The Hadwiger Conjecture for Unconditional Convex Bodies

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September 5, 2016

Abstract

We investigate the famous Hadwiger Conjecture, focusing on unconditional convex bodies. We establish some facts about covering with homotethic copies for L^p balls, and prove a property of outer normals to unconditional bodies. We also use a projection method to prove a relation between the bounded and unbounded versions of the conjecture.

1 Definitions and Background

Definition 1. *A convex body (closed and compact) in $\mathbb{R}^n$ is called unconditional if it is symmetric with respect to the coordinate hyperplanes.*

Definition 2. *For convex bodies K and T, let $N(K,T)$ be the minimum number of translates of T needed to cover K.*

Conjecture 1 (Hadwiger)**.** *For K a convex body in $\mathbb{R}^n$, $N(K, int(K)) \leq 2^n$.*

2 L^p balls

2.1 $2n$ translates

For B_p^n the unit ball in the L^p norm, we first show that $N(B_p^n, (1-\frac{1}{n})^{\frac{1}{p}} B_p^n) \leq 2n$, i.e. we can achieve a homothety ratio of $(1-\frac{1}{n})^{\frac{1}{p}}$ with just $2n$ translates. We start by determining the covering number $i(B_1^n) =$

$N(B_1^n, int(B_1^n))$ of the cross-polytope, B_1^n. First, it is impossible to cover it with less than $2n$ translates. If this was possible, we'd have two vertices (points of the form $(0,\dots,\pm 1,\dots,0)$), say v_1 and v_2 satisfying

$$\|v_1 - y\|_1 < 1, \|v_2 - y\|_1 < 1$$

for some $y \in \mathbb{R}^n$. But we have

$$\|v_1 - y\| + \|v_2 - y\| \geq \|(v_1 - y) + (y - v_2)\| = \|v_1 - v_2\| = 2,$$

so we need at least $2n$ translates. Let's suppose that translates of the form $\pm\lambda e_i$ work, where e_i are the standard basis vectors and $\lambda \in (0,1]$. We need that $\forall x \in \partial B_1^n \ \exists\ 1 \leq i \leq n$ s.t. $\|x - \lambda e_i\| < 1$. Since $\|x\| = 1\ \exists\ i$ s.t. $|x_i| \geq \frac{1}{n}$. Let $\lambda = \frac{1}{n}$, then the translate $\pm\frac{1}{n}e_i$ works, where the sign depends on that of x_i. Then

$$\|x - \frac{1}{n}e_i\| = |x_i - \frac{1}{n}| + 1 - |x_i| = 1 - \frac{1}{n} < 1 \text{ if } x_i > 0$$

and

$$\|x + \frac{1}{n}e_i\| = -x_i - \frac{1}{n} + 1 + x_i = 1 - \frac{1}{n} < 1 \text{ if } x_i < 0$$

Thus, $i(B_1^n) = 2n$, and, in particular, the homothety ratio $1 - \frac{1}{n}$ still works with $2n$ translates. For general p we may do a similar procedure. Assume translates of the form $\pm\lambda e_i$ work. We need that $\forall x \in \partial B_p^n \ \exists\ 1 \leq i \leq n$ s.t. $\|x - \lambda e_i\|_p < 1$. Since $\|x\|_p = 1$, $\exists\ i$ s.t. $|x| \geq n^{-\frac{1}{p}}$, so let us set $\lambda = n^{-\frac{1}{p}}$. Assuming wlog that $x_i > 0$, i.e. $x_i \in [n^{-\frac{1}{p}}, 1]$, we have

$$\|x - n^{-\frac{1}{p}}\|_p^p = |x_i - n^{-\frac{1}{p}}|^p + 1 - |x_i|^p = (x_i - n^{-\frac{1}{p}})^p + 1 - x_i^p.$$

Differentiating the above expression with respect to x_i we get

$$p(x_i - n^{-\frac{1}{p}})^{p-1} - px_i^{p-1}.$$

But $x_i > x_i - n^{-\frac{1}{p}}$, so the norm is strictly decreasing on $[n^{-\frac{1}{p}}, 1]$. Thus,

$$\|x - n^{-\frac{1}{p}}\|_p^p \leq 1 - n^{-p\frac{1}{p}} = 1 - \frac{1}{n} < 1$$

In particular, the homothety ratio $(1 - \frac{1}{n})^{\frac{1}{p}}$ works with $2n$ translates, so $N(B_p^n, (1 - \frac{1}{n})^{\frac{1}{p}} B_p^n) \leq 2n$.

