Nonlinear representation, backward SDEs, and application to the Principal-Agent problem

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Outline

(Static) Principal-Agent Problem

- Principal delegates management of output process X,
  only observes X

- Agent devotes effort $a \Longrightarrow X^a$, chooses optimal effort by

$$V_A := \max_a \mathbb{E}\, U_A\big(\qquad - c(a)\big)$$

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  only observes X
  pays salary defined by contract $\xi(X)$

- Agent devotes effort $a \Longrightarrow X^a$, chooses optimal effort by

$$V_A(\xi) := \max_a \mathbb{E}\, U_A\big(\xi(X^a) - c(a)\big) \quad \Longrightarrow \quad \hat{a}(\xi)$$

- Principal chooses optimal contract by solving

$$\max_\xi \mathbb{E}\, U_P\big(X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)})\big) \quad \text{under constraint} \quad V_A(\xi) \geq \rho$$

$\Longrightarrow$ Non-zero sum Stackelberg game

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$\Longrightarrow$ **Non-zero sum Stackelberg game**

Principal-Agent problem formulation

Agent problem :

$$V_0^A(\xi) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[K_T^\nu \xi(X) - \int_0^T K_t^\nu c_t(\nu_t)dt\Big], \quad K_t^\nu := e^{-\int_0^t k_s^\nu ds}$$

$\mathbb{P}\in\mathcal{P}$: weak solution of Output process for some ν valued in U :

$$dX_t = b_t(X,\nu_t)dt + \sigma_t(X,\nu_t)dW_t^{\mathbb{P}} \quad \mathbb{P}-\text{a.s.}$$

- Given solution $\mathbb{P}^*(\xi)$, Principal solves the optimization problem

$$V_0^P := \sup_{\xi\in\Xi_\rho} \mathbb{E}^{\mathbb{P}^*(\xi)}\Big[R_T U\big(\ell(X)-\xi(X)\big)\Big], \quad R_t := e^{-\int_0^t r_s ds}$$

$$\text{where} \quad \Xi_\rho := \big\{\xi(X) : V_0^A(\xi)\geq\rho\big\}$$

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Principal-Agent problem formulation : non-degeneracy

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$$where \;\; \Xi_\rho \;:=\; \big\{\xi(X) : \; V_0^A(\xi) \geq \rho\big\}$$

A subset of revealing contracts

- Path-dependent Hamiltonian for the Agent problem :

$$H_t(\omega, y, z, \gamma) := \sup_u \Big\{ (\sigma_t\lambda_t)(\omega, u)\cdot z + \tfrac{1}{2}\sigma_t\sigma_t^\top(\omega, u) : \gamma - k_t(\omega, u)y - c_t(\omega, u) \Big\}$$

- For $Y_0 \in \mathbb{R}$ and Z, Γ $\mathbb{F}^X$ $-$ prog meas, define

$$Y_t^{Z,\Gamma} = Y_0 + \int_0^T Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt, \mathcal{P} - \text{q.s.}$$

Proposition $V_A(Y_T^{Z,\Gamma}) = Y_0$. Moreover $\mathbb{P}^*$ is optimal iff

$$\nu_t^* = \operatorname*{Argmax}_{u \in U} H_t(Y_t, Z_t, \Gamma_t) = \hat{\nu}(Y_t, Z_t, \Gamma_t)$$

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Proof : classical verification argument !

For all $\mathbb{P} \in \mathcal{P}$, denote $J_A(\xi, \mathbb{P}) := \mathbb{E}^{\mathbb{P}}\big[K_T^\nu \xi - \int_0^T K_t^\nu c_t^\nu dt\big]$. Then

$$J_A\big(Y_T^{Z,\Gamma}, \mathbb{P}\big) = \mathbb{E}^{\mathbb{P}}\Big[K_T^\nu\Big\{Y_0 + \int_0^T Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(Y_t, Z_t, \Gamma_t)dt\Big\} - \int_0^T K_t^\nu c_t^\nu dt\Big]$$

