Control of Conditional Processes

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Credit

- **R.C.** with **Mathieu Laurière** and **Pierre Louis Lions**
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- **R.C.** with **Dan Lacker**
  Stochastic Analysis and Applications 2014: In Honour of Terry Lyons (2024) Springer Verlag
- **R.C.** with **Samuel Daudin**
  (in preparation)

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- D bounded open domain in $\mathbb{R}^d$, with smooth boundary ∂D;
- $\tau = \inf\{t > 0;\ X_t \notin D\}$ first exit time;
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Challenge: minimize over **open loop** and/or **Markovian** control processes

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- $N_t/N \to \mathbb{P}[\tau > t]$ probability that a typical individual is still alive at time t,
- $\frac{1}{N} \sum_{i=1}^{N_t} f(X_t^i) = \frac{1}{N} \sum_{i=1}^{N} f(X_t^i) \mathbf{1}_{\tau(X^i) > t} \to \mathbb{E}[f(X_t) \mathbf{1}_{\tau > t}]$.

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- $N_t/N \to \mathbb{P}[\tau > t]$ probability that a typical individual is still alive at time t,
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Optimization of the fitness of the individuals still alive naturally leads to the conditional control problem which we propose to study.

III. J. Math Approach: Feynman-Kac Relaxation

Soft killing instead of **hard killing**

$$J^V(\boldsymbol{\alpha}) = \int_0^T \frac{\mathbb{E}\left[f(X_t, \alpha_t)e^{-\int_0^t V(X_s)ds}\right]}{\mathbb{E}\left[e^{-\int_0^t V(X_s)ds}\right]} dt + \frac{\mathbb{E}\left[g(X_T)e^{-\int_0^T V(X_s)ds}\right]}{\mathbb{E}\left[e^{-\int_0^T V(X_s)ds}\right]}. \tag{1}$$

Original problem corresponds to $V = V^\infty$ given by:

$$V^\infty(x) = \begin{cases} 0 & \text{if } x \in D \\ \infty & \text{otherwise,} \end{cases} \tag{2}$$

in which case:

$$\int_0^t V^\infty(X_s)ds = \begin{cases} 0 & \text{if } X_s \in \overline{D},\ 0 \le s \le t \\ \infty & \text{if } X_s \notin \overline{D} \text{ for some } 0 \le s \le t, \end{cases} \tag{3}$$

so that:

$$e^{-\int_0^t V^\infty(X_s)ds} = \mathbf{1}_{[X_s \in \overline{D},\ 0 \le s \le t]} = \mathbf{1}_{[\tau_D \ge t]},$$

where τ_D is the first exit time of the domain D defined as:

$$\tau_D = \inf\{t \ge 0; X_t \notin \overline{D}\}.$$

Accordingly:

$$J^{V^\infty}(\boldsymbol{\alpha}) = \int_0^T \mathbb{E}\left[f(X_t, \alpha_t) \middle| \tau_D \ge t\right] dt + \mathbb{E}\left[g(X_T) \middle| \tau_D \ge T\right], \tag{4}$$

Approximation Procedure

Approximate V^∞ by $V^n = nV^1$ where

- $V^1(x) = \chi^\epsilon(d(x, D))$
- $d(x, D)$ denotes the distance from $x \in \mathbb{R}^d$ to the domain D,
- $\epsilon > 0$ is an arbitrary
- $$\chi^\epsilon(d) = \begin{cases} 0 & \text{if } d \leq 0 \\ \text{linear} & \text{if } 0 \leq d \leq \epsilon \\ 1 & \text{if } d \geq \epsilon. \end{cases} \tag{5}$$

Approximation Result

If $\mathbf{X} = (X_t)_{t\geq 0}$ satisfies $X_t = x_0 + \int_0^t \alpha_s ds + W_t$ **for some fixed** $\boldsymbol{\alpha}$ and $x_0 \in D$, then