2.2 2^n translates

The above approach uses translates that work for one extreme of the L^p balls ($p = 1$). We can also use the translates that work for the other extreme: L_∞, i.e. the cube. We know that translates in the directions of the diagonals give the covering number 2^n. Since the L^p balls are unconditional bodies, we may focus on just one quadrant, so let's assume wlog that all coordinates are non-negative. Assume a translate of the form $\boldsymbol{\epsilon} = (\epsilon, \ldots, \epsilon)$ covers the entire first quadrant of B_p^n, for some $\epsilon > 0$. To find the best possible epsilon (corresponding to the smallest possible homothety ratio), we must find the following:

$$\max_{x:x_i \geq 0\ \forall i} f(x) \text{ subject to g(x)=0 , where}$$
$$f(x) = \|(x_1, \ldots, x_n) - \boldsymbol{\epsilon}\|_p^p \text{ and}$$
$$g(x) = \|(x_1, \ldots, x_n)\|_p^p - 1.$$

We solve this problem with the method of Lagrange multipliers, first considering the boundary of B_p^n itself ($x_i > 0\ \forall i$), and then considering the boundary of the quadrant separately.

$$\nabla f = \mu \nabla g \text{ for some constant } \mu$$
$$(p(x_1 - \epsilon)^{p-1}, \ldots, p(x_n - \epsilon)^{p-1}) = \mu(px_1^{p-1}, \ldots, px_n^{p-1}),$$

where we have assumed that $x_i \geq \epsilon\ \forall i$ (the final result is the same for the case $0 < x_i < \epsilon$).We get that $(x_i - \epsilon)^{p-1} = \mu x_i^{p-1}$, or $x_i = \frac{\epsilon}{1-\mu^{\frac{1}{p-1}}}$ (assuming $p > 1$). We have that $\|x\|_p^p = 1 \Rightarrow n(\frac{\epsilon}{1-\mu^{\frac{1}{p-1}}})^p = 1 \Rightarrow \mu = (1 - \epsilon n^{\frac{1}{p}})^{p-1} \Rightarrow x_i = n^{-\frac{1}{p}}$. Then the corresponding distance (homothety ratio) is

$$\|(n^{-\frac{1}{p}}, \ldots, n^{-\frac{1}{p}}) - \boldsymbol{\epsilon}\|_p = n^{\frac{1}{p}}(n^{-\frac{1}{p}} - \epsilon) = 1 - \epsilon n^{\frac{1}{p}}$$

Now we examine the boundary of the quadrant. Suppose $x_i = 0$ for some i. Then $f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = \epsilon^p + \sum_{j \neq i}(x_j - \epsilon)^p$. We must now maximize the sum $\sum_{j \neq i}(x_j - \epsilon)^p$ where $\sum_{j\ neqi}(x_j)^p = 1$, but this is just the above Lagrange multiplier problem in $n-1$ dimensions, so the corresponding critical value is $f((n-1)^{-\frac{1}{p-1}}, \ldots, 0, \ldots, (n-1)^{-\frac{1}{p-1}}) = \epsilon^p + (1 - \epsilon(n-1)^{\frac{1}{p}})^p$. Notice that $n - 1 < n \Rightarrow \epsilon(n-1)^{\frac{1}{p}} < \epsilon n^{\frac{1}{p}} \Rightarrow (1 - \epsilon(n-1)^{\frac{1}{p}})^p > (1 - \epsilon n^{\frac{1}{p}})^p \Rightarrow$

$$\|((n-1)^{-\frac{1}{p-1}}, \ldots, 0, \ldots, (n-1)^{-\frac{1}{p-1}}) - \boldsymbol{\epsilon})\|_p > \|(n^{-\frac{1}{p}}, \ldots, n^{-\frac{1}{p}}) - \boldsymbol{\epsilon}\|_p.$$

