod}(S)$?
   iv) Can quasi-isometric relations between the curve and clique-curve graphs be leveraged to answer any of the above questions?
2) More generally: what other obstructions exist to embeddings of RAAGs into mapping class groups? As detailed above, there are well-known global

and local obstructions which do not suffice to totally obstruct embeddings. [11] introduces a third, which is also insufficient. Are there non-local and non-global obstructions, at either the graph- or group-theoretic levels?

There are, to be sure, many more questions than these to be asked about these topics, and much still to learn.

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