The Mathematics of Continuous Time Contract Theory

Nizar Touzi

Ecole Polytechnique, France

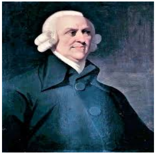
University of Michigan, April 3, 2018

Outline

1. Introduction to moral hazard
2. Mathematical models and examples
   - Static models in contract theory
   - Continuous time contract theory
   - Examples
3. General solution of the Principal-Agent problem
   - General formulation
   - From Stackelberg game to control problem
   - Proof through nonlinear representation

Moral hazard

Adam Smith (1723-1790), "Scottish economist, philosopher and author, moral philosopher, a pioneer of political economy"



Moral hazard is a major risk in economics

• Moral hazard : **Situation where an agent may benefit from an action whose cost is supported by others**

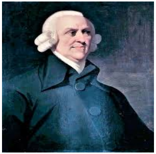
Should not count on agents' morality

Relationship between economic agents

- should instead be based on mutual interest
- each actor acts according to the incentives that are perceived

Moral hazard

Adam Smith (1723-1790), "Scottish economist, philosopher and author, moral philosopher, a pioneer of political economy"



Moral hazard is a major risk in economics

• Moral hazard : **Situation where an agent may benefit from an action whose cost is supported by others**

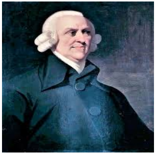
Should not count on agents' morality

Relationship between economic agents

- should instead be based on mutual interest
- each actor acts according to the incentives that are perceived

Moral hazard

Adam Smith (1723-1790), "Scottish economist, philosopher and author, moral philosopher, a pioneer of political economy"



Moral hazard is a major risk in economics

• Moral hazard : **Situation where an agent may benefit from an action whose cost is supported by others**

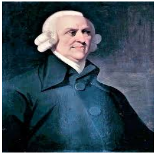
Should not count on agents' morality

Relationship between economic agents

- should instead be based on mutual interest
- each actor acts according to the incentives that are perceived

Moral hazard

Adam Smith (1723-1790), "Scottish economist, philosopher and author, moral philosopher, a pioneer of political economy"



Moral hazard is a major risk in economics

• Moral hazard : **Situation where an agent may benefit from an action whose cost is supported by others**

Should not count on agents' morality

Relationship between economic agents

- should instead be based on mutual interest
- each actor acts according to the incentives that are perceived

Moral hazard in real life

Moral hazard in real life

Moral hazard in real life

Moral hazard in real life

Resolution of moral hazard by contracting

In all above examples, moral hazard addressed in our everyday life :

- Employee receives a salary indexed on performance
- Driver pays insurance premium indexed on sinistrality record
- Manager compensation indexed on performance **AND** risk
- Regulation introduces taxes indexed on the polluting externalities of firms, subsidies for green transition, carbon market thus introducing a price for pollution

HOWEVER designing incentive regulation for some purpose can be misleading and subtle...

Resolution of moral hazard by contracting

In all above examples, moral hazard addressed in our everyday life :

- Employee receives a salary indexed on performance
- Driver pays insurance premium indexed on sinistrality record
- Manager compensation indexed on performance **AND** risk
- Regulation introduces taxes indexed on the polluting externalities of firms, subsidies for green transition, carbon market thus introducing a price for pollution

HOWEVER designing incentive regulation for some purpose can be misleading and subtle...

Moral hazard in insurance

- Fully insured drivers do not have incentive to increase caution

- Health insurance misleadingly lowers out-of-pocket expenses, and thus increases demand for medical services

- Un-employment insurance may give negative incentive to return to the job market

Balancing electricity grids systems by active demand

Peak-Time Rebates (PTR) : paying customers to reduce their electricity demand with respect to past reference consumption

$\Longrightarrow$ Make money out of nothing !