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&\qquad - \int_0^T K_t^\nu c_t^\nu dt\Big] \\
&= Y_0 + \mathbb{E}^{\mathbb{P}}\Big[\int_0^T K_t^\nu \Big\{-H_t(Y_t, Z_t, \Gamma_t) - k_t^\nu Y_t \\
&\qquad + b_t^\nu \cdot Z_t + \frac{1}{2}\sigma\sigma^\top : \Gamma_t - c_t^\nu\Big\}dt + Z_t \cdot \sigma_t^\nu dW_t^{\mathbb{P}}\Big]
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$$= Y_0 + \mathbb{E}^{\mathbb{P}}\Big[\int_0^T K_t^\nu\Big\{-H_t(Y_t, Z_t, \Gamma_t) - k_t^\nu Y_t + b_t^\nu \cdot Z_t + \frac{1}{2}\sigma\sigma^\top : \Gamma_t - c_t^\nu\Big\}dt\Big]$$

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&\qquad + b_t^\nu\cdot Z_t + \frac{1}{2}\sigma\sigma^\top : \Gamma_t - c_t^\nu\Big\}dt\Big] \\
&\leq Y_0 \quad \text{by definition of } H
\end{aligned}
$$

with equality iff ν^* achieves the max of Hamiltonian

Principal problem restricted to revealing contracts

Dynamics of the pair (X, Y) under "optimal response"

$$dX_t = \underbrace{\nabla_z H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)}_{b_t(X,\hat{\nu}(Y_t,Z_t,\Gamma_t))} dt + \underbrace{\left\{2\nabla_\gamma H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)\right\}^{\frac{1}{2}}}_{\sigma_t(X,\hat{\nu}(Y_t,Z_t,\Gamma_t))} dW_t$$

$$dY_t^{Z,\Gamma} = Z_t \cdot dX_t + \tfrac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt$$

$\Longrightarrow$ Principal's value function under revealing contracts :

$$V_P \geq V_0(X_0, Y_0) := \sup_{(Z,\Gamma)\in\mathcal{V}} \mathbb{E}\Big[U\big(\ell(X) - Y_T^{Z,\Gamma}\big)\Big], \quad \text{for all } Y_0 \geq \rho$$

where $\mathcal{V} := \Big\{(Z,\Gamma) : \; Z \in \mathbb{H}^2(\mathcal{P}) \text{ and } \; \mathcal{P}^*\big(Y_T^{Z,\Gamma}\big) \neq \emptyset\Big\}$

Reduction to standard control problem

Theorem (Cvitanić, Possamaï & NT '15)

Assume $\mathcal{V} \neq \emptyset$. Then

$$V_0^P = \sup_{Y_0 \geq \rho} V_0(X_0, Y_0)$$

Given maximizer $Y_0^\star$, the corresponding optimal controls $(Z^\star, \Gamma^\star)$ induce an optimal contract

$$\xi^\star = Y_0^\star + \int_0^T Z_t^\star \cdot dX_t + \frac{1}{2}\Gamma_t^\star : d\langle X\rangle_t - H_t(X, Y_t^{Z^\star,\Gamma^\star}, Z_t^\star, \Gamma_t^\star)dt$$

To prove the main result, it suffices that

$$\text{for all } \xi \in ?? \quad \exists\, (Y_0, Z, \Gamma) \quad \text{s.t.} \quad \xi = Y_T^{Z,\Gamma},\ \mathcal{P} - q.s.$$

OR, weaker sufficient condition :

$$\text{for all } \xi \in ?? \quad \exists\, (Y_0^n, Z^n, \Gamma^n) \quad \text{s.t.} \quad \text{``} Y_T^{Z^n,\Gamma^n} \longrightarrow \xi \text{''}$$

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From fully nonlinear HJB equation to semilinear

- $H_t(\omega, y, z, \gamma)$ non-decreasing and convex in γ, Then

$$H_t(\omega, y, z, \gamma) = \sup_{\sigma \geq 0} \left\{ \frac{1}{2}\sigma^2 : \gamma - H_t^*(\omega, y, z, \sigma) \right\}$$

