SYZYGIES OF THE SEGRE EMBEDDING VIA SIMPLICIAL COMPLEXES

LUKE KIERNAN AND RYAN CAPOUELLEZ
ADVISOR: ANDREW SNOWDEN

Abstract. We investigate the syzygies of the Segre embedding, by associating them with simplicial complexes. Using these complexes, we prove that the module of second syzygies is generated in degree 3, a familiar result from homology. We also prove a partial result about the module of third syzygies, and give explicit examples of nontrivial cycles in these complexes.

1. Background

1.1. The Segre Embedding.

Definition 1.1. Given $n, m \in \mathbb{N}$, the **Segre embedding** is the function $f_{n,m} : \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^m(\mathbb{C}) \to \mathbb{P}^{(n+1)(m+1)-1}(\mathbb{C})$ given by

$$([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) \mapsto [x_0 y_0 : \cdots : x_0 y_m : x_1 y_0 : \cdots : x_1 y_m : \ldots x_n y_m]$$

We can generalize to more than 2 projective spaces.

Definition 1.2. Given $n_1, \ldots n_k \in \mathbb{N}$, the **generalized Segre embedding** is the function $f_{n_1,\ldots n_k} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^s$ given by

$$([a_{1,i}]_{0 \le i \le n_1}, \ldots [a_{k,i}]_{0 \le i \le n_1}) \mapsto \left[\prod_{i=1}^{k} a_{i,j_i} \right]_{(0,\ldots 0) \le (j_1,\ldots j_k) \le (n_1,\ldots n_k)}$$

Here, s stands for $\prod_{i=1}^{k}(n_i+1) - 1$ and the $(j_1, \ldots j_k)$ are ordered lexographically.

Definition 1.3. Given $n_1, \ldots n_k \in \mathbb{N}$, let $x_{i,j} : \mathbb{C}^{n_i+1} \to \mathbb{C}$ denote the function

$$(a_0, a_1, \ldots a_n) \mapsto a_j$$

The i indicates which projective space the function $x_{i,j}$ corresponds to. If $n_1 = n_2 = 2$, $x_{1,1}$ and $x_{2,1}$ seem identical, but they operate on two different copies of $\mathbb{C}^3$.

Definition 1.4. The **coordinate ring** of the Segre embedding $f_{n_1,\ldots n_k}$ is the graded $\mathbb{C}$-module generated by $\{\prod_{i=1}^{k} x_{i,j_i}\}_{(0,\ldots 0) \le (j_1,\ldots j_k) \le (n_1,\ldots n_k)}$.

Equivalently, the coordinate ring is the sum of all monomials in the $x_{i,j}$, such the total degree in $\{x_{m,j}\}_{1 \le j \le n_m}$ is independent of m. (The grading is given by this degree: henceforth I will just use the word degree.) For example, if $n_1 = n_2 = 2$, the monomial $(x_{1,1}x_{2,2})(x_{1,2}x_{2,2}) = (x_{1,1}x_{1,2})(x_{2,2}^2)$ is in the coordinate ring, and of degree 2. $x_{1,1}x_{2,2}^2$ is not in the coordinate ring.

If $a < b$, then we can identify $\mathbb{P}^a$ with the subset of $\mathbb{P}^b$ such that the last $b - a$ homogeneous coordinates are zero. In a similar manner, if $(\forall i)(n_i \le m_i)$, then the coordinate ring of $f_{n_1,\ldots n_k}$ may be identified with a subset of the coordinate ring of $f_{m_1,\dots m_k}$. Letting $N = \max\{n_i | 1 \leq i \leq k\}$, the coordinate ring of the Segre embedding $f_{n_1,\dots n_k}$ is effectively a subset of the coordinate ring of $f_{N,\dots N}$, where there are k N's. Thus, it suffices to consider the case where $n_1 = n_2 = \cdots = n_k$. From now on, $C_{n,k}$ will denote the coordinate ring of $f_{n,\dots n}$, where there are k n's. $(C_{n,k})_d$ will denote the set of degree d elements of the coordinate ring.

Definition 1.5. Let $M_{n,k,d} \subset \mathrm{Mat}_{n\times k}(\mathbb{N}\cup\{0\})$ denote the set of all n by k matrices with non-negative entries, such that the sum of all entries in each column is d.

Theorem 1.6. *There is a natural bijection ϕ_d between monomials in $(C_{n,k})_d$ and $M_{n,k,d}$. Multiplication of monomials corresponds to addition of matrices: if $a \in (C_{n,k})_d$ and $b \in (C_{n,k})_f$, then $\phi_{d+f}(ab) = \phi_d(a) + \phi_f(b)$.*

Proof. For $(0,\dots 0) \leq (a_1,\dots a_k) \leq (n,\dots n)$, let

$$\phi_1\left(\prod_{i=1}^{n} x_{i,a_i}\right) = \begin{bmatrix} e_{a_1} & e_{a_2} & \dots & e_{a_k} \end{bmatrix}$$

Then define ϕ on higher degree elements so that $\phi_{d+f}(ab) = \phi_d(a) + \phi_f(b)$. The column sum condition on the matrices coincides with the equal degree condition on the elements of the graded ring. For the inverse, the (i,j) element of the matrix gives the exponent of $x_{j,i}$. □

1.2. **Syzygies and Minimal Free Resolutions.** The following definitions come from page 470 of [1] and page 3 of [2].

Definition 1.7. Given some graded R-module M, a **free resolution** of M is an exact chain complex of free graded R-modules

$$\mathcal{F} : \dots F_n \xrightarrow{\phi_n} \dots \xrightarrow{\phi_2} F_2 \xrightarrow{\phi_1} F_1 \xrightarrow{\phi_0} F_0$$

such that the ϕ_i are homogenous maps of degree 0 and $F_0/\mathrm{Im}(\phi_0) \cong M$. The module $\mathrm{Im}(\phi_i)$ is called the **module of ith syzygies** of M (or, sometimes, the **ith syzygy module** of M).