Baltimore Baseball Field
fined in 2013 by Federal Energy Regulation Commission (FERC)

Outline

1. Introduction to moral hazard
2. Mathematical models and examples
   - Static models in contract theory
   - Continuous time contract theory
   - Examples
3. General solution of the Principal-Agent problem
   - General formulation
   - From Stackelberg game to control problem
   - Proof through nonlinear representation

Delegation problem

X : value of an output process owned by Principal

Agent devotes effort a, thus impacting distribution of $X \Longrightarrow X^a$

- cost of effort $c(a)$
- compensation ξ : contract

Choose ξ so that Agent devotes effort in the interest of Principal

First best contract : Principal has full power, up to participation level R

$$\sup_{a,\xi} \left\{ \mathbb{E}\, U_A(\xi - c(a)) : \mathbb{E}\, U_P(-\xi + X^a) \geq R \right\}$$

Centralized economy : Social Planner organizes Principal-Agent relation

$$\sup_{a,\xi} \lambda\, \mathbb{E}\, U_A(\xi - c(a)) + (1-\lambda)\, \mathbb{E}\, U_P(-\xi + X^a)$$

Delegation problem

X : value of an output process owned by Principal

Agent devotes effort a, thus impacting distribution of $X \Longrightarrow X^a$

- cost of effort $c(a)$
- compensation ξ : contract

Choose ξ so that Agent devotes effort in the interest of Principal

First best contract : Principal has full power, up to participation level R

$$\sup_{a,\xi} \left\{ \mathbb{E}\, U_A(\xi - c(a)) : \mathbb{E}\, U_P(-\xi + X^a) \geq R \right\}$$

Centralized economy : Social Planner organizes Principal-Agent relation

$$\sup_{a,\xi} \lambda\, \mathbb{E}\, U_A(\xi - c(a)) + (1-\lambda)\, \mathbb{E}\, U_P(-\xi + X^a)$$

Delegation problem

X : value of an output process owned by Principal

Agent devotes effort a, thus impacting distribution of $X \Longrightarrow X^a$

- cost of effort $c(a)$
- compensation ξ : contract

Choose ξ so that Agent devotes effort in the interest of Principal

First best contract : Principal has full power, up to participation level R

$$\sup_{a,\xi} \left\{ \mathbb{E}\, U_A(\xi - c(a)) : \mathbb{E}\, U_P(-\xi + X^a) \geq R \right\}$$

Centralized economy : Social Planner organizes Principal-Agent relation

$$\sup_{a,\xi} \lambda\, \mathbb{E}\, U_A(\xi - c(a)) + (1-\lambda)\, \mathbb{E}\, U_P(-\xi + X^a)$$

Second best contract : Principal-Agent Problem

- Principal delegates management of output process X,
  only observes X

- Agent devotes effort $a \Longrightarrow X^a$, chooses optimal effort by

$$V_A := \max_a \mathbb{E}\, U_A(\qquad - c(a))$$

(Static) Principal-Agent Problem

- Principal delegates management of output process X,
  only observes X
  pays salary defined by contract $\xi(X)$

- Agent devotes effort $a \Longrightarrow X^a$, chooses optimal effort by

$$V_A(\xi) := \max_a \mathbb{E}\, U_A\big(\xi(X^a) - c(a)\big) \quad \Longrightarrow \quad \hat{a}(\xi)$$

- Principal chooses optimal contract by solving

$$\max_\xi \mathbb{E}\, U_P\big(X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)})\big) \quad \text{under constraint} \quad V_A(\xi) \geq R$$

$\Longrightarrow$ **Non-zero sum Stackelberg game**

Difficult to solve, ... restrict to affine $\xi(x) := a + bx$

(Static) Principal-Agent Problem

- Principal delegates management of output process X,
  only observes X
  pays salary defined by contract $\xi(X)$

- Agent devotes effort $a \Longrightarrow X^a$, chooses optimal effort by

$$V_A(\xi) := \max_a \mathbb{E}\, U_A\big(\xi(X^a) - c(a)\big) \quad \Longrightarrow \quad \hat{a}(\xi)$$

- Principal chooses optimal contract by solving

$$\max_\xi \mathbb{E}\, U_P\big(X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)})\big) \quad \text{under constraint} \quad V_A(\xi) \geq R$$

$\Longrightarrow$ Non-zero sum Stackelberg game

Difficult to solve, ... restrict to affine $\xi(x) := a + bx$

(Static) Principal-Agent Problem

- Principal delegates management of output process X,
  only observes X
  pays salary defined by contract $\xi(X)$

- Agent devotes effort $a \Longrightarrow X^a$, chooses optimal effort by

$$V_A(\xi) := \max_a \mathbb{E}\, U_A\big(\xi(X^a) - c(a)\big) \quad \Longrightarrow \quad \hat{a}(\xi)$$