Denote $k_t := H_t(Y_t, Z_t, \Gamma_t) - \frac{1}{2}\hat{\sigma}_t^2 : \Gamma_t + H_t^*(Y_t, Z_t, \hat{\sigma}_t) \geq 0$

Then, required representation $\xi = Y_T^{Z,\Gamma}$, $\mathcal{P}-$q.s. is **equivalent to**

$$\xi = Y_0 + \int_0^T Z_t \cdot dX_t + H_t^*(Y_t, Z_t, \hat{\sigma}_t)dt - \int_0^T k_t dt, \quad \mathcal{P} - \text{q.s.}$$

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Outline

1. The Principal-Agent problem
   - Formulation
   - Reduction to standard control problem
2. Linear and semilinear representation
3. Fully nonlinear representation
   - Revisiting random horizon backward SDEs
   - Random horizon 2nd order backward SDEs

Linear representation of random variables

Predictable Representation Property of BM

Theorem

For all $\xi \in \mathbb{L}^2(\mathcal{F}_T^W)$, there is a unique (Y, Z) $\mathbb{F}^W-$prog. meas. s.t.

$$\xi = Y_t + \int_t^T Z_s dW_s, \quad \mathbb{P}-a.s. \quad \mathbb{E}\Big[\int_0^T (|Y_t|^2 + |Z_t|^2) dt\Big] < \infty,$$

- For $\xi = g(W_T)$: Heat equation and Itô's formula
- True for $\xi = g(W_{t_1}, \ldots, W_{t_n})$... conclude by density argument

Heat equation with path dependent boundary condition

Connection with $W^{1,2}-$solution of Heat equation

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Semilinear representation of random variables

Here again, $\xi = \xi(W_s, s \le T)$. Let

$$f : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}, \quad \text{Lip in } (y,z), \text{ unif in } (t,\omega), \ \mathbb{E}\Big[\int_0^T |f_s^0|^2 ds\Big] < \infty$$

Theorem (Pardoux & Peng '92)

For all $\xi \in \mathbb{L}^2(\mathcal{F}_T^W)$, there is a unique $\mathbb{F}^W-$prog. meas. (Y, Z), $\mathbb{E}\big[\int_0^T (|Y_t|^2 + |Z_t|^2)dt\big] < \infty$, s.t

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad \mathbb{P} - a.s.$$

Unique fixed point for the Picard iteration

$$(Y, Z) \longmapsto Y'_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z'_s dW_s, \ 0 \le t \le T$$

$W^{1,2}-$type solution of semilinear heat equation with

path-dependent nonlinearity and boundary data

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$W^{1,2}-$**type solution of semilinear heat equation with**

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Backward SDE and semilinear PDE

Rewrite the backward SDE in differential form

$$dY_t = -f_t(Y_t, Z_t)dt + Z_t dW_t, \ \ t \le T, \quad \text{and} \, Y_T = \xi, \ \ \mathbb{P} - \text{a.s.}$$

In the Markovian case $\xi = g(W_T)$ and $f_t(y,z) = f(t, W_t, y, z) \Longrightarrow$

$Y_t = v(t, W_t)$, so that $dY_t = \partial_t v(t, W_t)dt + Dv(t, W_t)\cdot dW_t + \frac{1}{2}\Delta v(t, W_t)dt$

by Itô's formula. Direct identification yields

$$Z_t = Dv(t, W_t) \quad \text{and} \quad \partial_t v(t, W_t) + \frac{1}{2}\Delta v(t, W_t) = -f_t(v, Dv)$$

Backward SDE $\equiv$ path-dependent semilinear PDE with Sobolev-type of regularity [Barles & Lesigne '97]

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Outline

Towards fully nonlinear PDEs : probabilistic framework

In order to cover fully nonlinear PDEs, we need

quasi-sure stochastic analysis...