There is a natural bijection between the generators of $C_{n,k}$ and elements of $\{0,\dots n\}^k$: if $\lambda = (a_1,\dots a_k) \in \{0,\dots n\}^k$, let $x^\lambda = \prod_{i=1}^n x_{i,a_i}$ In contrast, let y_λ denote a formal symbol, and let $S = \mathbb{C}[y_\lambda]$ be the ring generated by the y_λ. Via the natural homomorphism from S to $C_{n,k}$, we may regard $C_{n,k}$ as an S-module. Also, $\phi : C_{n,k} \to S$ extends to a map $\widetilde{\phi} : S \to M_{n,k}$.

Example 2.6 of [2] provides a minimal free resolution of $\mathbb{C}$.

Let Δ be the simplicial complex with vertices labeled by the y_λ. For a vertex v, m_v will denote the corresponding label: for a face A, let $m_A = \prod_{v\in A} m_v$.

Let F_i be the free S-module whose basis is the set of faces of Δ having i vertices. Then there is graded complex of free S-modules

$$\mathscr{C}(\Delta) : \dots F_n \xrightarrow{\delta_n} F_{n-1} \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_0} F_0$$

The differential is given by $\delta_n A = \sum_{v\in A} (-1)^{\mathrm{pos}(n,A)}\, m_v\, (A\backslash\{v\})$, where pos(n, A) indicates the number of vertices of A preceding n in the ordering of the vertices. Instead of grading by elements of $\mathbb{Z}^{nk}$, as in [2], we grade by elements of $M_{n,k}$.

Definition 1.8. The **matrix degree** of gA is $\widetilde{\phi}(gm_A)$.

Essentially, [2] assigns weight e_i to the ith variable, whereas we assign weight $\phi(x^\lambda)$ to y_λ. The boundary operators still are homogeneous maps of degree 0: $m_v\,(A\backslash\{v\})$ and A have the same matrix degree. Hence for each $\xi \in M_{n,k}$, we get an associated chain complex

$$\mathscr{C}(\Delta)_\xi : \dots (F_n)_\xi \xrightarrow{\delta_n} (F_{n-1})_\xi \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_0} (F_0)_\xi$$

where $(F_i)_\xi$ indicates the set of elements of F_i of matrix degree ξ.

Theorem 1.9. *$\mathscr{C}(\Delta)$ is a minimal free resolution of $\mathbb{C}$.*

Proof. $\mathscr{C}(\Delta)$ is the Koszul complex of S, as described in **Example 2.6** of [2]. (The symbols in our polynomial ring S are indexed by $\{0,\dots n\}^k$ under lexographical order, instead of natural numbers, but it's otherwise it's the same situation.) □

Now, we need a way to compute the syzygies from this free resolution.

Theorem 1.10. *If $\mathcal{F} : \dots \to F_1 \to F_0$ is a minimal free resolution of an S-module M, then the ith syzygy of M has $\dim_{\mathbb{C}}(Tor_i^S(\mathbb{C}, M)_j)$ generators of degree j (and this set of generators is minimal).*

Proof. This combines **Corollary 1.5** and **Proposition 1.7** of [2]. Applying F_i to a minimal set of generators for F_i yields a minimal set of generators for the image of d_i. d_i preserves degree: there are $\dim_{\mathbb{C}}(\mathrm{Tor}_i^S(\mathbb{C}, M)_j)$ generators of F_i of degree j, so there are $\dim_{\mathbb{C}}(\mathrm{Tor}_i^S(\mathbb{C}, M)_j)$ generators of the image of d_i of degree j. □

Hence to compute the number of generators, we take the complex $\mathscr{C}(\Delta)$ and tensor over S with the coordiante ring $C_{n,k}$. This is still a graded complex: the matrix degree of $g \otimes_S hA$ is $\phi(g)\widetilde{\phi}(hm_A)$.

Definition 1.11. For $\xi \in M_{n,k,d}$, let Σ_ξ be the simplicial complex with vertices consisting of elements of $M_{n,k,d}$. The m-simplices are $m+1$ tuples of vertices $[B_0 ... B_m]$ such that $B_1 + \dots + B_m \leq \xi$, where $\leq$ indicates comparison entry-by-entry.

(Henceforth we will omit the condition that $\xi \in M_{n,k,d}$: in light of the bijection in **Theorem 1.6**, "the degree of ξ" refers to d, the sum of each column of ξ.) We will use $F_{\xi,m}$ to denote the free $\mathbb{C}$ module generated by the set of all m-simplices in Σ_ξ. These submodules form a chain complex:

$$\mathcal{G}_\xi : \dots G_{\xi,2} \to G_{\xi,1} \to G_{\xi,0} \to 0$$

with differentials given by $d([B_0 \dots B_m]) = \sum_{i=0}^m (-1)^{i-1}[B_0 \dots \widehat{B}_i \dots B_m]$, where $\widehat{B}_i$ indicates that B_i is omitted.

Theorem 1.12. *$(C_{n,k} \otimes_S \mathscr{C}(\Delta))_\xi$ is isomorphic to $\mathcal{G}_\xi$.*

Proof. Let

$$\tau(h \otimes_S gA) = [\widetilde{\phi}(v)]_{v \in A}$$

Note $h \otimes_S gA = g \cdot h \otimes_S A \in (C_{n,k} \otimes_S \mathscr{C}(\Delta))_\xi$ implies that

$$\phi(g \cdot h) + \sum_{v \in A} \widetilde{\phi}(m_v) = \xi$$

Solving for $g \cdot h$, we find $\phi(g \cdot h) = \xi - \sum_{v \in A} \widetilde{\phi}(m_v)$. Thus $g \cdot h$ is uniquely determined by ξ and A, so the function is injective. The fact that τ is surjective is immediate.