- Principal chooses optimal contract by solving

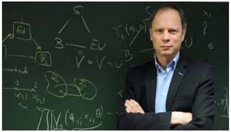
$$\max_\xi \mathbb{E}\, U_P\big(X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)})\big) \quad \text{under constraint} \quad V_A(\xi) \geq R$$

$\Longrightarrow$ **Non-zero sum Stackelberg game**

Difficult to solve, ... restrict to affine $\xi(x) := a + bx$

Contract theory at the heart of modern economic theory

Nobel Prize 2014 winner : Jean Tirole



Social interactions, incentives induced by well-designed contracts

- Industrial organization
- Organization theory
- Regulation

From static to dynamic Principal-Agent problem

Principal/Agent problem turns out to be

More accessible in continuous time [Holmström & Milgrom '85]

Nobel Prize 2016 winners : Oliver Hart and Bengt Holmström

Cvitanić & Zhang ('12, Book), Sannikov '08 : new approach
Throughout literature : simple drift control and no diffusion control

Holmström & Milgrom '85

- Agent solves the control problem :

$$V_0^A(\xi) := \sup_\alpha \mathbb{E}\Big[\xi \quad - \int_0^T \frac{1}{2}\alpha_t^2 dt\Big]$$

where $\alpha \in \{\mathbb{F}-\text{adapted valued in } \mathbb{R}\}$, and output process :

$$dX_t = \alpha_t dt + dW_t$$

- Given solution $\alpha^*(\xi)$, Principal solves the optimization problem

$$V_0^P := \sup_{\xi \in \Xi_R} \mathbb{E}\Big[X_T^{\alpha^*(\xi)} - \xi \quad \Big]$$

where $\Xi_R := \{\xi : \mathcal{F}_T^X - \text{measurable}, \text{ such that } V_0^A(\xi) \geq R\}$

Holmström & Milgrom '85

- Agent solves the control problem :

$$V_0^A(\xi) := \sup_{\alpha} \mathbb{E}\Big[\xi \quad - \int_0^T \frac{1}{2}\alpha_t^2 dt\Big]$$

where $\alpha \in \{\mathbb{F}-\text{adapted valued in } \mathbb{R}\}$, and output process :

$$dX_t = \alpha_t dt + dW_t$$

- Given solution $\alpha^*(\xi)$, Principal solves the optimization problem

$$V_0^P := \sup_{\xi \in \Xi_R} \mathbb{E}\left[X_T^{\alpha^*(\xi)} - \xi \quad \right]$$

where $\Xi_R := \{\xi : \mathcal{F}_T^X - \text{measurable}, \text{ such that } V_0^A(\xi) \geq R\}$

Holmström & Milgrom '85

- Agent solves the control problem :

$$V_0^A(\xi) := \sup_{\alpha} \mathbb{E}\Big[\xi(X) - \int_0^T \frac{1}{2}\alpha_t^2 dt\Big]$$

where $\alpha \in \{\mathbb{F}-\text{adapted valued in } \mathbb{R}\}$, and output process :

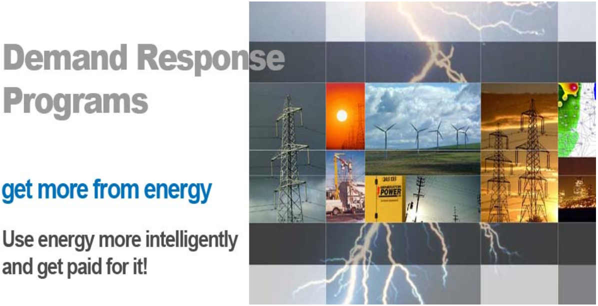
$$dX_t = \alpha_t dt + dW_t$$

- Given solution $\alpha^*(\xi)$, Principal solves the optimization problem

$$V_0^P := \sup_{\xi \in \Xi_R} \mathbb{E}\Big[X_T^{\alpha^*(\xi)} - \xi(X)\Big]$$

where $\Xi_R := \{\xi : \mathcal{F}_T^X - \text{measurable, such that } V_0^A(\xi) \geq R\}$

Application to Demand-Response programs



45 billions Euros for an objective of 200 millions smart meters

London Demand response program : participation induced a decrease of 4% customers annual bill, i.e. 20 GBP out of 500.

