MATHEMATICAL ASPECTS OF ARBITRAGE

IOANNIS KARATZAS

Columbia University

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SYNOPSIS

We set up a mathematical framework for investment in a financial market with a fixed number of assets, thought of as stocks. We discuss various forms of what is commonly called **"arbitrage"** in this context, and single out a particularly egregious form of this phenomenon.

It has long been observed that, ruling out *some* forms of arbitrage, is essential for being able to construct a theory for finance that can deal effectively with questions of financing liabilities, of portfolio optimization, and the like.

But which forms of arbitrage should be ruled out?
This turns out to be a tricky question. We run through a few options, listing their successes and shortcomings, **then focus on a specific choice.**

. We demonstrate that ruling out this – rather egregious, as it turns out – form of arbitrage, leads to a theory which is descriptive, flexible, very general, and simple.

1. SETUP FOR A FINANCIAL MARKET

We consider strictly positive, continuous random functions of time, or "processes" $X_1(t), \ldots, X_n(t)$, $0 \leq t < \infty$, representing the prices or the capitalizations of n given assets (e.g., stocks) across time. Then

$$X(t) = X_1(t) + \cdots + X_n(t)$$

is the total capitalization in the market consisting of these assets. Please think of capitalizations as prices, multiplied by the number of shares outstanding.

SMALL INVESTOR: Decides what proportions $\pi_1(t), \cdots, \pi_n(t)$ of current wealth $V(t) > 0$ to invest in the different assets at each time t, based upon the information, or "history", available up until that time, say $\mathfrak{F}(t) = \sigma(X_1(s), \ldots, X_n(s),\ 0 \leq s \leq t)$.

This results in a *Portfolio*

$$\pi(t) = \big(\pi_1(t), \cdots, \pi_n(t)\big), \qquad 0 \leq t < \infty$$

as above. The proportion

$$\pi_0(t) := 1 - \sum_{i=1}^{n} \pi_i(t)$$

gets invested in a money-market with zero interest rate (today's situation). If $\pi_0(t)$ is positive, this means hoarding of cash; if negative, it means borrowing cash at zero interest rate.

- We say that a given portfolio is a *Stock Portfolio*, if it never borrows from, or lends into, the money market:

$$\sum_{i=1}^{n} \pi_i(t) = 1, \qquad 0 \leq t < \infty.$$

2. WEALTH GENERATED BY A STRATEGY

The wealth, or "value", $V^\pi(\cdot) > 0$ corresponding to a given portfolio $\pi(\cdot)$ and starting with an initial fortune of $\$ 1$, satisfies then

$$\frac{V^\pi(t+\Delta t) - V^\pi(t)}{V^\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \, \frac{X_i(t+\Delta t) - X_i(t)}{X_i(t)}$$

or in differential form

$$\frac{\mathrm{d} V^\pi(t)}{V^\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \, \frac{\mathrm{d} X_i(t)}{X_i(t)} \, , \qquad V^\pi(0) = 1 \, .$$

In words: The instantaneous rate of return of the portfolio's value is the average, according to the weights $\pi_1(t), \cdots, \pi_n(t)$, of the instantaneous rates of return for the individual assets.

This value never vanishes; and if started with an initial capital of $\$ \, v$, wealth scales as

$$V^{v,\pi}(\cdot) = v \cdot V^\pi(\cdot) \, .$$

3. ARBITRAGE

Consider a real number $T > 0$ and any two portfolios $\pi(\cdot)$, $\rho(\cdot)$.

We shall say that $\pi(\cdot)$ is *arbitrage relative to* $\rho(\cdot)$ (or that $\pi(\cdot)$ *outperforms* or *dorminates* $\rho(\cdot)$) over the time horizon $[0, T]$, if

$$\mathbb{P}(V^{\pi}(T) \geq V^{\rho}(T)) = 1 \quad \text{and} \quad \mathbb{P}(V^{\pi}(T) > V^{\rho}(T)) > 0\,.$$

- We shall call such relative arbitrage *strong*, if

$$\mathbb{P}(V^{\pi}(T) > V^{\rho}(T)) = 1\,.$$

CASH PORTFOLIO: With $\kappa(\cdot) \equiv 0$, the portfolio that never touches the stock market and puts the initial capital of $1 under the proverbial mattress, we have $V^{\kappa}(\cdot) \equiv 1$.