It is a striaghtforward computation to show that τ commutes with the relevant differentials: we're essentially just "forgetting" about the h and the g in $g \otimes_S hA$ and then applying $\widetilde{\phi}$ to the vertices. $\square$

Theorem 1.13. *Fix some d. Then the ith homology of the simplicial complex Σ_ξ is trivial for all ξ of degree d if and only if the $(i+1)$th syzygy module of $C_{n,k}$ has no generators of degree d.*

Proof. This follows from **Theorem 1.10** and **Theorem 1.12**. $\square$

The above result will be our chief tool for investigating the syzygies.

Theorem 1.14. *In the decomposition of $Tor_i^S(C_{n,k}, \mathbb{C})$, it suffices to consider pieces of degree ξ such that the entries of ξ are ones and zeros.*

A detailed proof of the above claim is beyond the scope of this paper, but we'll give a general outline. $C_{n,k}$ is naturally a representation of $GL_n(\mathbb{C}) \times \cdots \times GL_n(\mathbb{C})$, where there are k copies of $GL_n(\mathbb{C})$. Tor is functorial, so $Tor_i^S(\mathbb{C}, C_{n,k})$ is a representation of $GL_n(\mathbb{C})^k$. Each $(Tor_i^S(C_{n,k}, \mathbb{C}))_\lambda$ is a subrepresentation. Let V_λ denote the subrepresentation associated to lambda. From the form of the Koszul complex, one can show that V_λ is a polynomial representation. Applying Schur-Weyl duality yields a correspondance between V_λ (as a $GL_n(\mathbb{C})^k$ module) and the 1^q weight space of V_λ (as an S_k module), where q is the degree of V_λ. This implies that the pieces of degree ξ, where the entries of ξ are ones and zeros, generate $Tor_i^S(C_{n,k}, \mathbb{C})$.

2. Second Syzygies

Now, we proceed to characterize the module of second syzygies. These results are well-known. However, our methods are different: we use explicit geometric constructions instead of tools from homological algebra.

Theorem 2.1. *The first homology of the complex Σ_ξ is nontrivial if and only if $\deg \xi = 3$: equivalently, the the module of second syzygies is generated by elements of degree 3.*

Much of the argument for the forward direction comes down to the following lemma:

Lemma 2.2. *If $\deg(\xi) \geq m$, given any $m-1$ vertices $v_1, \dots v_{m-1}$ in Σ_ξ, there exists a vertex w such that for all i wv_i is an edge.*

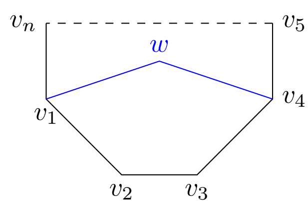
Proof. Consider the jth column of ξ and of the v_i. $\deg(\xi) \geq m$ implies that the jth column of the matrix ξ consists of a sum of at least distinct m unit vectors. By **Theorem 1.14**, we may assume all entries of ξ are either 1 or 0. The jth column of each v_i consists of exactly one unit vector, for a total of $m-1$ or fewer unit vectors between the $m-1$ vertices. By the Pigeonhole Principle, there exists some unit vector e_{a_j} that appears in the jth column of ξ but not among the jth columns of vertices. Set $w = \left[e_{a_1}, \dots e_{a_n}\right]$. Then for each i, $v_i + w \leq \xi$. $\square$

The strategy is to use **Lemma 2.2** to split large 1-cycles, reducing to 1-cycles of a small size. Then, show that the small 1-cycles are trivial.

Lemma 2.3. *If $\deg(\xi) \geq 3$, any 1-cycle can be expressed as a sum of 1-cycles with 5 edges (i.e. 1-simplices) or fewer.*

Proof. Suppose for sake of contradiction there are some 1-cycles that cannot be expressed as a sum of 1-cycles with 5 or fewer edges. Let C be a 1-cycle with the minimal number of vertices that cannot so be expressed. Label the vertices $v_1, \ldots v_n$. Of course, we must have $n > 5$. By **Lemma 2.2**, $\deg(\xi) \geq 3$ implies that there exists some vertex w such that $v_1 w$ and $w v_4$ are both 1-simplices in the simplicial complex. Let $C' = v_1 v_2 v_3 v_4 w$ and $C'' = v_1 w v_4 \ldots v_n$. $n > 5$ guarantees that the number of 1-simplices in C' and in C'' are smaller than the number of vertices in C. By the minimality of C, it follows that C' and C'' can be expressed in terms of 1-cycles with 5 or fewer edges. Thus $C = C' + C''$ can also be expressed in such a manner, a contradiction. Hence all 1-cycles that cannot be expressed as a sum of 1-cycles with 5 or fewer edges.

□



Lemma 2.4. *If* $\deg(\xi) \geq 4$*, all 1-cycles in* Σ_ξ *can be expressed as a sum of 1-cycles with 4 edges or fewer.*

Proof. By **Lemma 2.3**, it suffices to consider 1-cycles with 5 edges. Let C be a 1-cycle with vertices $v_1, \ldots v_5$. Apply **Lemma 2.2** with the three vertices v_1, v_2, and v_4. $\deg(\xi) \geq 4$ implies that there exists some vertex w such that wv_1, wv_2, and wv_4 are edges. Let $C' = v_2 v_3 v_4 w$, $C'' = v_4 v_5 v_1 w$, and $T = [v_1 v_2 w]$. Then $C = C' + C'' + \partial T$, so $C \equiv C' + C''$. C' and C'' each have 4 edges. □

With these lemmas, now we will show that for $\deg(\xi) \neq 3$, the module of second syzygies is trivial. For $\deg(\xi) \leq 2$, the simplicial complex is just too small: it has no 1-cycles. For $\deg(\xi) \geq 4$, by **Lemma 2.3** and **Lemma 2.4**, it suffices to show that 1-cycles with 4 edges are trivial. This requires considering specific vertices and specific cycles; for notational convenience, we will denote the vertex $\left[e_{a_1}, \ldots e_{a_k}\right]$ by $(a_1, \ldots a_k)$ or simply $a_1 \ldots a_k$.