Demand-Response programs in electricity tarification

• Output process : electricity demand under effort α

$$dX_t \quad = \quad -\alpha_t dt + dW_t$$

Agent reduces consumption at high peak times by solving

$$V_0^A(\xi) := \sup_{\alpha} \mathbb{E}\Big[\xi(X) + \int_0^T \big(f(X_t) - \frac{1}{2}\alpha_t^2\big) dt\Big]$$

• Given solution $\alpha^*(\xi)$, Principal solves the optimization problem

$$V_0^P \quad := \quad \sup_{\xi \in \Xi_R} \mathbb{E}^{\alpha^*}\Big[-\xi(X) - \int_0^T \big(\pi X_t - g(X_t)\big) dt\Big]$$

where $\Xi_R := \big\{\xi : \mathcal{F}_T^X - \text{measurable, such that } V_0^A(\xi) \geq R\big\}$

Demand-Response programmes with diffusion control

- Output process : electricity demand under effort α, β

$$dX_t \quad = \quad -\alpha_t dt + \beta_t^{-1} dW_t, \quad \beta = \text{Agent responsiveness}$$

Agent reduces consumption and volatility by solving

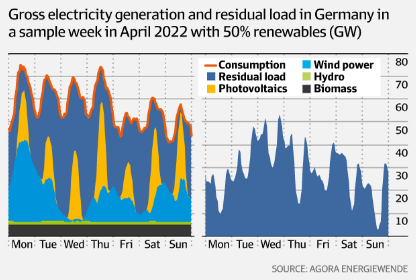
$$V_0^A(\xi) := \sup_{\alpha,\beta} \mathbb{E}\Big[\xi(X) + \int_0^T \Big(f(X_t) - \frac{1}{2}\alpha_t^2 - \frac{1}{2}\beta_t^2\Big) dt\Big]$$

- Given solution $(\alpha^*, \beta^*)(\xi)$, Principal solves

$$V_0^P \quad := \quad \sup_{\xi \in \Xi_R} \mathbb{E}^{\alpha^*,\beta^*}\Big[-\xi(X) - \int_0^T \big(\pi X_t - g(X_t)\big) dt\Big]$$

where $\Xi_R := \big\{\xi : \mathcal{F}_T^X - \text{measurable}, \text{ such that } V_0^A(\xi) \geq R\big\}$

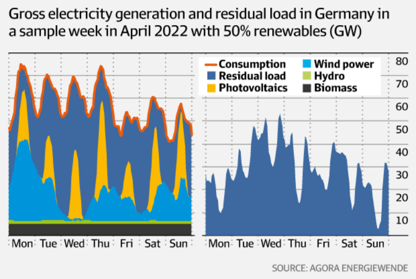
Another interpretation of volatility in electricity market



- Demand in excess of "free energy" exhibits high volatility
- High cost to match the supply to highly volatile excess demand

- Technological solution : acting on the supply side by using high capacity batteries for electricity storage
- Demand-Response : acting on the demand side by optimal contracting

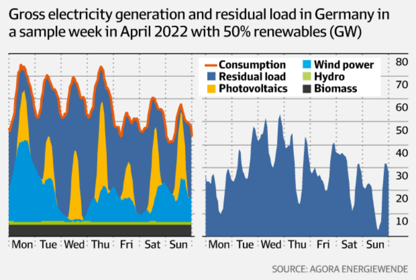
Another interpretation of volatility in electricity market



- Demand in excess of "free energy" exhibits high volatility
- High cost to match the supply to highly volatile excess demand

- Technological solution : acting on the supply side by using high capacity batteries for electricity storage
- Demand-Response : acting on the demand side by optimal contracting

Another interpretation of volatility in electricity market



- Demand in excess of "free energy" exhibits high volatility
- High cost to match the supply to highly volatile excess demand

- Technological solution : acting on the supply side by using high capacity batteries for electricity storage
- Demand-Response : acting on the demand side by optimal contracting

Another example : Portfolio managers compensation

- Output process : portfolio value under effort π_t

$$dX_t \quad = \quad \pi_t \frac{dS_t}{S_t} \;=\; \pi_t(\lambda dt + dW_t)$$

Portfolio manager chooses portfolio allocation by solving

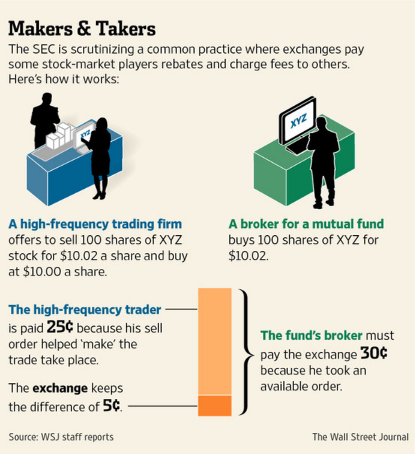
$$V_0^A(\xi) := \sup_{\alpha} \mathbb{E}\Big[\xi(X) - \int_0^T \frac{1}{2}\pi_t^2 dt\Big]$$