◇ for any bounded function g

$$\mathbb{E}[g(X_T) \mid \tau_D > T] = \lim_{n\to\infty} \frac{\mathbb{E}[g(X_T)e^{-n\int_0^T V^1(X_s)ds}]}{\mathbb{E}[e^{-n\int_0^T V^1(X_s)ds}]}. \tag{6}$$

◇ Similarly, if $\int_0^T \mathbb{E}[|f(X_t, \alpha_t)|]dt < \infty$, we also have:

$$\int_0^T \mathbb{E}[f(X_t, \alpha_t) \mid \tau_D > t]dt = \lim_{n\to\infty} \int_0^T \frac{\mathbb{E}[f(X_t, \alpha_t)e^{-n\int_0^t V^1(X_s)ds}]}{\mathbb{E}[e^{-n\int_0^t V^1(X_s)ds}]}. \tag{7}$$

Assumptions

- The running and terminal **cost functions** satisfy:
  - The action space A is a closed convex subset of $\mathbb{R}^d$;
  - The function g is continuous and bounded on $\mathbb{R}^d$;
  - For each $\alpha \in A$, the function $f(\cdot, \alpha)$ is continuous and bounded on $\mathbb{R}^d$.
  - For each $x \in \mathbb{R}^d$, the function $f(x, \cdot)$ is convex on A.

  Separable case:
  $$f(x, \alpha) = \frac{1}{2}|\alpha|^2 + \tilde{f}(x)$$
  for some measurable bounded $\tilde{f}$
- As for the **relaxation potential**
  - The function V is continuous on $\mathbb{R}^d$ and $0 \le V \le 1$.

First Deterministic Control Problem over a Space of Probabilities

Given $\alpha_t = \phi_t(X_t)$

$$dX_t = \phi_t(X_t)dt + dW_t$$

Corresponding cost $J^V(\boldsymbol{\alpha}) = J^{(1)}(\phi)$

$$J^{(1)}(\phi) = \int_0^T \int f(x, \phi_t(x))\mu_t(dx)\ dt + \int g(x)\mu_T(dx), \quad (8)$$

where we use the notation μ_t for the probability measure:

$$\mu_t(dy) = \frac{\mathbb{E}[\delta_{X_t}(dy)e^{-A_t}]}{\mathbb{E}[e^{-A_t}]}, \qquad 0 \leq t \leq T, \quad (9)$$

where $A_t = \int_0^t V(X_s)ds$.

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FPK Equation

The measure valued function $t \mapsto \mu_t$ satisfies the forward FPK equation:

$$\partial_t \mu_t = \frac{1}{2}\Delta\mu_t - \text{div}(\phi_t \mu_t) - (V- < \mu_t, V >)\mu_t,$$

in the sense of Schwartz distributions with $\mu_0 = \mu_0(dx)$ and

$$< \mu, V >= \int_{\mathbb{R}^d} V(x)\mu(dx).$$

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A Non-Local Superposition Principle

Known for Solutions of Standard FPK equations

Let us start with a couple $(\boldsymbol{\phi}, \boldsymbol{\mu})$ such that

- $\boldsymbol{\phi} = \big(\phi_t(x)\big)_{0 \le t \le T,\ x \in \mathbb{R}^d}$ is a $\mathbb{R}^d$-valued measurable function on $[0, T] \times \mathbb{R}^d$
- $\boldsymbol{\mu} = (\mu_t)_{0 \le t \le T}$ is a measurable flow of probability measures satisfying
  - $\int_0^T \int_{\mathbb{R}^d} |\phi_t(x)|^2 \mu_t(dx) dt < \infty$
  - the non-local FPK equation

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$$\partial_t\mu_t = \frac{1}{2}\Delta\mu_t - \mathrm{div}(\phi_t\mu_t) - (V- < \mu_t, V >)\mu_t,$$

Then there exists a weak solution $\mathbf{X} = (X_t)_{0\le t\le T}$ of the stochastic differential equation

$$dX_t = \phi_t(X_t)dt + dW_t$$

with $X_0 \sim \mu_0$ and such that

$$\mu_t(dy) = \frac{\mathbb{E}[\delta_{X_t}(dy)e^{-A_t}]}{\mathbb{E}[e^{-A_t}]}, \qquad 0 \le t \le T,$$

with $A_t = \int_0^t V(X_s)ds$. Moreover $\sup_{0\le t\le T} \mathbb{E}[|X_t|^2] < \infty$.