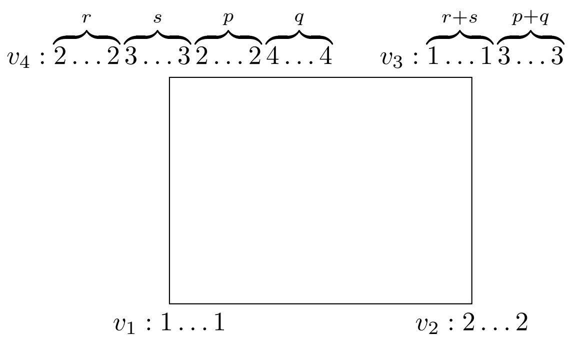
Let C be some 1-cycle with vertices $v_1, \ldots v_4$. If $\deg(\xi) \geq 5$, then apply **Lemma 2.2** to the four vertices of C: there exists some vertex w such that $v_1 w, \ldots v_4 w$ are edges. For $1 \leq i \leq 4$, let $T_i = [v_i v_{i+1} w]$, where the indices are mod 4. Then $C = \sum_{i=1}^{4} \partial T_i$.

If $\deg(\xi) = 4$, then we narrow down the possibilities. Apply **Theorem 1.14** and reorder each column of ξ so that the ones are at the top: then the rows of zeros may be ignored. Hence we may assume $n = 4$ and ξ consists of all 1's. Using the relevant group actions, turn v_1 into $1 \ldots 1$. (For example, if $v_1 = 314$, apply $(13) \times id \times (14) \in S_4 \times S_4 \times S_4$ to all vertices.) Note that because $v_1 v_2$ is an edge, v_2 cannot possibly involve 1. By a similar argument, we can use permutations to turn v_2 into $(2, \ldots 2)$, without affecting $v_1 = (1, \ldots 1)$.

Now, we turn to v_3. $v_3 v_2$ is an edge, so v_3 does not involve 2. That leaves only 1, 3, and 4. Neither v_1 nor v_2 involve a 3 or 4, so we may apply the permutation (34) to the coordinate that are 4. Reordering the coordinates does not change v_1

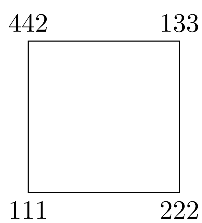
and v_2, so without loss of generality the coordinates occur in non-decreasing order. Thus $v_3 = \overbrace{1\dots1}^{a}\overbrace{3\dots3}^{b}$

Finally, we determine v_4. Considering the edges $v4v_1$ and v_4v_3, the first a coordinates of v_4 may use 2, 3, and 4. Note that between the first a coordinates of the other 3 vertices, we have only used 1 and 2. Thus, using the permutation (34), we may turn any 4's in this block into 3's. The edges v_4v_1 and v_4v_3 imply that the last b coordinates are among $\{2,4\}$. Reordering within the block of the first a coordinates or within the block of the last b coordinates does not affect the other vertices, so without loss of generality each block is non-decreasing. Hence $v_4 = \overbrace{2\dots2}^{r}\overbrace{3\dots3}^{s}\overbrace{2\dots2}^{p}\overbrace{4\dots4}^{q}$, where $r+s=a$ and $p+q=b$.

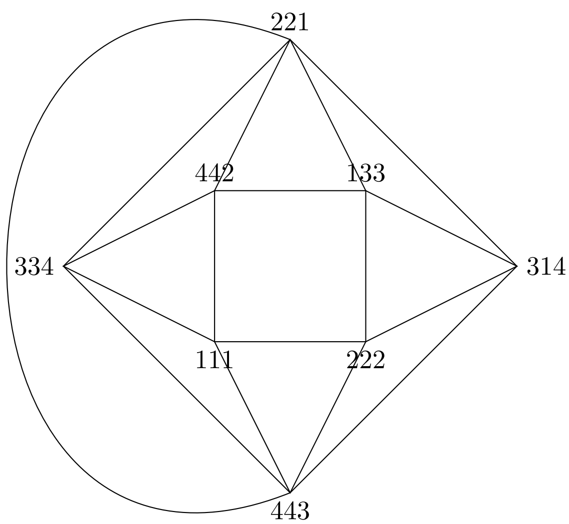


It turns out we can do a bit better than this: note that none of v_1, v_4, v_3 use 4 in the first $r+s+p$ coordinates. Let $w = \overbrace{4\dots4}^{r+s}\overbrace{4\dots4}^{p}\overbrace{2\dots2}^{q}$. Then v_1w, v_2w, and v_3w are all edges. The above cycle differs from $v_1v_2v_3w$ by the boundaries of the two triangles v_3v_4w and v_4v_1w. Thus, we may replace v_4 by w.

It suffices to show the 1-cycle below is trivial.



It can be expressed in terms of triangles as depicted below.



We have shown that if $\deg(\xi) \neq 3$, all 1-cycles are trivial. To finish the proof of **Theorem 2.1**, we now need to show that if $\deg(\xi) = 3$ then there exists a nontrival 1-cycle. The following result will also be useful for dealing with third syzygies, so we state it as a lemma.

Lemma 2.5. *Let Δ_l denote the free $\mathbb{Z}$-module on set of l-simplices in Σ_ξ, and let $d_{\deg(\xi)-1}$ denote the boundary map from $\Delta_{\deg(\xi)-1}$ to $\Delta_{\deg(\xi)-2}$. If $\deg(\xi) > 1$, then im$(d_{\deg(\xi)-1})$ is the free module generated by $\{d_{\deg(\xi)-1}(\sigma) | \sigma \in \Delta_{\deg(\xi)-1}\}$.*

Proof. Clearly, $\mathrm{im}(d_{\deg(\xi)-1})$ is generated by $\{d_{\deg(\xi)-1}(\sigma) | \sigma \in \Delta_{\deg(\xi)-1}\}$. It remains to be shown that there are no relations among these generators, i.e. that the kernel of $d_{\deg(\xi)-1} : \Delta_{\deg(\xi)-1} \to \Delta_{\deg(\xi)-2}$ is trivial.