- Given solution $\pi^*(\xi)$, investor solves the optimization problem

$$V_0^P \quad := \quad \sup_{\xi \in \Xi_R} \mathbb{E}^{\pi^*} U_P\big(-\xi(X) + X_T\big)$$

where $\Xi_R := \big\{\xi : \mathcal{F}_T^X - \text{measurable}, \text{ such that } V_0^A(\xi) \geq R\big\}$

Yet, another example : Makers-Takers fees... towards Fintech



Modeling Makers-Takers fees

- Fundamental price $\{S_t\}_{t\geq 0}$, **Market Maker** posts bid-ask prices

$$p_t^b = S_t - \delta_t^b \quad \text{and} \quad p_t^a = S_t + \delta_t^a$$

- MM inventory $q_t = N_t^b - N_t^a$, where N_t^b, N_t^a point process with unit jumps representing the arrivals of bid-ask order with intensities

$$\lambda_t^b = \lambda(\delta_t^b) \;\text{ and }\; \lambda_t^a = \lambda(\delta_t^a), \;\text{ with }\; \lambda(x) = Ae^{-kx}$$

- MM chooses the departure from fundamental price :

$$V_A(\xi) \;:=\; \sup_{\delta^b,\delta^a} \mathbb{E}\, U_A\Big(\xi + \int_0^T p_t^a dN_t^a - p_t^b dN_t^b - q_t S_t\Big)$$

- Given optimal response $\delta^*(\xi)$, **Platform** chooses optimal contract

$$V_P \;=\; \sup_{\xi\in\Xi_R} \mathbb{E}\, U_P\big(-\xi + c(N_T^a + N_T^b)\big)$$

Modeling Makers-Takers fees

- Fundamental price $\{S_t\}_{t\geq 0}$, **Market Maker** posts bid-ask prices

$$p_t^b = S_t - \delta_t^b \quad \text{and} \quad p_t^a = S_t + \delta_t^a$$

- MM inventory $q_t = N_t^b - N_t^a$, where N_t^b, N_t^a point process with unit jumps representing the arrivals of bid-ask order with intensities

$$\lambda_t^b = \lambda(\delta_t^b) \;\; \text{and} \;\; \lambda_t^a = \lambda(\delta_t^a), \;\; \text{with} \;\; \lambda(x) = Ae^{-kx}$$

- MM chooses the departure from fundamental price :

$$V_A(\xi) \;\; := \;\; \sup_{\delta^b,\delta^a} \mathbb{E}\, U_A\Big(\xi + \int_0^T p_t^a dN_t^a - p_t^b dN_t^b - q_t S_t\Big)$$

- Given optimal response $\delta^*(\xi)$, **Platform** chooses optimal contract

$$V_P \;\; = \;\; \sup_{\xi \in \Xi_R} \mathbb{E}\, U_P\big(-\xi + c(N_T^a + N_T^b)\big)$$

Modeling Makers-Takers fees

- Fundamental price $\{S_t\}_{t\geq 0}$, **Market Maker** posts bid-ask prices

$$p^b_t = S_t - \delta^b_t \quad \text{and} \quad p^a_t = S_t + \delta^a_t$$

- MM inventory $q_t = N^b_t - N^a_t$, where N^b_t, N^a_t point process with unit jumps representing the arrivals of bid-ask order with intensities

$$\lambda^b_t = \lambda(\delta^b_t) \;\text{ and }\; \lambda^a_t = \lambda(\delta^a_t), \quad \text{with} \quad \lambda(x) = Ae^{-kx}$$

- MM chooses the departure from fundamental price :

$$V_A(\xi) \;:=\; \sup_{\delta^b,\delta^a} \mathbb{E}\, U_A\Big(\xi + \int_0^T p^a_t dN^a_t - p^b_t dN^b_t - q_t S_t\Big)$$