When $\rho(\cdot) \equiv \kappa(\cdot)$, this leads to the usual definition of arbitrage, with respect to cash.

. EXAMPLE OF ARBITRAGE (somewhat egregious)

Nothing seems to capture the general idea of arbitrage better than this story:

A finance professor (*not* a practitioner; the distinction is important here) and a normal person walk down the street. The second guy sees a $100 bill on the street, and stoops to pick it up.

"Don't do that!" the finance professor screams. *"It is absolutely impossible for a $100 bill to be lying on the street; for if it were, someone would have picked it up already"*.

. EXAMPLE OF ARBITRAGE (BERNOULLI model)

- A BIT MORE SERIOUSLY:

Suppose time is discrete, and there is only one day: $t = 0$ is "morning" (when you read your paper and call your broker), and $t = 1$ is "evening" (when you find out how you fared that day).

- There is one risky asset with prices $X(0) = x > 0$ and $X(1) = x\,B$, where B is a BERNOULLI random variable

$$\mathbb{P}(B = u) = p \in (0, 1) \quad \text{and} \quad \mathbb{P}(B = d) = 1 - p$$

for some $u > d > 0$.

The alternative is a money market, where \$1 in the morning becomes \$(1+R) in the evening for some $R \geq 0$.

• If you hold y shares of the asset and z units of the money market, then the value of your strategy moves from

$$V(0) = x \cdot y + 1 \cdot z$$

in the morning to

$$V(1) = x\,B \cdot y + (1+R) \cdot z$$

in the evening, after prices have been announced in late afternoon.

. It is easy to see that this caricature of a model is free of arbitrage, *if and only if*

$$d < 1 + R < u \,.$$

Neither the stock dominates the money market $(1+R \leq d)$, nor the other way round $(1+R \geq u)$.

When such domination exists, so does arbitrage.

4. THE MARKET PORTFOLIO

Consider now the relative weights of the various assets, in terms of capitalization:

$$\mu_i(t) := \frac{X_i(t)}{X(t)} = \frac{X_i(t)}{X_1(t) + \ldots + X_n(t)}, \qquad i = 1, \ldots, n.$$

These are strictly positive quantities, add up to 1, and are determined on the basis of the capitalizations of the market's various assets at any given time. They constitute a portfolio with strictly positive weights, which we call the *market portfolio* $\mu(\cdot)$.

This portfolio invests in each stock in proportion to its relative capitalization. Put differently: $\mu(\cdot)$ buys one share of each asset at time $t = 0$ and holds on to it. Ends up *"owning the entire market"*, in proportion of course to the initial fortune:

$$V^{\mu}(\cdot) = \frac{X(t)}{X(0)}, \qquad V^{v,\mu}(\cdot) = v \cdot \frac{X(t)}{X(0)}.$$

Here is how these market weights typically look like, ordered from largest to smallest, in the S&P market with $n = 500$.

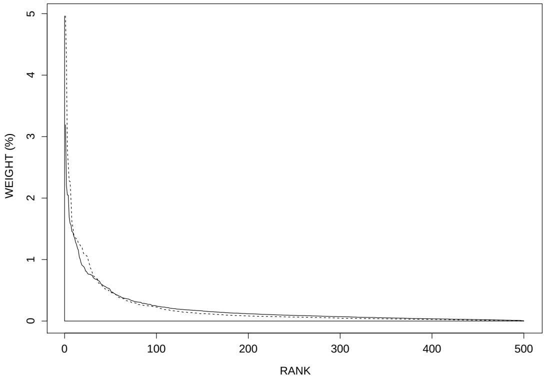

Figure 1: Capital Distribution for the S&P 500 Index. Dec. 30, 1997 (solid line), and Dec. 29, 1999 (broken line). From E.R. FERNHOLZ (2002) *Stochastic Portfolio Theory*, page 70.