Take the dual. Let $\Phi : \Delta \to \Delta^*$ be the dual isomorphorism. The boundary map as in homology $d_{\deg(\xi)-1} : \Delta_{\deg(\xi)-1} \to \Delta_{\deg(\xi)-2}$ becomes the boundary map as in cohomology $d^*_{\deg(\xi)-1} : \Delta^*_{\deg(\xi)-2} \to \Delta^*_{\deg(\xi)-1}$. Consider some $\deg(\xi) - 2$ simplex $[v_0 \dots v_{\deg(\xi)-2}]$. Then $\sum_{i=0}^{\deg(\xi)-2} v_i \leq \xi$. Note that the sum of each column in ξ is $\deg(\xi)$. The sum of each column in $\sum_{i=0}^{\deg(\xi)-2}$ is $\deg(\xi) - 1$. Therefore there exists a unique $w \in \Sigma_\xi$ such that $w + \sum_{i=0}^{\deg(\xi)-2} = \xi$. Geometrically, there is a unique $\deg(\xi) - 1$ simplex $\sigma = [v_0 \dots v_{\deg(\xi)-2} w]$ such that the $\deg(\xi) - 2$ simplex $[v_0 \dots v_{\deg(\xi)-2}]$ is a face of σ. Therefore

$$d^*_{\deg(\xi)-1}\left(\Phi([v_0 \dots v_{\deg(\xi)-2}])\right) = \pm\Phi([v_0 \dots v_{\deg(\xi)-2} w])$$

This implies that $d^*_{\deg(\xi)-1}$ is surjective.

$$\begin{array}{ccc} \Delta^*_{\deg(\xi)-1} & \xleftarrow{d^*_{\deg(\xi)-1}} & \Delta^*_{\deg(\xi)-2} \\ \downarrow{\scriptstyle \Phi^{-1}} & & \downarrow{\scriptstyle \Phi^{-1}} \\ \Delta_{\deg(\xi)-1} & \xrightarrow{d_{\deg(\xi)-1}} & \Delta_{\deg(\xi)-2} \end{array}$$

Suppose that $a \in \ker(d_{\deg(\xi)-1})$. Trace a back to an element of $\Delta^*_{\deg(\xi)-2}$: Φ^{-1} is an isomorphism and $d^*_{\deg(\xi)-1}$ is surjective. There exists $b \in \Delta^*_{\deg(\xi)-2}$ such that $a = \Phi^{-1}(d^*_{\deg(\xi)-1}(b))$. By the commutativity of the above diagram,

$$\Phi^{-1}(b) = d_{\deg(\xi)-1}(\Phi^{-1}(d^*_{\deg(\xi)-1}(b))) = d_{\deg(\xi)-1}(a) = 0$$

However, Φ is an isomorphism: we must have $b = 0$ and thus $a = 0$.

□

This lemma immediately yields a useful criteria for identifying nontrivial cycles: if C is a $\deg(\xi)-2$ cycle in Σ_ξ, then we only need to check if C is in the free module generated by $\{d_{\deg(\xi)-1}(\sigma) | \sigma \in \Delta_{\deg(\xi)-1}\}$.

Theorem 2.6. *If* $\deg(\xi) = 3$*, then* Σ_ξ *contains a nontrival 1-cycle.*

Proof. First, we addresss $k = 2$. Without loss of generality, $n = 3$ and all entries of ξ are ones. Let $v_1 = 11$, $v_2 = 22$, $v_2 = 13$, and $v_4 = 32$. Let $C = \sum_{i=1}^4 [v_i v_{i+1}]$, where the indices are mod 4). Apply the above lemma: $\deg(\xi) = 3$, and C is a $3-2=1$ cycle. No three vertices in C form a triangle: neither v_1v_3 nor v_2v_4 are edges. Therefore, C is not in the free module generated by the boundaries of all triangles in Σ_ξ. By **Corollary 2.6.1**, C is a nontrivial 1-cycle.

If $k > 2$, then simply repeat the last digit $k-1$ times: $v_1 = 11\dots1$, $v_2 = 22\dots2$, $v_3 = 13\dots3$, and $v_4 = 32\dots2$. No three vertices in this cycle form a triangle, so for the same reasons, this cycle is nontrivial. □

That finishes the proof of **Theorem 2.1**. It turns out that when $\deg(\xi) = 3$, all nontrivial 1-cycles in Σ_ξ are expressible in terms of those with 4 edges.

Theorem 2.7. *When* $\deg(\xi) = 3$*, all nontrivial 1-cycles in* Σ_ξ *are expressible in terms of those with 4 edges.*

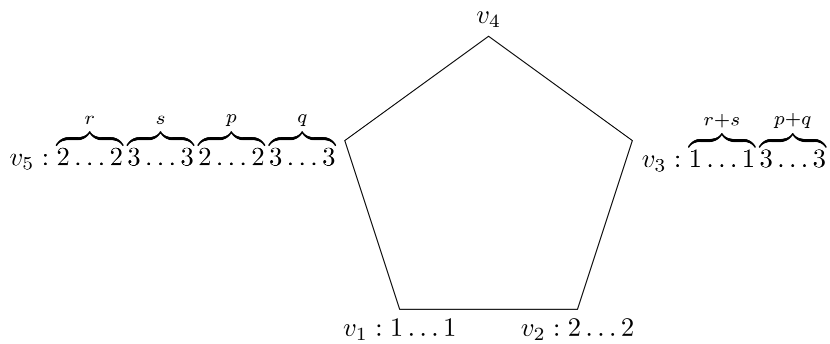
Proof. By **Lemma 2.3**, $\deg(\xi) \geq 3$ implies that all 1-cycle are expressible in terms of those with 5 or fewer edges. All 1-cycles with 3 or fewer edges are trivial, so it remains to be shown that all 1-cycles with 5 edges are expressible in terms of those with 4 edges. Using the group actions, we restrict the possibilities to one 1-cycle of length 5. Then, we express that 1-cycle in terms of 1-cycles of length 4.