Existence of an Optimal Control

Ideas from Optimal Transport

Notation: $\mathbb{A}^{(2)}(\mu_0)$ set of couples $(\boldsymbol{\theta}, \boldsymbol{\mu})$, $\boldsymbol{\theta} = (\theta_t)_{0\le t\le T}$, $\boldsymbol{\mu} = (\mu_t)_{0\le t\le T}$, for which θ_t is absolutely continuous with respect to μ_t and there exists a measurable function $[0, T] \times \mathbb{R}^d \ni (t, x) \mapsto \phi_t(x) \in \mathbb{R}^d$ such that

$$\frac{d\theta_t}{d\mu_t}(x) = \phi_t(x) \qquad \text{and} \qquad \int_0^T \int_{\mathbb{R}^d} \phi_t(x)^2 \mu_t(dx)dt < \infty.$$

If $(\boldsymbol{\theta}, \boldsymbol{\mu}) \in \mathbb{A}^{(2)}(\mu_0)$, the **superposition principle** implies that there exists a process $\mathbf{X} = (X_t)_{0\le t\le T}$ satisfying $dX_t = \phi_t(X_t)dt + dW_t$, and such that the probability measures μ_t are given by

$$\mu_t(dx) = \frac{\mathbb{E}\left[\delta_{X_t}(dx)e^{-\int_0^t V(X_s)ds}\right]}{\mathbb{E}\left[e^{-\int_0^t V(X_s)ds}\right]}.$$

Define the functional J by

$$J(\boldsymbol{\theta}, \boldsymbol{\mu}) = \begin{cases} \int_0^T \int f(x, \phi_t(x))\mu_t(dx)\, dt + \int g(x)\mu_T(dx), & \text{if } (\boldsymbol{\theta}, \boldsymbol{\mu}) \in \mathbb{A}^{(2)} \\ \infty & \text{otherwise.} \end{cases} \tag{10}$$

Theorem:

There exists a couple $(\boldsymbol{\theta}, \boldsymbol{\mu}) = (\phi_t, \mu_t)_{0\le t\le T} \in \mathbb{A}^{(2)}$ minimizing $J(\boldsymbol{\phi}, \boldsymbol{\mu})$.

Characterizaton of the Optimal Control

The Adjoint PDE

If $\boldsymbol{\phi} = (\phi_t)_{0 \le t \le T}$ is a bounded feedback function and if $\boldsymbol{\mu} = (\mu_t)_{0 \le \le T}$ is the solution of the corresponding FPK equation, the adjoint equation reads

$$0 = \partial_t u + \frac{1}{2}\Delta_x u + \phi_t \cdot \nabla_x u - (V - < \mu, V >)u + V < \mu, u > + \frac{1}{2}|\phi_t|^2 + \tilde{f}$$

Assumption:

$$K_\phi := \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E} \int_t^T |\phi_s(X_s^{t,x})|^2 ds < \infty$$

where $\mathbf{X}^{t,x} = (X_s^{t,x})_{t \le s \le T}$ satisfies the state equation $dX_s = \phi_s(X_s)ds + dW_s$ over the interval $[t, T]$ with initial condition $X_t = x$. Clearly satisfied when

- ϕ is bounded.
- $\phi \in L^q([0, T]; L^p(\mathbb{R}^d))$ for some $p \ge 2, \quad q > 2, \quad \frac{d}{p} + \frac{2}{q} < 1$.