5. MARTINGALES AND ALL THAT

Consider a positive, continuous process $Z(\cdot)$, adapted to the flow of information $\mathbb{F} = \{\mathfrak{F}(t)\}_{0 \le t < \infty}$. It is called

- Martingale, if $\mathbb{E}^{\mathbb{P}}\big[\, Z(t) \,\big|\, \mathfrak{F}(s) \,\big] \;=\; Z(s)$
- Supermartingale, if $\mathbb{E}^{\mathbb{P}}\big[\, Z(t) \,\big|\, \mathfrak{F}(s) \,\big] \;\le\; Z(s)$

holds w.p.1., for every $0 \le s < t < \infty$. Here $\mathbb{E}^{\mathbb{P}}$ denotes the integral (expectation) with the respect to the underlying probability measure $\mathbb{P}$.

Very important notions in Probability Theory. Depend crucially both on the flow of information, and on the underlying probability measure.

Their "localizations" are also very important. Nonnegative local martingales are supermartingales. And bounded local martingales are martingales.

6. NUMÉRAIRE PORTFOLIO

Suppose that there exists a portfolio $\nu(\cdot)$ such that

$$\frac{V^\pi(\cdot)}{V^\nu(\cdot)} \quad \text{is a supermartingale, for every portfolio } \pi(\cdot)\,.$$

That is, for every portfolio $\pi(\cdot)$ and $0 \le s < t < \infty$, we have

$$\mathbb{E}^{\mathbb{P}}\left[\frac{V^\pi(t)}{V^\nu(t)} \,\middle|\, \mathfrak{F}(s) \right] \le \frac{V^\pi(s)}{V^\nu(s)}\,, \qquad \mathbb{P}-\text{a.e.}$$

. We say then, that $\nu(\cdot)$ has the *"numéraire property"*.

(Think of a "Gold Standard", whose value cannot be beaten systematically over time by any other investment strategy.)

When it exists, a numéraire portfolio has several interesting optimality properties; some of these are information-theoretic and go back all the way to Kelly (1956), Breiman (1961), Thorp (1969), as well as to Cover (1984) and Long (1990). We'll discuss them presently.

. Foremost among these properties, is the maximization of asymptotic rate of growth from investment: for *any portfolio* $\pi(\cdot)$ we have

$$\limsup_{T \to \infty} \frac{1}{T} \log \left(\frac{V^{\pi}(T)}{V^{\nu}(T)} \right) \leq 0 \,, \qquad \text{a.e.}$$

To wit, its growth rate cannot be surpassed.

. A numéraire portfolio $\nu(\cdot)$ also maximizes expected relative log-utility from investment: for any given $T \in (0, \infty)$ and portfolio $\pi(\cdot)$, we have

$$\mathbb{E}^{\mathbb{P}} \left[\log \left(\frac{V^{\pi}(T)}{V^{\nu}(T)} \right) \right] \leq 0 \,.$$

- In order to determine whether a portfolio $\nu(\cdot)$ with the numéraire property exists, and then describe it, one has to impose considerable structure on the dynamics of asset capitalizations.

In other words, we shall need to adopt a "model" for them; and then *take this model seriously.* We shall see how presently.

For the time being, we claim that:

ARBITRAGE CANNOT EXIST OVER

ANY GIVEN TIME HORIZON OF FINITE LENGTH,

RELATIVE TO A STRATEGY $\nu(\cdot)$ WITH THE

NUMÉRAIRE PROPERTY

(UNDER SOME EQUIVALENT $\widehat{\mathbb{P}} \sim \mathbb{P}$.)

¶ Indeed, suppose $\nu(\cdot)$ has the numéraire property. If $\pi(\cdot)$ were an arbitrage opportunity relative to $\nu(\cdot)$ over some time-horizon $[0, T]$, then we would have

$$\mathbb{P}\left(\frac{V^{\pi}(T)}{V^{\nu}(T)} \geq 1\right) = 1 \quad \text{and} \quad \mathbb{P}\left(\frac{V^{\pi}(T)}{V^{\nu}(T)} > 1\right) > 0$$

from the definition of arbitrage, thus also

$$\mathbb{E}^{\mathbb{P}}\left[\frac{V^{\pi}(T)}{V^{\nu}(T)}\right] \leq \frac{V^{\pi}(0)}{V^{\nu}(0)} = 1$$

from the supermartingale property; but this is absurd.

Easy to see that this same argument works if ν has the *"extended numéraire property"*, that is, the numéraire property under some equivalent probability measure $\widehat{\mathbb{P}} \sim \mathbb{P}$.