Without loss of generality, $n = 3$ and

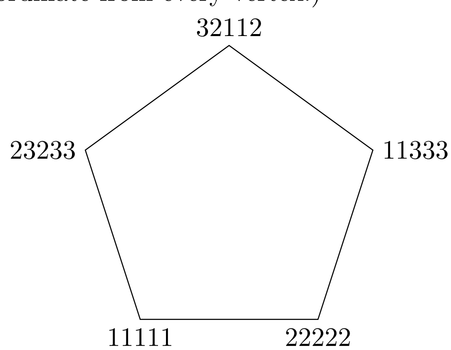
$$\xi = \begin{bmatrix} 1 & \dots & 1 \\ 1 & \dots & 1 \\ 1 & \dots & 1 \end{bmatrix}$$

for a total of k columns. Let C be a 1-cycle with 5 edges. Label the vertices $v_1, \dots v_5$. It suffices to consider cases where $v_1 = 1\dots1$ and $v_2 = 2\dots2$. v_2v_3 is an edge, so v_3 must have only 1 and 3 as coordinates. Reordering, $v_3 = \overbrace{1\dots1}^{a}\overbrace{3\dots3}^{b}$. Now consider v_5. v_5v_1 is an edge, so v_5 must only have 2 and 3 as coordinates.

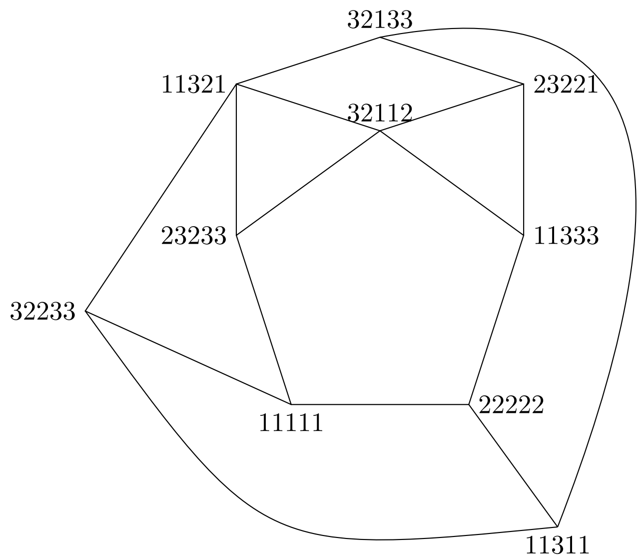
Reordering within the first a coordinates, and within the last b coordinates, $v_5 = \overbrace{2\dots2}^{r}\overbrace{3\dots3}^{s}\overbrace{2\dots2}^{p}\overbrace{3\dots3}^{q}$, where $r+s=a$ and $p+q=b$.



That leaves v_4. With the exception of the last p coordinates, each coordinate of v_3 is distinct from the corresponding coordinate of v_5. There are only 3 possibilities, so from the edges v_3v_4 and v_4v_5, all but the last p coordinates of v_4 are determined. And each of the last p coordinates must be 1 or 2. Hence, we find $v_4 = \overbrace{2\dots2}^{r}\overbrace{3\dots3}^{s}\overbrace{2\dots2}^{p}\overbrace{1\dots1}^{l}\overbrace{2\dots2}^{m}$, where $l+m=p$. That is 5 blocks of coordinates, so all 1-cycles of length 5 are produced from the one below, through duplication and/or deletion. (If we were to have $l=0$ we would just delete the 4th coordinate from every vertex.)



Below is a depiction of how to break this up into squares and triangles. We are interested in the equivalence class of this 5-cycle in homology, so triangles don't matter. The basic approach is to use squares and triangles to get a pair of vertices that match in most coordinates (e.g. 32133 and 3223), so that there is greater freedom in choosing things adjacent to both of them.



□

3. Third Syzygies

[3] proves that the module of third syzygies is nontrivial if and only if $\deg \xi = 4$. We proved a partial result in this direction. The idea is to create a process akin to separating off a pentagon in the treatment of 1-cycles. First, show that one can "split off" some small piece from an arbitrary 2-cycle, in such a way that the 2-cycle becomes "simpler." Second, show that each small piece is homologically trivial. We have shown that all "small pieces" of a certain type are homologically trivial.

This definition is a more restrictive version of the one found in [3].

Definition 3.1. A 2-cycle C (in Σ_ξ) is a **UFO** if and only if there exists vertices $\{v_i\}_{1\le i\le q}, u_1, u_2$ such that $C = \sum_{i=1}^{q}([v_i v_{i+1} u_1] + [v_i v_{i+1} u_2])$, where the indices are mod q.

A UFO is the simplest type of 2-cycle: it is a 1-cycle, with two extra vertices attached to the top and the bottom. See Figure 1 for an example: the triangular faces make up the 2-cycle, but they're not filled in so that one can see the full structure. Given a UFO η, we refer to $\sum_{i=1}^{q}[v_i v_{i+1}]$ as "the 1-cycle of η." u_1 and u_2 will be referred to as the "top and bottom" vertices of the UFO. (If there are multiple valid ways to assign v_i and u_1, u_2 to the vertices, then just choose one such labeling.)

Lemma 3.2. *When $\deg(\xi) \ge 5$, all UFOs in Σ_ξ can be expressed in terms of those with a 1-cycle of length at most 5.*

Proof. Let w_1 and w_2 be the top and bottom vertices of the UFO. Let $v_1, \dots v_n$ the the vertices around the 1-cycle (in clockwise order, say), where $n > 5$.