Theorem:

For each feedback function ϕ satisfying the above assumption, for each continuous flow $\boldsymbol{\mu} = (\mu_t)_{0 \le t \le T}$ of probability measures on $\mathbb{R}^d$, the adjoint PDE admits a unique solution in the sense of viscosity. *Uniqueness in the class of bounded continuous functions.*

Regularity of the Co-State

A First A-Priori Bound.

$$\|u_t\|_\infty \le \frac{e^t}{2}\left(e^{2(T-t)}-1\right)\|\tilde{f}\|_\infty + \frac{e^{2T}}{2}K_\phi + e^T\|g\|_\infty .$$

A Second A-Priori Bound. (using the fact that $\tilde{u}$ is a solution of the adjoint PDE for a smooth $\tilde{\phi}$ and a smooth running cost $\tilde{F}$.)

$$\int_0^T \int_{\mathbb{R}^d} |\nabla \tilde{u}_t(x)|^2 \mu_t(dx)dt \le 4e^{4T}\|\tilde{F}\|_\infty^2 \left(1 + \frac{3}{2}T + \|\tilde{\phi} - \phi\|^2_{L^2(\boldsymbol{\mu})}\right)$$

Theorem:

If $(\phi, \boldsymbol{\mu}) \in \mathbb{A}^{(2)}(\mu_0)$ is such that ϕ is bounded, the viscosity solution of the adjoint equation is a bounded continuous function on $\mathbb{R}^d$ whose first order derivatives in $x \in \mathbb{R}^d$ in the sense of distributions are functions in $L^2([0,T]\times\mathbb{R}^d, \boldsymbol{\mu})$ and $L^2_{loc}([0,T]\times\mathbb{R}^d, dt\,dx)$.

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If $\boldsymbol{\beta} = (\beta_t)_{0 \le t \le T}$ is another bounded measurable feedback control function, we have:

$$\frac{d}{d\epsilon} J(\boldsymbol{\phi} + \epsilon \boldsymbol{\beta})\Big|_{\epsilon=0} = \int_0^T < \beta_t(\nabla u_t + \phi_t), \mu_t > \; dt.$$

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As a result, if ϕ is a critical point, then

$$\phi_t(x) = -\nabla u_t(x), \qquad \mu_t - a.s.\ x \in \mathbb{R}^d, \quad a.e.\ t \in [0, T].$$

Characterization of the Optimum

The optimal control $\phi_t(x) = -\nabla u_t(x)$ is obtained from the (unique) solution of the the **forward backward non-local PDE system**:

$$\begin{cases} \partial_t \mu = \frac{1}{2}\Delta_x \mu + \mathbf{div}_x(\nabla_x u\ \mu) - (V - < \mu, V >)\mu \\ 0 = \partial_t u + \frac{1}{2}\Delta_x u - \frac{1}{2}|\nabla_x u|^2 - (V - < \mu, V >)u + V < \mu, u > + \tilde{f} \end{cases}$$

on the support of μ.

Proposition:

For each continuous flow $\hat{\boldsymbol{\mu}} = (\hat{\mu}_t)_{0 \le t \le T}$ of probability measures on $\mathbb{R}^d$, the second PDE (above) admits a unique solution in the sense of viscosity which is continuously differentiable with uniformly bounded first derivatives. Moreover, this solution is actually a classical solution when $\tilde{f}$ and g are three times differentiable with bounded derivatives.

The Open Loop Problem

Mimicking Argument: Gyongi, Brunick-Shreve

Given

- $\boldsymbol{\alpha} = (\alpha_t)_{0 \le t \le T}$ general adapted **open loop** control process
- corresponding state process $\mathbf{X} = (X_t)_{0 \le t \le T}$ satisfying $dX_t = \alpha_t dt + dW_t$
- additive functional $A_t = \int_0^t V(X_s) ds$

one can find

- a state process $\hat{\mathbf{X}} = (\hat{X}_t)_{0 \le t \le T}$ satisfying $d\hat{X}_t = \psi_t(\hat{X}_t, \hat{A}_t) dt + d\hat{W}_t$
- with additive functional $\hat{A}_t = \int_0^t V(\hat{X}_s) ds$
- the deterministic (feedback) function ψ_t given by
  $\psi_t(x, a) = \mathbb{E}[\alpha_t \,|\, X_t = x, \; A_t = a]$
- (X_t, A_t) has the same distribution as $(\hat{X}_t, \hat{A}_t)$, $\qquad t \in [0, T]$