7. "YOU CANNOT BEAT THE MARKET"

MOST OF MODERN FINANCE IS PREDICATED ON THE ASSUMPTION THAT ARBITRAGE RELATIVE TO CASH $\kappa(\cdot) \equiv 0$ DOES **NOT** EXIST. [1]

(Remember the finance professor of our silly little story.)

. It is also posited very often that

ARBITRAGE DOES NOT EXIST

RELATIVE TO THE MARKET PORTFOLIO $\mu(\cdot)$.

("You cannot beat the market.")

[1] In philosophical terms: an "Apophatic" approach.

8. EQUIVALENT MARTINGALE MEASURE

In fact, most of modern finance rests on an *even stronger – and very normative – assumption:*

That there should **exist** [2] an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ under which the asset capitalization processes $X_1(\cdot), \cdots, X_n(\cdot)$ are martingales – at least locally.

Such a measure is known as *"Equivalent Martingale Measure"* (EMM). The notion bears a striking similarity to the DE FINETTI (1937, 74) theory of coherent subjective probabilities and inference.

[2] In philosophical terms: a "Cataphatic" approach.

- Under such an equivalent martingale measure, the value

$$V^{\rho}(\cdot) \equiv V^{\rho}(\cdot)/V^{\kappa}(\cdot)$$

of **any** portfolio $\rho(\cdot)$ is a positive local martingale, thus also a supermartingale.

. Portfolios, whose values are in fact *martingales* under some EMM, will play an important role in a minute.

¶ Bernoulli Model (cont'd)

Recall that $X(0) = x > 0$, that $B = X(1)/X(0)$ is Bernoulli

$$\mathbb{P}(B = u) = p \in (0,1) \quad \text{and} \quad \mathbb{P}(B = d) = 1 - p\,,$$

and that arbitrage is absent *if and only if:* $d < 1 + R < u$.

An equivalent probability measure $\mathbb{Q}$ assigns

$$\mathbb{Q}(B = u) = q \in (0,1) \qquad \text{and} \qquad \mathbb{Q}(B = d) = 1 - q\,.$$

• Suppose $\mathbb{Q}$ equates the expected return from the stock to that from the money market (*risk neutrality*), namely

$$1 + R \;=\; \mathbb{E}^{\mathbb{Q}}(B) \;=\; u \cdot q + d \cdot (1 - q)\,, \quad \text{to wit:} \quad q = \frac{(1+R) - d}{u - d}\,.$$

A fussy way to express this, is to say that the *discounted prices* $(1+R)^{-t}X(t)$, $t = 0, 1$ *have the martingale property:*

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{X(1)}{1+R}\right) \;=\; X(0)\,.$$

9. E.M.M. PROSCRIBES ARBITRAGE

PROPOSITION: *EVERY PORTFOLIO $\rho(\cdot)$, WHOSE VALUE $V^{\rho}(\cdot)$ IS A* **MARTINGALE** *UNDER* **SOME** *E.M.M. $\mathbb{Q} \sim \mathbb{P}$, HAS THE (GENERALIZED) NUMÉRAIRE PROPERTY.*

- MARKET: This property certainly holds for the *market portfolio* $\mu(\cdot)$, whose value $V^{\mu}(\cdot)$ is proportional to the total market capitalization

$$X(\cdot) = X_1(\cdot) + \cdots + X_n(\cdot),$$

and therefore a martingale under *any* probability measure $\mathbb{Q} \sim \mathbb{P}$ that makes $X_1(\cdot), \cdots, X_n(\cdot)$ martingales: (sum of martingales is a martingale).

- CASH: It also holds for $\kappa(\cdot) \equiv 0$, the *cash strategy* that keeps the initial dollar under the proverbial mattress and "generates" $V^{\kappa}(\cdot) \equiv 1$: a constant is a (very solid) martingale.

- Thus, from this Proposition and from the claim on slide 15:

Assuming an EMM exists, arbitrage is impossible relative to *any* portfolio $\rho(\cdot)$, whose value process $V^{\rho}(\cdot)$ is a martingale — under some EMM.

In particular:

IF AN EMM EXISTS, THEN ARBITRAGE IS IMPOSSIBLE RELATIVE TO THE MARKET PORTFOLIO

(OR TO CASH).