- Given optimal response $\delta^*(\xi)$, **Platform** chooses optimal contract

$$V_P \;=\; \sup_{\xi \in \Xi_R} \mathbb{E}\, U_P\big(-\xi + c(N^a_T + N^b_T)\big)$$

Modeling Makers-Takers fees

- Fundamental price $\{S_t\}_{t\geq 0}$, **Market Maker** posts bid-ask prices

$$p_t^b = S_t - \delta_t^b \quad \text{and} \quad p_t^a = S_t + \delta_t^a$$

- MM inventory $q_t = N_t^b - N_t^a$, where N_t^b, N_t^a point process with unit jumps representing the arrivals of bid-ask order with intensities

$$\lambda_t^b = \lambda(\delta_t^b) \;\; \text{and} \;\; \lambda_t^a = \lambda(\delta_t^a), \;\; \text{with} \;\; \lambda(x) = Ae^{-kx}$$

- MM chooses the departure from fundamental price :

$$V_A(\xi) \;\; := \;\; \sup_{\delta^b,\delta^a} \mathbb{E}\, U_A\Big(\xi + \int_0^T p_t^a dN_t^a - p_t^b dN_t^b - q_t S_t\Big)$$

- Given optimal response $\delta^*(\xi)$, **Platform** chooses optimal contract

$$V_P \;\; = \;\; \sup_{\xi\in\Xi_R} \mathbb{E}\, U_P\big(-\xi + c(N_T^a + N_T^b)\big)$$

Outline

Continuous time Principal-Agent problem

Fix $\Omega := C^0([0,T],\mathbb{R}^d)$, paths space for output process X, and let

$$V_0^A(\xi) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[K_T\xi - \int_0^T K_t c_t(\nu_t^{\mathbb{P}})dt\Big], \quad K_t := e^{-\int_0^t k_s^\nu ds}$$

where $\mathbb{P}\in\mathcal{P}$ weak solution of Output process :

$$dX_t = b_t(X,\nu_t)dt + \sigma_t(X,\nu_t)dW_t^{\mathbb{P}} \quad \mathbb{P}-\text{a.s.}$$

- Given solution $\mathbb{P}^*(\xi)$, Principal solves the optimization problem

$$V_0^P := \sup_{\xi\in\Xi_R} \mathbb{E}^{\mathbb{P}^*(\xi)}\Big[H_T U(\ell(X)-\xi(X))\Big], \quad H_t := e^{-\int_0^t h_s ds}$$

where $\Xi_R := \{\xi(X_.)$, such that $V_0^A(\xi)\geq R\}$

Principal-Agent problem : more natural formulation

Fix $\Omega := C^0([0,T],\mathbb{R}^d)$, paths space for output process X, and let

$$V_0^A(\xi) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[K_T\xi - \int_0^T K_t c_t(\nu_t^{\mathbb{P}})dt\Big], \quad K_t^{\nu} := e^{-\int_0^t k_s^{\nu} ds}$$

where $\mathbb{P}\in\mathcal{P}$ weak solution of Output process :

$$dX_t \;=\; \sigma_t(X,\beta_t)\big[\lambda_t(X,\alpha_t)dt + dW_t^{\mathbb{P}}\big] \;\; \mathbb{P}-\text{a.s.}$$

- Given solution $\mathbb{P}^*(\xi)$, Principal solves the optimization problem

$$V_0^P \;:=\; \sup_{\xi\in\Xi_R} \mathbb{E}^{\mathbb{P}^*(\xi)}\Big[H_T U(\ell(X)-\xi(X))\Big], \quad H_t := e^{-\int_0^t h_s ds}$$

where $\Xi_R := \{\xi(X_.),\text{ such that } V_0^A(\xi)\geq R\}$

Principal-Agent problem : more natural formulation

Fix $\Omega := C^0([0,T],\mathbb{R}^d)$, paths space for output process X, and let

$$V_0^A(\xi) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[K_T\xi - \int_0^T K_t c_t(\nu_t^{\mathbb{P}})dt\Big], \quad K_t^\nu := e^{-\int_0^t k_s^\nu ds}$$

where $\mathbb{P}\in\mathcal{P}$ weak solution of Output process :

$$dX_t = \sigma_t(\beta_t)\big[\lambda_t(\alpha_t)dt + dW_t^{\mathbb{P}}\big], \quad \mathbb{P}-\text{a.s.}$$