The Open Loop Optimization Problem

$$
\begin{aligned}
J^V(\boldsymbol{\alpha}) &= \int_0^T \frac{\mathbb{E}^{\mathbb{P}}\left[f(X_t,\alpha_t)e^{-A_t}\right]}{\mathbb{E}^{\mathbb{P}}\left[e^{-A_t}\right]}dt + \frac{\mathbb{E}^{\mathbb{P}}\left[g(X_T)e^{-A_T}\right]}{\mathbb{E}^{\mathbb{P}}\left[e^{-A_T}\right]} \\
&= \int_0^T \frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}[f(X_t,\alpha_t)|X_t,A_t]e^{-A_t}\right]}{\mathbb{E}^{\mathbb{P}}\left[e^{-A_t}\right]}dt + \frac{\mathbb{E}^{\mathbb{P}}\left[g(X_T)e^{-A_T}\right]}{\mathbb{E}^{\mathbb{P}}\left[e^{-A_T}\right]} \\
&\geq \int_0^T \frac{\mathbb{E}^{\mathbb{P}}\left[f(X_t,\mathbb{E}[\alpha_t|X_t,A_t])e^{-A_t}\right]}{\mathbb{E}^{\mathbb{P}}\left[e^{-A_t}\right]}dt + \frac{\mathbb{E}^{\mathbb{P}}\left[g(X_T)e^{-A_T}\right]}{\mathbb{E}^{\mathbb{P}}\left[e^{-A_T}\right]} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by Jensen's inequality} \\
&= \int_0^T \frac{\mathbb{E}^{\hat{\mathbb{P}}}\left[f(\hat{X}_t,\psi_t(\hat{X}_t,\hat{A}_t))e^{-\hat{A}_t}\right]}{\mathbb{E}^{\hat{\mathbb{P}}}\left[e^{-\hat{A}_t}\right]}dt + \frac{\mathbb{E}^{\hat{\mathbb{P}}}\left[g(\hat{X}_T)e^{-\hat{A}_T}\right]}{\mathbb{E}^{\hat{\mathbb{P}}}\left[e^{-\hat{A}_T}\right]} \\
&= J^V(\hat{\boldsymbol{\alpha}}),
\end{aligned}
$$

with $\hat{\alpha}_t = \psi_t(\hat{X}_t, \hat{A}_t)$. Consequently:

$$
\inf_{\boldsymbol{\alpha}} J^V(\boldsymbol{\alpha}) = \inf_{\psi} \int_0^T \frac{\mathbb{E}\left[f\big(\hat{X}_t,\psi_t(\hat{X}_t,\hat{A}_t)\big)e^{-\hat{A}_t}\right]}{\mathbb{E}\left[e^{-\hat{A}_t}\right]}dt + \frac{\mathbb{E}\left[g(\hat{X}_T)e^{-\hat{A}_T}\right]}{\mathbb{E}\left[e^{-\hat{A}_T}\right]}.
$$

Value function over ALL open loop controls is the same as over FEEDBACK functions of (X_t, A_t)

A Second Deterministic Control Problem over Probabilities

Given $\alpha_t = \psi_t(X_t, A_t)$

$$\begin{cases} dX_t &= \psi_t(X_t, A_t)dt + dW_t \\ dA_t &= V(X_t)dt. \end{cases}$$

Corresponding cost $J^V(\boldsymbol{\alpha}) = J^{(2)}(\psi)$

$$J^{(2)}(\psi) = \int_0^T \frac{\mathbb{E}\left[f(X_t, \psi_t(X_t, A_t))e^{-A_t}\right]}{\mathbb{E}\left[e^{-A_t}\right]} dt + \frac{\mathbb{E}\left[g(X_T)e^{-A_T}\right]}{\mathbb{E}\left[e^{-A_T}\right]}$$

$$= \int_0^T \left(\int \mu_t(dx, da) f(x, \psi_t(x, a))\right) dt + \int \mu_T(dx, da) g(x)$$

where μ_t is the Gibbs probability measure:

$$\mu_t(dx, da) = \frac{\mathbb{E}[\delta_{(X_t, A_t)}(dx, da)e^{-A_t}]}{\mathbb{E}[e^{-A_t}]}, \qquad 0 \le t \le T.$$