Since the right hand side corresponds to the only critical value given to us by the original Lagrange problem, we know that the absolute maximum must occur one of the coordinate hyperplanes. We may continue to inductively assume that a greater distance occurs at a point where some (new) $x_j = 0$, eventually getting that the absolute maximum must occur at one of the basis vectors e_i. Let's assume wlog that it occurs at e_1. Then the absolute maximum of f in the first quadrant is

$$f(1, 0, \dots, 0) = (n-1)\epsilon^p + (1-\epsilon)^p$$

We need the right hand side to satisfy $(n-1)\epsilon^p + (1-\epsilon)^p < 1$. Notice that for $p \geq 2$, $(n-1)\epsilon^p + (1-\epsilon)^p \leq (n-1)\epsilon^2 + (1-\epsilon)^2 = n\epsilon^2 - 2\epsilon + 1$. Trying $\epsilon = \frac{1}{n}$ (as a guess), we get that

$$\|(x_1, \dots, x_n) - \boldsymbol{\epsilon}\|_p \leq (1 - \frac{1}{n})^{\frac{1}{p}}.$$

In particular, the homothety ratio $(1-\frac{1}{n})^{\frac{1}{p}}$ works with 2^n translates for $p \geq 2$, so $N(B_p^n, (1-\frac{1}{n})^{\frac{1}{p}} B_p^n) \leq 2^n$. With the above (non-optimal) guess, we have gotten the same homothety ratio, but with more translates.

2.3 Upper-Semi Continuity of $N(K, \lambda K)$

We have the following, due to [1]:

Theorem 1. *Let L and K be convex bodies such that $d_{BM}(L, K) < 1 + \frac{\epsilon}{\lambda}$ and $N(K, \lambda K) = A$. Then $N(L, (\lambda + \epsilon)L) \leq A$.*

We may use this theorem to find a homothety ratio for large p. For $p \leq q$, we have that $n^{\frac{1}{q}-\frac{1}{p}}\|\cdot\|_p \leq \|\cdot\|_q \leq \|\cdot\|_p$, i.e. $d(B_p^n, B_q^n) \leq n^{\frac{1}{p}-\frac{1}{q}}$. Thus $d(B_p^n, B_\infty^n) \leq n^{\frac{1}{p}}$. We also have that $N(B_\infty^n, \frac{1}{2}B_\infty^n) = 2^n$ (this is the best possible homothety ratio for 2^n translates). Then, if we want $N(B_p^n, \lambda B_p^n) \leq 2^n$, it suffices to have $n^{\frac{1}{p}} < 1 + 2(\lambda - \frac{1}{2}) = 2\lambda$, i.e. $p > \frac{\log n}{\log 2\lambda}$. For example, we may try to reproduce the above results by taking $\lambda = (1 - \frac{1}{n})^{\frac{1}{p}}$. Then $p > \frac{\log n}{\log 2 + \frac{1}{p}\log(1-\frac{1}{n})} \Rightarrow p > \frac{\log \frac{n^2}{n-1}}{\log 2}$. Figure 1 shows how this expression grows with n.