$\Omega := \{\omega \in C^0(\mathbb{R}_+, \mathbb{R}^d) : \omega(0) = 0\}$

X : canonical process, i.e. $X_t(\omega) := \omega(t)$

$\mathcal{F}_t := \sigma(X_s, s \leq t)$, $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$

$\langle X \rangle$: quadratic variation process (defined on $\mathbb{R}_+ \times \omega$)

$$\langle X \rangle_t := X_t^2 - \int_0^t 2X_s dX_s = \mathbb{P}-\lim_{|\pi|\to 0} \sum_{n\geq 1} \left| X_{t\wedge t_n^\pi} - X_{t\wedge t_{n-1}^\pi} \right|^2$$

for all semimartingale probability measure $\mathbb{P}$ on Ω, and

$$\hat{\sigma}_t^2 := \overline{\lim}_{h\searrow 0} \frac{\langle X \rangle_{t+h} - \langle X \rangle_t}{h}$$

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Semimartingale measures on canonical space

$\mathcal{P}^W$: collection of all semimartingale measures $\mathbb{P}$ such that

$$dX_t = b_t dt + \sigma_t dW_t, \quad \mathbb{P}-\text{a.s.}$$

for some $\mathbb{F}-$processes b and σ, and $\mathbb{P}-$Brownian motion W

Class of prob. meas. on Ω : $\mathcal{P} \subset \mathcal{P}^W$
$\longrightarrow$ sufficiently rich (to satisfy DP properties...)
$\longrightarrow$ if not, enrich it...

- $\mathcal{P}-$quasi-surely MEANS $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}$

Semimartingale measures on canonical space

$\mathcal{P}^W$: collection of all semimartingale measures $\mathbb{P}$ such that

$$dX_t = b_t dt + \sigma_t dW_t, \quad \mathbb{P} - \text{a.s.}$$

for some $\mathbb{F}-$processes b and σ, and $\mathbb{P}-$Brownian motion W

Class of prob. meas. on Ω : $\mathcal{P} \subset \mathcal{P}^W$
$\longrightarrow$ sufficiently rich (to satisfy DP properties...)
$\longrightarrow$ if not, enrich it...

- $\mathcal{P}-$quasi-surely MEANS $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}$

Nonlinear expectation operators

- $\mathcal{P}^0$: subset of local martingale measures, i.e.

$$dX_t = \sigma_t dW_t, \quad \mathbb{P} - \text{a.s. for all} \quad \mathbb{P} \in \mathcal{P}^0$$

$\Longrightarrow$ Nonlinear expectation $\mathcal{E}_t := \sup_{\mathbb{P}\in\mathcal{P}^0} \mathbb{E}_t^{\mathbb{P}}$

- $\mathcal{P}^L(\mathbb{P}) := \Big\{ \mathbb{Q} = \mathcal{E}\big(\int_0^\cdot \lambda_t \cdot dW_t \big) \cdot \mathbb{P} : \; \|\lambda\|_{\mathbb{L}^\infty} \le L \Big\}$

$\Longrightarrow$ Nonlinear expectation $\mathcal{E}_t^{\mathbb{P}} := \sup_{\mathbb{P}'\in\mathcal{P}^L(\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}$

- $\mathcal{P}^L := \cup_{\mathbb{P}\in\mathcal{P}^0} \mathcal{P}^L(\mathbb{P})$

$\Longrightarrow$ Nonlinear expectation $\mathcal{E}_t^L := \sup_{\mathbb{P}\in\mathcal{P}^L} \mathbb{E}_t^{\mathbb{P}}$

Nonlinearity

Assumptions $F : \mathbb{R}_+ \times \omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d_+ \longrightarrow \mathbb{R}$ satisfies

(C1$_L$) Lipschitz in $(y, \sigma z)$:

$$\left|F(., y, z, \sigma) - F(., y', z', \sigma)\right| \le L \left(|y - y'| + \left|\sigma(z - z')\right|\right.$$

(C2$_\mu$) Monotone in y :

$$(y - y') \cdot \left[F(., y, .) - F(., y', .)\right] \le -\mu \, |y - y'|^2$$

Denote $f_t(y, z) := F_t\big(y, z, \widehat{\sigma}_t\big)$ and $f^0_t := f_t(0, 0)$

Wellposedness of random horizon backward SDEs

τ : stopping time, ξ is $\mathcal{F}_\tau-$meas., consider the backward SDE :

$$Y_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} f_s(Y_s, Z_s)ds - \int_{t\wedge\tau}^{\tau} (Z_s \cdot dX_s + dN_s), \quad \mathbb{P}-\text{a.s.}$$

where N martingale with $\langle N, X\rangle = 0$, $\mathbb{P}-$a.s.