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FPK Equation

The measure valued function $t \mapsto \mu_t^{(1)}$ satisfies the forward FPK equation:

$$\partial_t \mu = \frac{1}{2}\Delta_x \mu - \mathrm{div}_x(\psi_t \mu) - V\partial_a \mu - (V - <\mu, V>)\mu,$$

in the sense of Schwartz distributions with $\mu_0 = \mu_0(dx) \otimes \delta_0(da)$ and

$$<\mu, V> = \int_{\mathbb{R}^d} \int_{[0,T]} V(x)\mu(dx, da).$$

.

The Corresponding Optimization Problem

Need to redo the entire analysis

- Existence of Optima
- Solution of the Adjoint PDE
- A priori bounds
- Regularity of co-states

The adjoint PDE is degenerate, approach via vanishing viscosity solutions

The Adjoint PDE and the Forward-Backward System

Rewriting FPK and the adjoint equations for the minimizer (still n the case of separable running cost) **PDE System**

$$\begin{cases} \partial_t \mu = \frac{1}{2}\Delta_x \mu + \mathbf{div}_x(\nabla_x u \ \mu) - V\partial_a \mu - (V- < \mu, V >)\mu \\ 0 = \partial_t u + \frac{1}{2}\Delta_x u - \frac{1}{2}|\nabla_x u|^2 + V\partial_a u - (V- < \mu, V >)u + V < \mu, u > + \tilde{f}. \end{cases}$$

Summary

Comparison of the solutions of the conditional control problems

- close loop case (feedback functions of (t, x))
  - $\alpha_t = \phi_t(X_t)$, state $\mu_t(dx) = \mathbb{E}_{x_0}[\delta_{X_t}(dx)e^{-A_t}]/\mathbb{E}_{x_0}[e^{-A_t}]$
- general open loop case (feedback functions of (t, x, a) because of the mimicking result)
  - $\alpha_t = \psi_t(X_t, A_t)$, state $\mu_t(dx, da) = \mathbb{E}_{x_0}[\delta_{(X_t,A_t)}(dx, da)e^{-A_t}]/\mathbb{E}_{x_0}[e^{-A_t}]$

Dynamics (FPK) + Adjoint Equation

Closed loop case PDE System (1)

$$\begin{cases} \partial_t\mu = \frac{1}{2}\Delta\mu + \text{div}_x(\nabla_x u\ \mu) - (V - <\mu, V>)\mu \\ 0 = \partial_t u + \frac{1}{2}\Delta_x u - \frac{1}{2}|\nabla_x u|^2 - (V - <\mu, V>)u + V <\mu, u> + \tilde{f}. \end{cases}$$

Open loop case (*after mimicking*) **PDE System (2)**

$$\begin{cases} \partial_t\mu = \frac{1}{2}\Delta_x\mu + \text{div}_x(\nabla_x u\ \mu) - V\partial_a\mu - (V - <\mu, V>)\mu \\ 0 = \partial_t u + \frac{1}{2}\Delta_x u - \frac{1}{2}|\nabla_x u|^2 + V\partial_a u - (V - <\mu, V>)u + V <\mu, u> + \tilde{f}. \end{cases}$$

They are **identical** once we notice that u in (2) **does not depend upon** a, implying that **the first marginal of** μ solves the FPK in (1) !

Equality of the Value Functions

Once we get to that point, **simple consequence of Jensen's inequality**.

Samuel Daudin

Back to the Original Hard Killing Model

Main obstacle to the proof of the **equality of the value functions.**

No Mimicking Theorem Available

A New Mimicking Theorem

Let $x_0 \in D$ and let us assume that $\mathbf{X} = (X_t)_{0 \le t \le T}$ is an Itô process of the form

$$X_t = x_0 + \int_0^t \alpha_s \, ds + W_t$$

where

- $\mathbf{W} = (W_t)_{0 \le t \le T}$ is a Wiener process
- $\boldsymbol{\alpha} = (\alpha_t)_{0 \le t \le T}$ is a bounded progressively measurable process.