¶ *Proof of Proposition:* A bit of calculus reveals that, for any portfolios $\pi(\cdot)$ and $\rho(\cdot)$, we have:

$$\mathrm{d}\left(\frac{V^\pi(t)}{V^\rho(t)}\right) = \left(\frac{V^\pi(t)}{V^\rho(t)}\right) \sum_{i=1}^{n} \frac{\pi_i(t) - \rho_i(t)}{X_i(t)}\, \mathrm{d}\widehat{X}_i(t)\,, \qquad (1)$$

where we have introduced the "shifted asset processes"

$$\widehat{X}_i(\cdot) := X_i(\cdot) - \int_0^{\cdot} \frac{\mathrm{d}\langle X_i, V^\rho\rangle(t)}{V^\rho(t)}\,.$$

Suppose that the $X_i(\cdot)$'s are (local) martingales w.r.t some EMM $\mathbb{Q} \sim \mathbb{P}$, and $V^\rho(\cdot)$ is a martingale w.r.t this $\mathbb{Q}$.

Then these $\widehat{X}_i(\cdot)$'s are (local) martingales *under a new probability measure* $\widehat{\mathbb{P}}_\rho \sim \mathbb{Q}$, thus $\widehat{\mathbb{P}}_\rho \sim \mathbb{P}$. Back into (1), this implies that the ratio

$$V^\pi(\cdot) \,/\, V^\rho(\cdot)$$

is a positive (local) martingale, thus also a supermartingale, under this measure $\widehat{\mathbb{P}}_\rho \sim \mathbb{P}$: the numéraire property of $\rho(\cdot)$ under $\widehat{\mathbb{P}}_\rho$.

A beautiful mathematical theory for Finance can be constructed on the basis of this normative assumption, the existence of an EMM. For instance, see the monographs

- I. Karatzas & S.E. Shreve *"Methods of Mathematical Finance"* (1998),
- H. Föllmer & A. Schied *"Stochastic Finance"* (2004),
- F. Delbaen & W. Schachermayer *"The Mathematics of Arbitrage"* (2006).

One of the big successes of this theory is the answer it provides to the problem of hedging liabilities.

¶ 10. HEDGING UNDER A UNIQUE E.M.M.

Suppose the investor faces a **liability**, a payment he has to make at a given time $T \in (0, \infty)$. The size $Y(\omega)$ of this liability is not known in advance; rather, it is a random variable $Y : \Omega \to [0, \infty)$ whose value is revealed only at time $t = T$.

For example, a function $Y(\omega) = f(\mathcal{X}(T, \omega))$ of the vector of asset prices

$$\mathcal{X}(T, \omega) = \big(X_1(T, \omega), \ldots, X_n(T, \omega)\big)'$$

at that time into the future.
For instance, a "call option" (the right, *not obligation,* to exchange cash for stock) with exercise price $q > 0$, i.e.,

$$Y(\omega) = \big(X_1(T, \omega) - q\big)^+ .$$

How much is this liability worth at time $t = 0$?

One way to measure this worth, is in terms of the **smallest amount** of initial capital (fortune) $v > 0$ that the investor has to set aside at time $t = 0$ so that, by proper investment in the market's assets, this amount can grow by time $t = T$ to a level that is enough to cover the liability *without risk,* i.e., with probability one.

THEOREM: Harrison & Pliska (1981). *If there exists a* **unique** *Equivalent Martingale Measure* $\mathbb{Q}$ *, then the quantity*

$$\mathfrak{H}^Y := \inf \left\{ v > 0 \, : \, \exists \text{ strategy } \pi(\cdot) \text{ s.t. } \mathbb{P}(V^{v,\pi}(T) \geq Y) = 1 \right\}$$

is given by

$$\mathfrak{H}^Y = \mathbb{E}^{\mathbb{Q}}(Y) \, .$$

Yes, the expectation that actuaries have been using heuristically since the time of the BERNOULLI s, *but now under the EMM*. Also: *"unique no-arbitrage price"*.

BLACK & SCHOLES

- For instance, suppose $n = 1$ and postulate the SAMUELSON stock-price dynamics

$$\frac{\mathrm{d}X(t)}{X(t)} = \beta(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)\,, \qquad X(0) = x > 0\,.$$

Here $W(\cdot)$ is standard BROWNian motion, σ is a positive number, $\beta(\cdot)$ a "well-behaved" random process, $X(0) = x > 0$ the asset price at time $t = 0$.