- Given solution $\mathbb{P}^*(\xi)$, Principal solves the optimization problem

$$V_0^P := \sup_{\xi\in\Xi_R} \mathbb{E}^{\mathbb{P}^*(\xi)}\Big[K_T U(\ell - \xi)\Big], \quad H_t := e^{-\int_0^t h_s ds}$$

where $\Xi_R := \{\xi, \text{ such that } V_0^A(\xi) \geq R\}$

Main result

- Path-dependent Hamiltonian for the Agent problem :

$$H_t(\omega, y, z, \gamma) := \sup_u \Big\{ b_t(\omega, u) \cdot z + \frac{1}{2}\sigma_t\sigma_t^\top(\omega, u) : \gamma - k_t(\omega, a, b)y - c_t(\omega, a, b) \Big\}$$

Theorem

Under some technical conditions, the optimal contract is of the form $\xi^* = Y_T^{Z^*,\Gamma^*}$, *with*

$$Y_t = Y_0^* + \int_0^t Z_s^* \cdot dX_s + \frac{1}{2}\Gamma_s^* : d\langle X \rangle_s - H_s(Y_s^{Z^*,\Gamma^*}, Z_s^*, \Gamma_s^*)ds$$

where (Y_0^*, Z^*, Γ^*) *is the solution of a* **standard stochastic control problem**

The stochastic control problem

$\bar{X} := (X, Y)$ satisfies $d\bar{X}_t = \bar{\mu}(\bar{X}_t, Z_t, \Gamma_t)dt + \bar{\sigma}(\bar{X}_t, Z_t, \Gamma_t)dW_t$:

$$dX_t = \nabla_z H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt + \left\{2\nabla_\gamma H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)\right\}^{\frac{1}{2}} dW_t$$
$$dY_t^{Z,\Gamma} = Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt$$

$$V_P = \sup_{Y_0 \geq R} V_0(X_0, Y_0), \text{ where } \quad V_0(\bar{X}_0) := \sup_{Z,\Gamma} \mathbb{E}\left[U(\ell(X_T) - Y_T^{Z,\Gamma})\right]$$

$V_0(\bar{X}_0) = V(0, \bar{X}_0)$; V solution of Hamilton-Jacobi-Bellman eq.

$$\begin{aligned} \partial_t V_0 + \sup_{z,\gamma}\left\{\bar{\mu}(.,z,\gamma)DV_0 + \frac{1}{2}\overline{\sigma\sigma}^\top(.,z,\gamma) : D^2V_0\right]\right\} &= 0 \\ V_0(T,x,y) &= U(\ell(x) - y) \end{aligned}$$

The stochastic control problem

$\bar{X} := (X, Y)$ satisfies $d\bar{X}_t = \bar{\mu}(\bar{X}_t, Z_t, \Gamma_t)dt + \bar{\sigma}(\bar{X}_t, Z_t, \Gamma_t)dW_t$:

$$dX_t = \nabla_z H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt + \{2\nabla_\gamma H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)\}^{\frac{1}{2}} dW_t$$

$$dY_t^{Z,\Gamma} = Z_t \cdot dX_t + \tfrac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt$$

$$V_P = \sup_{Y_0 \geq R} V_0(X_0, Y_0), \text{ where } \quad V_0(\bar{X}_0) := \sup_{Z,\Gamma} \mathbb{E}\Big[U(\ell(X_T) - Y_T^{Z,\Gamma})\Big]$$

$V_0(\bar{X}_0) = V(0, \bar{X}_0)$; V solution of Hamilton-Jacobi-Bellman eq.

$$\partial_t V_0 + \sup_{z,\gamma}\Big\{\bar{\mu}(., z, \gamma)DV_0 + \tfrac{1}{2}\overline{\sigma\sigma}^\top(., z, \gamma) : D^2 V_0\Big]\Big\} = 0$$

$$V_0(T, x, y) = U(\ell(x) - y)$$

Sufficient conditions

To prove the main result, it suffices that

$$\text{for all } \xi \in ?? \quad \exists\,(Y_0, Z, \Gamma) \quad \text{s.t.} \quad \xi = Y_T^{Z,\Gamma}, \ \mathbb{P}-\text{a.s. for all } \mathbb{P} \in \mathcal{P}$$

where we recall that

$$Y_t = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2}\Gamma_s : d\langle X\rangle_s - H_s(Y_s^{Z,\Gamma}, Z_s, \Gamma_s)ds$$