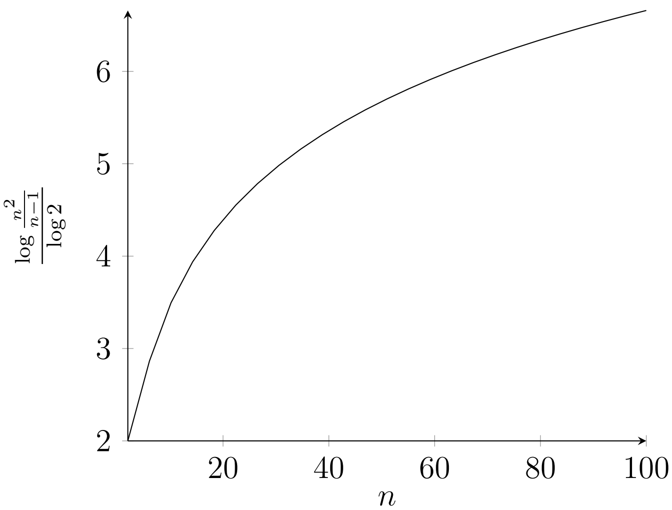


Figure 1: For a fixed n, it is enough to choose any p greater than the value of the above graph at n to guarantee that 2^n translates work for a homothety ratio of $(1-\frac{1}{n})^{\frac{1}{p}}$

3 Outer Normals

Suppose K is an unconditional body. For $x \in \partial K$, v an outer normal at x means $(v,y) \leq (v,x) \; \forall \; y \in K$. If K is strictly convex, then $(v, y-x) < 0$ for $y \neq x$ in K. By unconditionality, for $1 \leq i \leq n$, the point $x^i = (x_1, x_2, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n) \in K$. $(v, x^i - x) = (v, -2x_i e_i) = -2x_i v_i < 0 \Rightarrow x_i v_i > 0$ for $x_i \neq 0 \Rightarrow x_i \neq 0 \to v_i \neq 0 \Rightarrow v_i = 0 \to x_i = 0$. Now, suppose K is smooth, so there is only one outer normal v at each $x \in \partial K$. We claim that then $x_i = 0 \to v_i = 0$. Suppose not, so $\exists\; x, v, i$ such that $x_i = 0$ but $v_i \neq 0$. $\forall y \in K, (v,y) \leq (v,x) = (\mathrm{Proj}_{e_i^\perp}(v), x)$ since $x_i = 0$. But $(\mathrm{Proj}_{e_i^\perp}(v), y) = (v, \mathrm{Proj}_{e_i^\perp}(y)) \leq (v,x)$. Therefore, $\mathrm{Proj}_{e_i^\perp}(v)$ is also an outer normal to K at x, which contradicts K being smooth. Thus, we have proved the following:

Theorem 2. *For K unconditional, smooth, and strictly convex, for $x \in \partial K$ and v the outer normal to K at x, $x_i = 0 \leftrightarrow v_i = 0$.*

4 Projections

4.1 Bounded Bodies

Given a (bounded) convex body K, we may assume that $0 \in K$ (doesn't change covering number). Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}$ (where we think of $\mathbb{R}^n$ as the orthogonal direct sum $\mathbb{R}^1 \oplus \mathbb{R}^{n-1}$ and project parallel to $\mathbb{R}^1$). We hope to achieve that if N translates of $int(\pi(K))$ are enough to cover $\pi(K)$, then at most 2 translates of $int(K)$ per translate of $\pi(K)$ are needed to cover K, i.e. the covering number of K is at most $2N$. Continuing this argument inductively (we may keep projecting down to the dimension below), we eventually reach R^1. This final projection must be a bounded segment, which has a covering number of 2, so we get that $N(K, int(K)) \leq 2^n$. Specifically, we conjecture that if $y \in \partial\pi(K)$ satisfies $y \in x + int(\pi(K))$ for some translate $x \in R^{n-1}$, then $\forall z$ s.t. $\pi(z) = y$, $z \in x \pm \epsilon e_1 + int(K)$ for some $\epsilon > 0$, where $e_1 = (1, 0, \ldots, 0)$

Conjecture 2. *The above conjecture works for unconditional bodies.*

Theorem 3 (Originally formulated by Boltyanski and Soltan, see [2])**.** *Suppose $K \in \mathbb{R}^n$ is closed and bounded. Suppose there are no lines parallel to $\mathbb{R}^1$ intersecting K at only one point (perhaps we can achieve this by rotating K). Then:*

$$i(K) \leq 2i(\pi(K))$$

Proof. See Appendix; slight modification of the proof found in [2]. □

4.2 Equivalence of Bounded and Unbounded Conjectures

Suppose $K \in \mathbb{R}^n$ is a closed convex body. Let $c(K) = N(K, int(K))$, $h(K) = \min_{N \in \mathbb{N}} \{n : K \in \bigcup_{i=0}^{N} (x_i + \lambda_i K)$ where $x_i \in \mathbb{R}^n, \lambda_i \in (0, 1)\ \forall i\}$. Clearly, $c(K) \leq h(K)$, and by compactness they are equal in the case that K is bounded. We attempt to establish the equivalence of the following two conjectures:

Conjecture 3 (Hadwiger)**.** *For K bounded, $c(K) = h(K) \leq 2^n$.*

Conjecture 4. *For K unbounded, if $c(K)$ is finite, then $c(K) \leq h(K) \leq 2^{n-1}$.*

Proposition 1. *Conjecture 4 $\rightarrow$ Conjecture 3.*

Proof. Suppose $K \in \mathbb{R}^n$ is bounded. Consider the (unbounded) cylinder $\tilde{K} := K \times [0, \infty] \in \mathbb{R}^{n+1}$. First, we claim that $h(\tilde{K}) \leq h(K)$, in particular that $h(\tilde{K})$ is finite. For suppose that $K \in \bigcup_{i=0}^{n}(x_i + \lambda_i K)$, then clearly $\tilde{K} \in \bigcup_{i=0}^{n}(x_i - \epsilon e_{n+1} + \lambda_i \tilde{K})\ \forall\ \epsilon > 0$, where e_{n+1} is the $(n+1)^{\text{st}}$ standard basis vector. This is because $\partial\tilde{K}$ projects exactly down to ∂K, except for the bottom face of the cylinder. Thus, if $x \in \partial K$ satisfies $x \in x_i + \lambda_i K$, then $\forall\ k > 0$, $x + ke_n \in x_i + \lambda_i \tilde{K}$. To cover the bottom face it suffices to shift all of our translates down any amount. Now, by our assumption $h(\tilde{K}) \leq 2^{(n+1)-1} = 2^n$. Thus, for some $\tilde{x}_i$ and λ_i, $\tilde{K} \in \bigcup_{i=0}^{2^n}(\tilde{x}_i + \lambda_i \tilde{K})$. Each $\tilde{x}_i$ is of the form $\tilde{x}_i = x_i \pm \epsilon_i e_{n+1}$ for $x_i \in \mathbb{R}^n$. Any point $x \in \partial K$ is also in $\partial\tilde{K}$, with last coordinate 0. Clearly, if $x \in \tilde{x}_i + \lambda_i \tilde{K}$, then $x \in x_i + \lambda_i K$. Thus, $K \in \bigcup_{i=0}^{2^n}(x_i + \lambda_i K)$, so $h(K) \leq 2^n$. $\square$

We conjecture that the reverse of the above proposition holds as well.

5 Acknowledgements

I would like to thank Boaz Slomka and Beatrice-Helen Vritsiou for advising me during the summer, and the NSF and the University of Michigan for the opportunity.

References

[1] M. Naszodi. *On the Constant of Homothety for Covering a Convex Set with its Smaller Copies (1991).*

[2] V. Boltyanski and P. Soltan. Combinatorial Geometry of Various Classes of Convex Sets (1978).

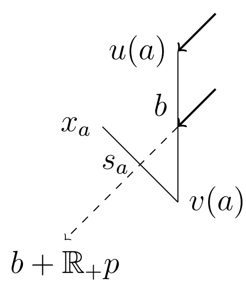
A Proof of Theorem 3

Proof. Let $p_1, \dots, p_m$ be directions in $\mathbb{R}^{n-1}$ illuminating $\pi(K)$. Let e_1 be the standard basis in $\mathbb{R}^1$. We claim that for big enough $\lambda > 0$, the directions

$$p_1 + \lambda e_1, \dots, p_m + \lambda e_1, p_1 - \lambda e_1, \dots, p_m - \lambda e_1$$

illuminate K.