Take $Y = (X(T) - q)^+$. Then it can be seen that a unique EMM indeed exists, and the quantity

$$\mathfrak{H}^Y = \mathbb{E}^{\mathbb{Q}}(Y)$$

is given by the famous BLACK & SCHOLES (1973) expression

$$H(T,x) = x \cdot \Phi\big(m_+(T,x)\big) - q\,e^{-rT} \cdot \Phi\big(m_-(T,x)\big)\,.$$

In this expression

$$H(T,x) \;=\; x \cdot \Phi\big(m_+(T,x)\big) - q\, e^{-rT} \cdot \Phi\big(m_-(T,x)\big)$$

we have denoted by

$$\Phi(\cdot) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\cdot} e^{\,-\xi^2/2}\,\mathrm{d}\xi$$

the cumulative standard GAUSSian distribution function,

$$m_\pm(T,x) \;=\; \frac{1}{\sigma\sqrt{T}} \left[\log\left(\frac{x}{q}\right) + \left(r \pm \frac{\sigma^2}{2}\right) T \right].$$

• In terms of this function $H(\cdot\,,\cdot)$ we can also compute the "hedging", portfolio $\pi_*(\cdot)$, which holds the number of shares

$$\frac{\partial H}{\partial x}\big(T-t, X(t)\big)\,, \qquad 0 \le t \le T$$

in the stock and has the "replication" property $V^{\,H(T,x),\,\pi_*}(T) = Y$.

Analogous computation in our BERNOULLI example.

¶ 11. AND YET....

Despite its great successes, this notion of Equivalent Martingale Measure (EMM) has its problems.

- A severe theoretical [3] issue, is that it amounts to a **very strong** NORMATIVE assumption:

Two different models can easily have the exact same characteristics (mean rates of return, and covariations), while one of them admits an EMM and the other does not.

[3] Rather: Epistemological.

- Another severe issue, both theoretical and *very* practical this time, is as follows: While a plausible assumption over "short" horizons (duration measured in weeks or months), the existence of an EMM is just a lousy assumption over "long" time horizons (duration measured in years or decades), leading to paradoxa and to internal contradictions.

To the best of my knowledge, E.R. Fernholz and E. Platen were the first to have noticed *and* written about this.

It is also quite clear that W. Buffett was very aware of such shortcomings, [4] and that he acted upon them.

[4] In the context of pricing long-term put options. I owe this piece of information to Peter Carr.

12. STOCHASTIC PORTFOLIO THEORY

This is a mathematical theory of Finance that does not rely on normative assumptions such as absence of arbitrage, let alone on the existence of an EMM. It was introduced, and developed very considerably, by E. Robert Fernholz (2002), building on the classic work of Markowitz (1952).

It provides simple, **descriptive** conditions under which arbitrage with respect to the market portfolio may exist, and then constructs very simple portfolio rules that can implement it.

It also provides conditions under which arbitrage is not possible; but these are not couched in terms of the existence of EMM either. They are rather based on quantities that are either observable, or can be measured with reasonable precision. (We shall not have time to go into this part of the theory today.)

13. THE "BENCHMARK" APPROACH

This is a theory for finance developed by ECKHARD PLATEN, based on the numéraire portfolio that we mentioned a few slides back.

Let's give a glimpse of the very basic facts about portfolios, before turning to this notion and developing it.

In order to do all this, we'll need a bit more structure than we have afforded up to now.

14. RATES OF RETURN, GROWTH, AND (CO)VARIATION

Equity market framework (Bachelier, Samuelson...)

$$\frac{\mathrm{d}X_i(t)}{X_i(t)} = \alpha_i(t)\,\mathrm{d}t + \sum_{k=1}^{K} \sigma_{ik}(t)\,\mathrm{d}W_k(t)\,, \qquad i = 1, \dots, n\,,$$

with $W(\cdot) = (W_1(\cdot), \cdots, W_K(\cdot))'$ a vector Brownian motion.

(Just a bit more general conceptually, than the model we adopted a moment ago, for the discussion of the Black & Scholes formula. With very small modifications, affords extension to the much more general setting of continuous semimnartingales.)