OR, weaker sufficient condition :

$$\text{for all } \xi \in ?? \quad \exists\,(Y_0^n, Z^n, \Gamma^n) \quad \text{s.t.} \quad \text{“}Y_T^{Z^n,\Gamma^n} \longrightarrow \xi\text{”}$$

Sufficient conditions

To prove the main result, it suffices that

$$\text{for all } \xi \in ?? \quad \exists\, (Y_0, Z, \Gamma) \quad \text{s.t. } \; \xi = Y_T^{Z,\Gamma}, \; \mathbb{P}-\text{a.s. for all } \mathbb{P} \in \mathcal{P}$$

where we recall that

$$Y_t = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2}\Gamma_s : d\langle X\rangle_s - H_s(Y_s^{Z,\Gamma}, Z_s, \Gamma_s)ds$$

OR, weaker sufficient condition :

$$\text{for all } \xi \in ?? \quad \exists\, (Y_0^n, Z^n, \Gamma^n) \quad \text{s.t. } \; \text{“} Y_T^{Z^n,\Gamma^n} \longrightarrow \xi \text{”}$$

Intuitions for the representation problem

Solve $\xi = Y_T^{Z,\Gamma}$, $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}$, where

$$Y_t = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2}\Gamma_s : d\langle X\rangle_s - H_s(Y_s^{Z,\Gamma}, Z_s, \Gamma_s)ds$$

• $H \equiv 0$: then $\Gamma \equiv 0$, $\mathcal{P} = \{\text{Wiener measure}\}$, and the problem reduces to

Path-dependent heat equation

• H linear in γ : then Γ is arbitrary, and $\mathcal{P}$ dominated family of measures...

Path-dependent semilinear parabolic equation

• General case :

Path-dependent fully nonlinear parabolic equation (HJB)

Extension to multiple agents

- 1 Principal and N possibly interacting Agents,
- 1 Principal and continuum mean field interacting Agents,

- N possibly interacting Principals (or even continuum) and 1 Agent

- Another open problem in economics since Holström & Milgrom : Limited liability restriction

$\Longrightarrow$ State constraint in the final stochastic control problem

Adverse selection

- Agent preferences U and c depend on a parameter $\theta \in \Theta$ called "type", which is unknown to Principal

- Principal only knows distribution of types $\mu(d\theta)$ on Θ

$\Longrightarrow$ design tarification so as to push Agent to "reveal its type"

Example : car dealers offer different qualities for the same model...

Adverse selection : work in progress...

Principal offers a menu of contracts $\{\xi^\theta, \theta \in \Theta\}$.

Problem of Agent θ choosing contract $\xi^{\theta'}$ is :

$$V_A(\theta, \xi^{\theta'}) := \sup_\nu \mathbb{E}^\nu \Big[\xi^{\theta'} - \int_0^T c(\theta, \nu_t) dt \Big]$$

Principal's problem is

$$\sup_{\{\xi^\theta\} \in \mathrm{ICC}} \int_\Theta \mathbb{E}^{\nu^*(\xi^\theta)} \big[U_P(X_T - \xi^\theta) \big] \, \mu(d\theta)$$

where $\{\xi^\theta\} \in$ ICC if satisfies participation constraint and

incentive compatibility : $V_A(\theta, \xi^{\theta'}) \le V_A(\theta, \xi^\theta)$ for all $\theta, \theta' \in \Theta$

Adverse selection : work in progress...

Principal offers a menu of contracts $\{\xi^\theta, \theta \in \Theta\}$.

Problem of Agent θ choosing contract $\xi^{\theta'}$ is :

$$V_A(\theta, \xi^{\theta'}) := \sup_\nu \mathbb{E}^\nu\Big[\xi^{\theta'} - \int_0^T c(\theta, \nu_t)dt\Big]$$

Principal's problem is

$$\sup_{\{\xi^\theta\}\in \mathrm{ICC}} \int_\Theta \mathbb{E}^{\nu^*(\xi^\theta)}\big[U_P(X_T - \xi^\theta)\big]\ \mu(d\theta)$$

where $\{\xi^\theta\} \in$ ICC if satisfies participation constraint and

incentive compatibility : $V_A(\theta, \xi^{\theta'}) \leq V_A(\theta, \xi^\theta)$ for all $\theta, \theta' \in \Theta$

Social interactions : a new challenge to mathematics

Smart grids and smart cities