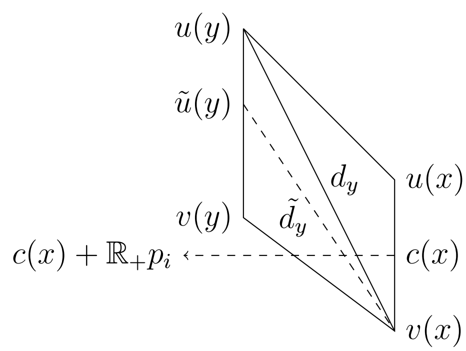
By our assumption, for $a \in K$ the line $l_a \parallel \mathbb{R}^1$ intersects K along a segment, call it $I(a)$. Let $u(a)$ be the "top" of this segment, and $v(a)$ the "bottom" (where x is "above" y if $x - y = ke_1$ for $k > 0$). Let $U = \{u(a) : a \in K\}$, $V = \{v(a) : a \in K\}$. For $a \in \partial K \setminus (U \cup V)$, if a direction $p \in \mathbb{R}^n$ illuminates $u(a)$, then it illuminates $I(a) \setminus \{v(a)\}$. Indeed, the illuminating condition guarantees that $x_a = u(a) + \lambda p \in int(K)$ for some $\lambda > 0$. Thus by convexity the segment s_a between x_a and $v(a)$ is in K, with every point except $v(a)$ an interior point. For $b \in I(a) \setminus \{v(a)\}$, the half-line $b + \mathbb{R}_+ p$ must intersect s_a at some point in $int(K)$. Thus, p illuminates b:



Similarly, if a direction illuminates $v(a)$, then it illuminates $I(a) \setminus \{v(a)\}$. Thus, it is enough to show that $2m$ directions are needed to illuminate $U \cup V$.

We first illuminate the set $U^0 \cup V^0$, where $U^0 := U \cap \pi^{-1}(\partial(\pi(K))$ and $V^0 := V \cap \pi^{-1}(\partial(\pi(K))$. The directions $p_1, \dots, p_m$ illuminate $\pi(K)$. By compactness, we can find closed sets $F_1, \dots, F_m$ such that $F_1 \cup \dots \cup F_m = \partial\pi(K)$ and direction p_i illuminates F_i. For any $x \in F_i$, we may again define $u(x)$ and $v(x)$ as the top and bottom points of $I(x)$, respectively, where $I(x)$ is the intersection of $l_x \parallel \mathbb{R}^1$ with K. Let $U_i = \{u(x) : x \in F_i\}$, $V_i = \{v(x) : x \in F_i\}$. Then $U_1 \cup \dots \cup U_m \cup V_1 \cup \dots \cup V_m = U^0 \cup V^0$. Our goal is to illuminate all of $cl(U^0 \cup V^0)$, and use a compactness argument to argue that only a finite number of illumination vectors (in particular, $2m$) actually suffices.

Let us first illuminate the points $v(x)$, for $x \in \partial\pi(K)$. Since $x \in F_i$ is illuminated by p_i, there exists some $y \in int(\pi(K))$ such that $y = x + \lambda p_i$ for some $\lambda > 0$. By assumption, K contains the segment $I(y)$, and $y \in int(\pi(K))$ means that $I(y) \setminus \{v(y), u(y)\} \subset int(K)$. The quadrilateral with vertices $u(y), u(x), v(x), v(y)$ is in K by convexity. Wlog let's say the longer diagonal d_y is the segment between $u(y)$ and $v(x)$. Since $d(y)$ is the longer diagonal, $(u(y), e_1) > (c(x), e_1)$, where $c(x) = \frac{u(x)+v(x)}{2}$. Thus, there exists a point $\tilde{u}(y) = u(y) - \delta e_1 \in int(K)$, with $\tilde{(}u(y), e_1) > (c(x), e_1)$, for some $\delta > 0$. Let $\tilde{d_y}$ be the segment connecting $\tilde{u}(y)$ and $v(x)$. Every point of this segment except $v(x)$ is an interior point of K by convexity. Since the half line $c(x) + \mathbb{R}_+ p_i$ and $\tilde{d_y}$ are co-planar, they must intersect at a point $c(x) + \lambda_x p_i \in int(K)$ for some $\lambda_x > 0$:

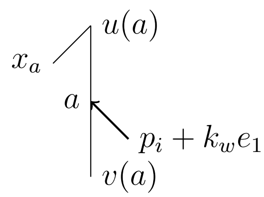


Convexity implies that for $0 < \lambda \leq \lambda_x$, $c(x) + \lambda p_i \in int(K)$. In particular, for $\lambda \leq \lambda_x$, $c(x) + \lambda p_i$ illuminates $c(x)$, which implies that $(c(x) - v(x)) + \lambda p_i$ illuminates $v(x)$. Equivalently, the vector $\frac{1}{\lambda}(c(x) - v(x)) + p_i$ illuminates $v(x)$ for $\lambda \geq \lambda_x$. $c(x) - v(x) = \mu e_1$ for some $\mu > 0$, so we have shown that for some $k_x > 0$, $p_i + k e_1$ illuminates $v(x)$ for $k \geq k_x$. We can similarly illuminate all points $u(x)$, obtaining that vectors of the form $p_i - j e_1$ illuminate $u(x)$ for $j \geq j_x > 0$.

Let us now illuminate points $w \in cl(U_0 \cup V_0) \setminus (U_0 \cup V_0)$. By definition, these points must be accumulation points of $U_0 \cup V_0$. We first claim that such points w must project to the boundary of $\pi(K)$, that is, for w an accumulation point, $\pi(w) \in \partial(\pi(K))$. Suppose, for a contradiction, that $\pi(w) \in int(\pi(K))$, so there exists an open ball $B_\epsilon(\pi(w))) \subset int(\pi(K)$, for some $\epsilon > 0$. By continuity of the projection function π, $\pi^{-1}(B_\epsilon(\pi(w))$ is an open neighborhood of w. Moreover, this neighborhood cannot contain points in $U_0 \cup V_0$, else $B_\epsilon(\pi(w))$ would contain a point in $\partial(\pi(K))$. But this means that w is an isolated point of $cl(U_0 \cup V_0)$, contradicting our assumption.

Thus, these accumulation points w can be one of two types. First, w may be equal to a point $u(x)$ or $v(x)$, for some $x \in \partial(\pi(K))$. In this case, we have already illuminated by the method above. Otherwise, $w \in I(x) \setminus \{u(x), v(x)\}$ for some $x \in \partial(\pi(K))$. But, by our first argument, the vectors illuminating $u(x)$ and $v(x)$ will then both illuminate w.

We have shown that for every point $w \in cl(U_0 \cup V_0)$, vectors of the form $p_i \pm ke_1$ illuminate w, for $k \geq k_w$, and $1 \leq i \leq m$. Since illuminating directions illuminate an open subset of ∂K, for each such w there exists a neighborhood $\tilde{A}_w \subset \partial K$ such that (wlog) a vector of the form $p_i + k_w e_1$ illuminates all of $\tilde{A}_w$. Now consider a point $a \in \tilde{A}_w$. If $a = v(x)$ for some $x \in \partial(K)$, we know that we may increase k_w without bound and stil illuminate a, i.e. $p_i + ke_1$ illuminates a for $k \geq k_w$. Else, a is in the interior of the segment $I(a)$. By the illuminating condition there exists a point $x_a \in intK$ such that $x_a = a + \lambda(p_i + k_w e_1)$, for some $\lambda > 0$. But then the segment between x_a and $u(a)$ is fully contained in K, with every point except (possibly) $u(a)$ an interior point. Thus, any vector of the form $p_i + ke_1$ for $k_w \leq k < \infty$ illuminates a (and all of $\tilde{A}_w$):