Then there exist

- a (deterministic) bounded measurable function $\tilde{\alpha} : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$
- a weak solution $\tilde{\mathbf{X}} = (\tilde{X}_t)_{0 \le t \le T}$ of the SDE

$$\tilde{X}_t = x_0 + \int_0^t \tilde{\alpha}(s, \tilde{X}_s) \, ds + \tilde{W}_t$$

such that

- $\mathcal{L}(X_t \,|\, \tau(X) > t) = \mathcal{L}(\tilde{X}_t \,|\, \tau(\tilde{X}) > t)$ for all $t \in [0, T]$
- and we may choose:

$$\tilde{\alpha}(t, x) = \mathbf{1}_D(x) \mathbb{E}[\alpha_t \mid X_{t \wedge \tau(X)} = x].$$

Main Technical Tool

If $b : [0, \infty) \times \mathbb{R}^d \ni (t, x) \mapsto b(t, x) \in \mathbb{R}^d$ is a bounded measurable function, the SDE

$$dX_t = b(t, X_t) dt + \mathbf{1}_D(X_t) dW_t$$

is well posed (i.e. existence and uniqueness of a weak solution hold for every initial condition $x \in \mathbb{R}^d$)

Equality of the Value FunctionsI

$$J^{\tau}(\boldsymbol{\alpha}) = \int_0^T \frac{\mathbb{E}^{\mathbb{P}}\left[f(X_{t\wedge\tau(X)}, \alpha_t)\mathbf{1}_{\tau(X)>t}\right]}{\mathbb{E}^{\mathbb{P}}[\tau(X) > t]} dt + \frac{\mathbb{E}^{\mathbb{P}}[g(X_T)\mathbf{1}_{\tau(X)>T}]}{\mathbb{P}[\tau(X) > T]}$$

$$= \int_0^T \frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}[f(X_{t\wedge\tau(X)}, \alpha_t)|X_{t\wedge\tau(X)}]\mathbf{1}_{\tau(X)>t}\right]}{\mathbb{E}^{\mathbb{P}}[\tau(X) > t]} dt + \frac{\mathbb{E}^{\mathbb{P}}\left[g(X_T)\mathbf{1}_{\tau(X)>T}\right]}{\mathbb{P}[\tau(X) > T]}$$

as $\mathbf{1}_{\tau(X)>t}$ is measurable with respect to $X_{t\wedge\tau(X)}$

$$J^{\tau}(\boldsymbol{\alpha}) \geq \int_0^T \frac{\mathbb{E}^{\mathbb{P}}\left[f\left(X_{t\wedge\tau(X)}, \mathbb{E}^{\mathbb{P}}[\alpha_t|X_{t\wedge\tau(X)}]\right)\mathbf{1}_{\tau(X)>t}\right]}{\mathbb{P}[\tau(X) > t]} dt + \frac{\mathbb{E}^{\mathbb{P}}\left[g(X_T)\mathbf{1}_{\tau(X)>T}\right]}{\mathbb{P}[\tau(X) > T]}$$

convexity of f in the variable α, so

$$J^{\tau}(\boldsymbol{\alpha}) \geq \int_0^T \frac{\mathbb{E}^{\mathbb{P}}\left[f\left(X_{t\wedge\tau(X)}, \tilde{\alpha}(t, X_{t\wedge\tau(X)})\mathbf{1}_{\tau(X)>t}\right)\right]}{\mathbb{P}[\tau(X) > t]} dt + \frac{\mathbb{E}^{\mathbb{P}}\left[g(X_T\mathbf{1}_{\tau(X)>T}\right]}{\mathbb{P}[\tau(X) > T]}$$