This framework allows us to think in terms of
Mean Return rates $\alpha(\cdot) = (\alpha_1(\cdot), \ldots, \alpha_n(\cdot))'$,
Volatility rates $\sigma(\cdot) = (\sigma_{ik}(\cdot))_{1 \le i \le n,\, 1 \le k \le K}$,
and an $(n \times n)-$matrix

$$c(\cdot) \,=\, \sigma(\cdot)\,\sigma'(\cdot)$$

of *Covariation Rates*

$$c_{ij}(t) \,:=\, \sum_{k=1}^{K} \sigma_{ik}(t)\sigma_{jk}(t) \,=\, \frac{1}{X_i(t)X_j(t)} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\,\langle X_i, X_j\rangle(t)\,.$$

Both processes $c(\cdot)$ and $\alpha(\cdot)$ are assumed to be $\mathbb{F}-$adapted and locally integrable.

In logarithmic form,

$$\frac{\mathrm{d}X_i(t)}{X_i(t)} = \alpha_i(t)\,\mathrm{d}t + \sum_{k=1}^{K} \sigma_{ik}(t)\,\mathrm{d}W_k(t)$$

becomes

$$\mathrm{d}\log X_i(t) = \gamma_i(t)\,\mathrm{d}t + \sum_{k=1}^{K} \sigma_{ik}(t)\,\mathrm{d}W_k(t)\,,$$

where the *growth rate* of the i^{th} asset is

$$\gamma_i(t) := \alpha_i(t) - \frac{1}{2}\,c_{ii}(t)$$

for each $i = 1, \cdots, n$.

- Analogous equation for the value of a portfolio:

$$\frac{\mathrm{d}V^{\pi}(t)}{V^{\pi}(t)} = \alpha^{\pi}(t)\,\mathrm{d}t + \sum_{k=1}^{K} \sigma_k^{\pi}(t)\,\mathrm{d}W_k(t)\,, \quad V^{\pi}(0) = 1\,,$$

where

$$\alpha^{\pi}(t) := \sum_{i=1}^{n} \pi_i(t)\,\alpha_i(t)\,, \qquad \sigma_k^{\pi}(t) := \sum_{i=1}^{n} \pi_i(t)\,\sigma_{ik}(t)\,.$$

Similarly, in logarithmic form

$$\mathrm{d} \log V^{\pi}(t) = \gamma^{\pi}(t)\,\mathrm{d}t + \sum_{k=1}^{K} \sigma_k^{\pi}(t)\,\mathrm{d}W_k(t)\,,$$

where the growth rate generated by the portfolio $\pi(\cdot)$ is

$$\gamma^{\pi}(t) = \pi'(t)\alpha(t) - \frac{1}{2}\,\pi'(t)\,c(t)\,\pi(t)\,.$$

We note that, in order for both integrals (ordinary, and stochastic), to make sense, we need here the local integrability

$$\int_0^T \Big(\big|\pi'(t)\alpha(t)\big| + \pi'(t)\,c(t)\,\pi(t) \Big) \mathrm{d}t < \infty\,, \qquad \forall \ \ T \in (0, \infty)\,.$$

15. MAXIMIZING THE LOCAL GROWTH RATE

(A) Now let us try to **maximize this rate of growth,** over all portfolios. In other words, try to find a portfolio $\nu(\cdot)$ with

$$\boxed{\nu(t) = \arg\max_{\xi \in \mathbb{R}^n} \Big(\xi' \alpha(t) - \frac{1}{2}\, \xi' c(t)\, \xi \Big) .} \qquad (2)$$

And of course the local integrability condition right above, i.e.,

$$\int_0^T \nu'(t)\, c(t)\, \nu(t)\, \mathrm{d}t < \infty , \qquad \forall \;\; T \in (0, \infty) .$$

(B) The first-order **structure condition** for this maximization is

$$\alpha(\cdot) = c(\cdot)\, \nu(\cdot) . \qquad (3)$$

(C) It turns out that a portfolio with this property (3) has also the **numéraire property** we encountered already:

$$\frac{V^\pi(\cdot)}{V^\nu(\cdot)} \quad \text{is a supermartingale, for every portfolio} \quad \pi(\cdot)\,. \tag{4}$$