Theorem (Y. Lin , Z. Ren, NT & J. Yang '17)

Let $\|\xi\|_{\mathcal{L}^q_{\rho,\tau}(\mathbb{P})} < \infty$, $\overline{f}^{q,\mathbb{P}}_{\rho,\tau} := \mathcal{E}^{\mathbb{P}}\Big[\Big(\int_0^\tau \big|e^{\rho t} f_t^0\big|^2 ds\Big)^{\frac{q}{2}}\Big]^{\frac{1}{q}} < \infty$, *for some* $\rho > -\mu$, $q > 1$. *Then the BSDE has a unique solution with*

$$\|Y\|_{\mathcal{D}^p_{\eta,\tau}(\mathbb{P})} + \|Z\|_{\mathcal{H}^p_{\eta,\tau}(\mathbb{P})} + \|N\|_{\mathcal{N}^p_{\eta,\tau}(\mathbb{P})} \le C\Big(\|\xi\|_{\mathcal{L}^q_{\rho,\tau}(\mathbb{P})} + \overline{f}^{q,\mathbb{P}}_{\rho,\tau}\Big)$$

for all $p \in (1, q)$ *and* $\eta \in [-\mu, \rho)$

Darling & Pardoux '97 : $\bar{\rho} := \rho + \frac{L^2}{2}$, $\mathbb{E}^{\mathbb{P}}$, instead of ρ, $\mathcal{E}^{\mathbb{P}}$.

Norms

We have used the notations

$$\|\xi\|^p_{\mathcal{L}^q_{\rho,\tau}(\mathbb{P})} := \mathcal{E}^{\mathbb{P}}\Big[\big|e^{\rho\tau}\xi\big|^q\Big]$$

$$\|Y\|^p_{\mathcal{D}^p_{\eta,\tau}(\mathbb{P})} := \mathcal{E}^{\mathbb{P}}\Big[\sup_{t\le\tau}\big|e^{\eta t}Y_t\big|^p\Big]$$

$$\|Z\|^p_{\mathcal{H}^p_{\eta,\tau}(\mathbb{P})} := \mathcal{E}^{\mathbb{P}}\Big[\Big(\int_0^\tau \big|e^{\eta t}\widehat{\sigma}_t^{\mathrm{T}}Z_t\big|^2 dt\Big)^{\frac{p}{2}}\Big]$$

$$\|N\|^p_{\mathcal{N}^p_{\eta,\tau}(\mathbb{P})} := \mathcal{E}^{\mathbb{P}}\Big[\Big(\int_0^\tau e^{2\eta t}\, d[N]_t\Big)^{\frac{p}{2}}\Big]$$

Random horizon reflected backward SDEs

Find (Y, Z) such that :

$$Y_{\cdot\wedge\tau} = \xi + \int_{\cdot\wedge\tau}^{\tau} f_s(Y_s, Z_s)ds - (Z_s \cdot dX_s + dU_s), \quad Y \geq S, \quad \mathbb{P}-\text{a.s.}$$
$$\text{and } \mathbb{E}^{\mathbb{P}}\Big[\int_0^{t\wedge\tau} (1 \wedge (Y_{r-} - S_{r-}))dU_r\Big] = 0, \text{ for all } t \geq 0,$$

where $U_{\wedge t}$ is a càdlàg $\mathbb{P}-$supermartingale, for all $t \geq 0$, starting from $U_0 = 0$, orthogonal to X, i.e. $[X, U] = 0$

Theorem (Y. Lin , Z. Ren, NT & J. Yang '17)

Assume further that S càdlàg, $\mathbb{F}^{+,\mathbb{P}}-$adapted, $\|S^+\|_{\mathcal{D}^q_{\rho,\tau}(\mathbb{P})} < \infty$. Then, the reflected BSDE has a unique solution $(Y, Z) \in \mathcal{D}^p_{\eta,\tau}(\mathbb{P}) \times \mathcal{H}^p_{\eta,\tau}(\mathbb{P})$, for all $p \in (1, q)$ and $\eta \in [-\mu, \rho)$.