$$= \int_0^T \frac{\mathbb{E}^{\mathbb{P}}\left[f\left(X_t, \tilde{\alpha}(t, X_t)\mathbf{1}_{\tau(X)>t}\right)\right]}{\mathbb{P}[\tau(X) > t]} dt + \frac{\mathbb{E}^{\mathbb{P}}\left[g(X_T\mathbf{1}_{\tau(X)>T}\right]}{\mathbb{P}[\tau(X) > T]}$$

$$= \int_0^T \mathbb{E}^{\mathbb{P}}\left[f\left(X_t, \tilde{\alpha}(t, X_t) \,|\, \tau(X) > t\right]dt + \mathbb{E}^{\mathbb{P}}\left[g(X_T) \,|\, \tau(X) > T\right]$$

$$= \int_0^T \mathbb{E}^{\tilde{\mathbb{P}}}\left[f\left(\tilde{X}_t, \tilde{\alpha}(t, \tilde{X}_t) \,|\, \tau(\tilde{X}) > t\right]dt + \mathbb{E}^{\mathbb{P}}\left[g(\tilde{X}_T) \,|\, \tau(\tilde{X}) > T\right]$$

What's Next?

A Lot of Questions

- **Minimization** of

$$J(\phi) = \int_0^T < f(\,\cdot\,, \phi_t(\cdot)), \mu_t > \; dt + \; < g, \mu_T >$$

for $\mu_t = \mathcal{L}(X_t | \tau > t)$

- **F-P-K equation**

$$\partial_t \mu_t = \frac{1}{2}\Delta\mu_t - \mathrm{div}_x(\phi_t \mu_t) + h_{\phi,\mu_0}(t)\mu_t,$$

where

  - $h_{\phi,\mu_0}(t) = \frac{f_{\phi,\mu_0}(t)}{1 - F_{\phi,\mu_0}(t)}$ is the hazard rate of the hitting time τ

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for $\mu_t = \mathcal{L}(X_t | \tau > t)$
- **F-P-K equation**
$$\partial_t \mu_t = \frac{1}{2} \Delta \mu_t - \mathrm{div}_x(\phi_t \mu_t) + h_{\phi,\mu_0}(t) \mu_t,$$
where
  - $h_{\phi,\mu_0}(t) = \frac{f_{\phi,\mu_0}(t)}{1 - F_{\phi,\mu_0}(t)}$ is the hazard rate of the hitting time τ
  - $F_{\phi,\mu_0}(t) = \mathbb{P}[\tau \leq t]$
  - $f_{\phi,\mu_0}(t)$ is the density
- **Adjoint equation ???**
$$\partial_t u = -\frac{1}{2} \Delta_x u - \phi_t \cdot \nabla_x u - h_{\phi,\mu_0}(t) u - f(\cdot, \phi_t(\cdot))$$

Pontriagin Maximum Principle if Hazard Rate Exists & is Bounded

- If $\boldsymbol{\beta} = (\beta_t)_{0 \le t \le T}$ is another bounded measurable feedback control"
  - $\frac{d}{d\epsilon}\mathbb{P}[\tau^{\phi+\epsilon\beta} > t]\Big|_{\epsilon=0} = \mathbb{E}\Big[\mathbf{1}_{\tau^\phi > t}\int_0^t \beta_s(X_s^\phi)dW_s\Big].$

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  - $\lambda_t = \lim_{\epsilon\searrow 0}\frac{\mu_t^{\phi+\epsilon\beta}-\mu_t^\phi}{\epsilon}$ exists as a mass 0 finite signed measure λ_t which satisfies

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- **Gateaux derivative of the objective function ???**

$$\frac{d}{d\epsilon}J(\phi+\epsilon\boldsymbol{\beta})\Big|_{\epsilon=0} = \int_0^T <\beta_t(\nabla u_t + \phi_t), \mu_t> \ dt.$$ **presumably not**

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- if $\boldsymbol{\phi}$ is a **critical point**, **do we still have?**

$$\phi_t(x) = -\nabla u_t(x), \qquad \mu_t - a.s.\ x\in\mathbb{R}^d, \quad a.e.\ t\in[0,T].$$ **presumably not**