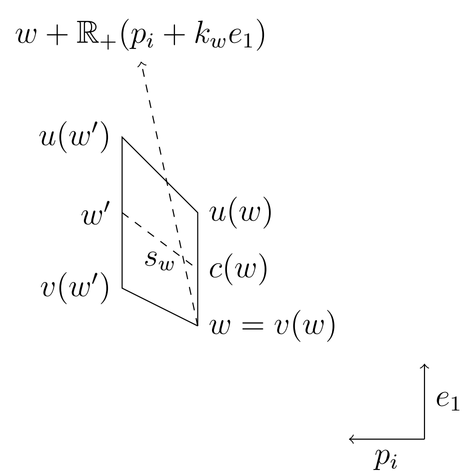


Thus, we know that for all $w \in cl(U_0 \cup V_0)$ there exists a neighborhood $\tilde{A}_w \in \partial K$ covered by vectors of the form $p_i \pm ke_1$ for $k \geq k_w$. The sets $A_w = \tilde{A}_w \cup cl(U_0 \cup V_0)$ are open in $cl(U_0 \cup V_0)$, and thus form an open cover. By compactness there exits a finite subcover $A_{w_1}, \ldots, A_{w_N}$. Let $\hat{k} = \max\limits_{1 \leq j \leq N} \{k_{w_j}\}$. Then the $2m$ vectors

$$p_1 + \hat{k}e_1, \ldots, p_m + \hat{k}e_1, p_1 - \hat{k}e_1, \ldots, p_m - \hat{k}e_1$$

illuminate $U_0 \cup V_0$.

It remains to illuminate $U_1 \cup V_1 = U \cup V \setminus cl(U_0 \cup V_0)$, that is, points $w \in (U \cup V)$ such that $\pi(w) \in int(\pi(K))$. We shall illuminate $cl(U_1 \cup V_1)$ and once again use a compactness argument. Consider a point $w \in cl(V_1)$. If $w \in cl(V_0)$, we have already illuminated w. Else, $\pi(w) \in int(\pi(K))$. There is a vertical segment $I(w) \in K$, with $I(w) \setminus \{u(w), v(w)\} \subset int(K)$, so the vector $+e_1$ illuminates w. Moreover, e_1 illuminates an open neighborhood $A_w \subset \partial K$ around w, where A_w may be chosen small enough so that $\pi(A_w) \subset int(\pi(K))$. ∂K has affine dimension $n-1$, so open sets in ∂K must also be of dimension $n-1$. Choose any vector p_i illuminating $\pi(K)$. We claim that the plane P_w^i containing w and parallel to $\text{span}\{p_i, e_i\}$ must contain another point $w' \in A_w$, where w' can be written as $w' = w + ap_i \pm be_i$, for $a, b > 0$. If $n = 2$, this is obvious, as then any plane P_w^i is all of $\mathbb{R}^2$. For $n \geq 3$, A_w and $\pi(A_w)$ must both be of dimension at least 2, so any "vertical" plane $P = \text{span}\{e_1, v\}$ for $v \in \mathbb{R}^{n-1}$ intersects A_w in a segment. The point $c(w) = \frac{u(w)+v(w)}{2} \in int(K)$, so the segment s_w connecting w' to $c(w)$ has all points except w in $int(K)$. But then for some $k_w > 0$, the half line $w + \mathbb{R}_+(p_i + k_w e_1)$ intersects $int(s_w)$, so for $k \geq k_w$, the vector $p_i + ke_1$ illuminates w:



We have now shown that for each $w \in V_1$ (similar argument for $w \in U \setminus U_0$), that vectors of the form $p_i \pm ke_1$ illuminate w, for $k \geq k_w$. By another compactness argument, we find that there exists a constant $\bar{k} > 0$ such that

$$p_1 + \bar{k}e_1, \ldots, p_m + \bar{k}e_1, p_1 - \bar{k}e_1, \ldots, p_m - \bar{k}e_1$$

illuminate $cl(U_1 \cup V_1)$. Finally, take $\lambda = \max\{\hat{k}, \bar{k}\}$, and then the vectors

$$p_1 + \lambda e_1, \ldots, p_m + \lambda e_1, p_1 - \lambda e_1, \ldots, p_m - \lambda e_1$$

illuminate ∂K, so the theorem is proved. $\square$