Random horizon 2^{nd}order backward SDE

For a stop. time τ, and $\mathcal{F}_\tau-$measurable ξ :

$$Y_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} F_s(Y_s, Z_s, \hat{\sigma}_s)ds - \int_{t\wedge\tau}^{\tau} Z_s \cdot dX_s + \int_{t\wedge\tau}^{\tau} dK_s, \quad \mathcal{P} - \text{q.s.}$$

K non-decreasing, $K_0 = 0$, and minimal in the sense

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\int_{t_1\wedge\tau}^{t_2\wedge\tau} dK_t\Big] = 0, \quad \text{for all} \quad t_1 \le t_2$$

Remark Deterministic finite horizon $\tau = T$: $(C2)_\mu$ not needed
Soner, NT & Zhang '14
Possamaï, Tan & Zhou '16

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Connection with fully nonlinear PDEs

Rewrite the 2BSDE in differential form

$$dY_t = -F_t(Y_t, Z_t, \hat{\sigma}_t)dt + Z_t \cdot dX_t - dK_t, \;\; t \leq \tau, \quad \text{and}\, Y_T = \xi, \;\; \mathcal{P} - \text{q.s.}$$

Markovian case $\xi = g(X_T)$ and $f_t(y, z) = f(t, X_t, y, z) \Longrightarrow Y_t = v(t, X_t)$

$$dY_t = \partial_t v(t, X_t)dt + Dv(t, X_t) \cdot dX_t + \frac{1}{2}\mathrm{Tr}\big[\hat{\sigma}_t^2 D^2 v(t, X_t)\big]dt, \;\; \mathcal{P} - \text{q.s.}$$

by Itô's formula. Direct identification yields

$$Z_t = Dv(t, X_t) \quad \text{and} \quad \partial_t v(t, X_t) + \frac{1}{2}\mathrm{Tr}\big[\hat{\sigma}_t^2 D^2 v(t, X_t)\big] \leq -F_t(v, Dv, \hat{\sigma}_t)$$

Finally, the minimality condition on K implies the fully nonlinear PDE

$$\partial_t v(t, X_t) + \sup_{\sigma}\Big\{\frac{1}{2}\mathrm{Tr}\big[\sigma^2 D^2 v(t, X_t)\big] + F_t(v, Dv, \sigma)\Big\} = 0$$

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Wellposedness of random horizon 2$^{\text{nd}}$order backward SDE

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$$Y \in \mathcal{D}^p_{\eta,\tau}(\mathcal{P}^L), \quad Z \in \mathcal{H}^p_{\eta,\tau}(\mathcal{P}^L) \quad \textit{for all} \quad \eta \in [\mu, \rho), \; p \in [1, q)$$

$$\|\xi\|^p_{\mathcal{L}^q_{\rho,\tau}(\mathcal{P})} := \mathcal{E}^L\Big[\big|e^{\rho\tau}\xi\big|^q\Big]$$

$$\|Y\|^p_{\mathcal{D}^p_{\eta,\tau}(\mathcal{P})} := \mathcal{E}^L\Big[\sup_{t\le\tau}\big|e^{\eta t}Y_t\big|^p\Big]$$

$$\|Z\|^p_{\mathcal{H}^p_{\eta,\tau}(\mathcal{P})} := \mathcal{E}^L\Big[\Big(\int_0^\tau \big|e^{\eta t}\widehat{\sigma}^{\mathrm{T}}_t Z_t\big|^2 dt\Big)^{\frac{p}{2}}\Big]$$