Without loss of generality the entries of ξ are ones and zeros. Thus, there are no "empty faces" in Σ_ξ; if AB, BC, and AC are all edges (and $\deg(\xi) \ge 3$), then ABC is a triangle in Σ_ξ. We follow the procedure of **Lemma 2.3**, except that for each additional vertex we insert, we also require each new vertex to have an edge to

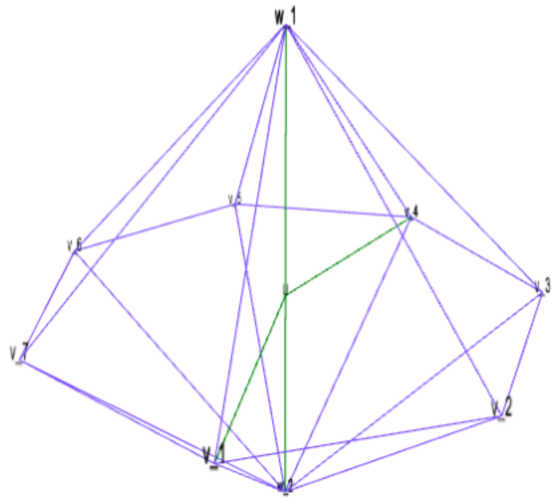
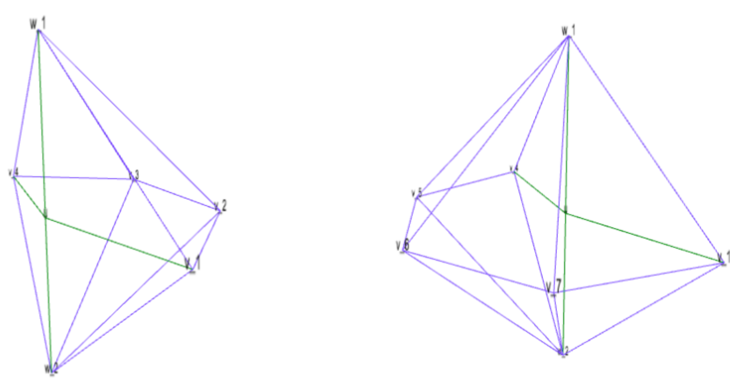


FIGURE 1. A UFO with 1-cycle of length 7

w_1 and to w_2. We then split the original UFO in to 2 smaller UFOs, corresponding to the 2 smaller 1-cycles produced in the proof of **Lemma 2.3**.

By **Lemma 2.2**, $\deg(\xi) \geq 5$ implies that there exists some vertex u such that v_1u, v_4u, w_1u, and w_2u are all edges in Σ_ξ. This implies that v_1uw_1, w_1uv_4, v_4uw_2, and w_2uv_1 are all triangles in Σ_ξ. Hence, we may split up the original UFO into 2 UFOs: one with 1-cycle v_1, v_2, v_3, v_4, u and another with 1-cycle $v_4, v_5, \ldots v_n, v_1, u$. When $n = 7$, these pieces are depicted in Figure 2a and Figure 2b, respectively.

□



(A) The first "piece" of the UFO (B) The second "piece" of the UFO

FIGURE 2. The decomposition of the UFO in Figure 1 into two smaller pieces

Recall that **Lemma 2.3** required $\deg(\xi) \geq 3$ for all 1-cycles to be expressible in terms of those of length at most 5. **Lemma 2.4** requires $\deg(\xi) \geq 5$ in order for

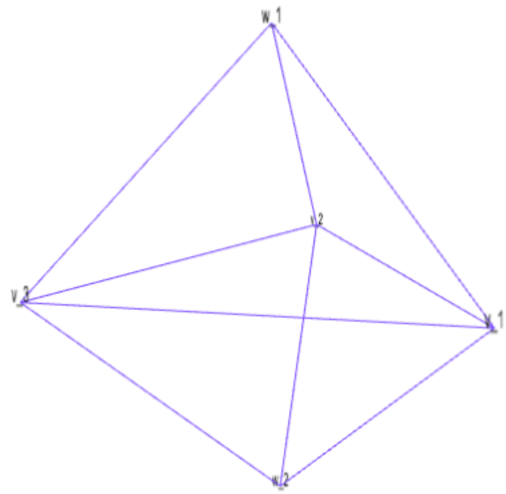


FIGURE 3. Decomposition of a UFO with a 1-cycle of length 3

all UFOs to be expressible in terms of those with 1-cycle of length at most 5. Note that $5 = 3 + 2$. In order to apply **Lemma 2.2** we must increase the degree by 2, so we get edges to the two vertices w_1 and w_2 that aren't in the 1-cycle.

Lemma 2.4 required $\deg(\xi) \geq 4$. If we increase $\deg(\xi)$ by 2, then we get a corresponding result for UFOs.

Lemma 3.3. *If* $\deg(\xi) \geq 6$, *all UFOs in* Σ_ξ *can be expressed in terms of those with a 1-cycle of length at most 4.*

Theorem 3.4. *If* $\deg(\xi) \geq 7$, *then all UFOs are homologically trivial.*

Proof. By **Lemma 3.3**, all UFOs in Σ_ξ can be expressed in terms of those with a 1-cycle of length at most 4.

If $n = 3$, the UFO is the boundary of $[v_1v_2v_3w_1] + [v_1v_2v_3w_2]$, as depicted in Figure 3.

If $n = 4$, note that the UFO has 6 vertices and $\deg(\xi) \geq 7$. Applying **Lemma 2.2**, there exists some vertex $u \in \Sigma_\xi$ such that $uv_1, \dots uv_4$, uw_1, uw_4 are all edges in Σ_ξ. Then the UFO is the boundary of $\sum_{i=1}^{4}[v_iv_{i+1}w_1u] + \sum_{i=1}^{4}[v_iv_{i+1}w_2u]$, where the indices of the v_i are mod 4, as depicted in Figure 4.

□

However, decomposing an arbitrary 2-cycle into these small pieces has resisted our efforts. How does one "split off" a UFO from an arbitrary 2-cycle in such a way that the remaining structure is "simpler" or "smaller?" A 1-cycle is a sum of polygons, each of which is described by the number of sides. 2-cycles resist such an easy description. "Spheres" (trivial fundamental group) are the equivalent of polygons, but "spheres" exhibit much more variety than polygons.

We also found an example of a nontrivial 2-cycle in Σ_ξ, when $n = 4$, $k = 2$, and ξ is all 1's. This is depicted in Figure 5. All the shaded triangles are oriented the same direction. The arrows serve to indicate that edges along the top and are

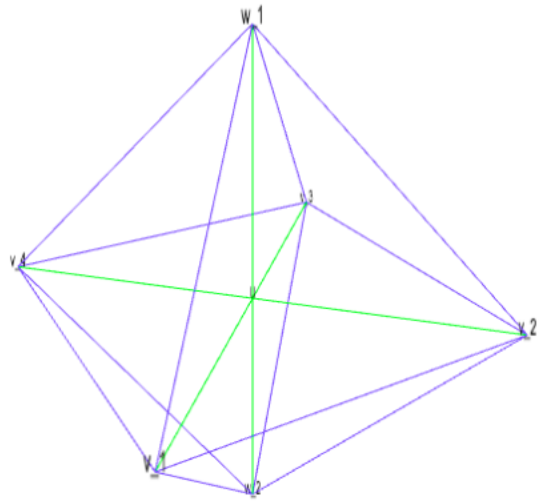


FIGURE 4. Decomposition of a UFO with a 1-cycle of length 4

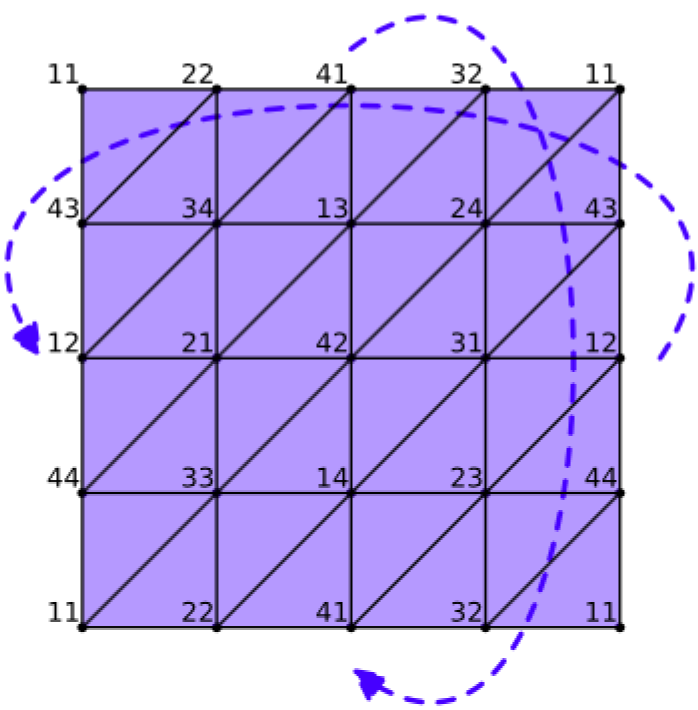


FIGURE 5. A nontrivial 2-cycle

the same as those along the bottom: the 2-cycle in Figure 5 is homeomorphic to a torus. Once again, 11 stands for the vertex $(e_1, e_1) \in \Sigma_\xi$.

Theorem 3.5. *The 2-cycle (in Σ_ξ) depicted in Figure 5 is nontrivial in homology.*

Proof. Clearly the sum of triangles in Figure 5 has no boundary, so it is a 2-cycle. Apply **Corollary 2.6.1**. The 2-cycle contains the face 11, 22, 33. That face is in one and only one tetrahedron: 11, 22, 33, 44. However, taking the boundary of that tetrahedron yields the face 11, 22, 44 (and 3 other triangle faces), which is not in the 2-cycle depicted in Figure 5. □

In finding this 2-cycle, the critical insight is that for each edge v_1v_2, there are exactly 2 other edges w_1w_2 such that $v_1v_2w_1w_2$ is a tetrahedron. The edge 11, 22 forms a tetrahedron with 33, 44, which also forms a tetrahedron with 12, 21. Repeating in this fashion, we get back after 4 pairs of edges, after 4 different tetrahedra. Make note of the vertices 11, 22, 33, 44 in the bottom-left corner of Figure 5, followed by 44, 33, 12, 12 above it, and so on, continuing upward. Do the same thing with the two edges 11, 44 and 22, 33 to get the tetrahedra along the bottom, toward the right. We found a way to choose two faces from each tetrahedron in such a way that the boundaries cancel.

While one could replace the segment from 11 to 33 by one from 22 to 44, doing so amounts to swapping the pair of faces in the tetrahedron with vertices 11, 22, 33, 44 for the opposite faces. Thus, they're equivalent in homology. Some arbitrary choices are involved in the process of constructing this 2-cycle: there are multiple 2-cycles of this type. However, one can show that they're all the same as the 2-cycle depicted, up to relabeling in the usual fashion.

References

[1] David Eisenbud. *Commutative Algebra: with a view toward algebraic geometry.* Vol. 150. Springer Science & Business Media, 2013.

[2] David Eisenbud. *The geometry of syzygies: a second course in algebraic geometry and commutative algebra.* Vol. 229. Springer Science & Business Media, 2005.

[3] Elena Rubei. "On syzygies of Segre embeddings". In: *Proceedings of the American Mathematical Society* 130.12 (2002), pp. 3483–3493.