(D) It turns out that the supermartingale numéraire property is equivalent to the **maximization of expected relative log-utility** as already seen: for every $T \in (0, \infty)$ and portfolio $\pi(\cdot)$, we have

$$\mathbb{E}^{\mathbb{P}}\left[\log\left(\frac{V^\pi(T)}{V^\nu(T)}\right)\right] \leq 0\,.$$

(E) It also turns out that the existence of a portfolio $\nu(\cdot)$ as in (3) implies the existence of a **"local martingale deflator"** for the market: a positive local martingale of the form

$$Z(t) = \exp\left(-\int_0^t \vartheta'(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t \left|\vartheta(s)\right|^2 \mathrm{d}s\right), \quad 0 \le t < \infty$$

with the property that all products

$$Z(\cdot)\, X_i(\cdot)\,, \quad i = 1, \cdots, n \quad \text{are local martingales.}$$

Indeed, all it takes in order to do this, is to set select right above

$$\vartheta(\cdot) = \sigma'(\cdot)\,\nu(\cdot)\,.$$

(Please note that with $n = 1$ and $\sigma > 0$, we get the "signal-to-noise" ratio $\nu = \alpha/\sigma^2(\cdot)$ and the so-called "Sharpe ratio" $\vartheta = \alpha/\sigma$.)

(F) It turns out that the existence of a local martingale deflator as above, *makes the entire market* **viable:** It becomes then impossible to finance in it non-trivial liabilities, or consumption streams, starting with initial capital that is positive but arbitrarily near zero.

This very egregious form of arbitrage, is thus proscribed. [5]

(G) Finally, it turns out that viability is equivalent to the following **boundedness in probability** requirement, for every $T \in (0, \infty)$:

$$\lim_{m \to \infty} \sup_{\pi} \mathbb{P}\big(V^{\pi}(T) > m\big) = 0\,, \qquad \forall \;\; T \in (0, \infty).$$

[5] Ruling out only the even more egregious possibility, of financing something non-trivial starting out with capital **exactly** equal to zero, turns out to be not good enough.

16. AN ENTIRE THEORY FOR FINANCE

In fact, all the above conditions (A)-(G) turn out to be equivalent. We have the following result.

FUNDAMENTAL THEOREM: *For the financial market considered here, the following are equivalent:*

(a) There is a portfolio ν that maximizes the rate of growth.
(b) There is a portfolio ν that satisfies the condition $\alpha = c\nu$.
(c) There is a portfolio ν with the numéraire property.
(d) There is a relatively log-optimal portfolio ν.
(e) There is a local martingale deflator.
(f) The market is viable.
(g) The boundedness-in-probability condition

$$\lim_{m \to \infty} \sup_{\pi} \mathbb{P}\big(V^{\pi}(T) > m\big) = 0, \qquad \forall \quad T \in (0, \infty).$$

is satisfied for every $T \in (0, \infty)$.

This is the cornerstone result for an entire theory for finance that can deal with hedging liabilities, portfolio optimization, equilibrium, constraints, infinitely-many assets as in zero-coupon bond markets, et cetera. [6]

Without ever needing to resort to the existence of an EMM.

But also without ever needing to proscribe the existence of an EMM: the theory admits it as a special case, when justified.

This theory "accommodates" the outperformance of one portfolio (including cash, or the market), by another. It only needs to proscribe those very egregious forms of arbitrage, that can "threaten" the viability of the market.

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[6] Amazing things can happen if any one – therefore all – of the above conditions fail; e.g., wealth processes that start out with ZERO initial capital and grow, monotonically and inexorably, to fantastic levels of wealth.

17. AN ARBITRARY NUMBER OF ASSETS

But it gets better. With proper functional-analytic attention, and under an appropriate version of the "structure condition $\alpha = c\nu$, this theory can deal with an infinity – even an uncountable infinity – of assets.

This is not idle speculation, or mere desire for generality. Zero-coupon bonds can have **arbitrary maturities** into the future; and understanding markets of such bonds calls for **exactly** such a theory. [7]

In a final bit of serendipity, it is PRECISELY the analogue of the structure condition that allows such a theory to be developed.

And this analogue is a generalization of the familiar HEATH-JARROW-MORTON volatility-to-drift condition.

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[7] And for a theory of stochastic integration w.r.t. an arbitrary collection of semimartingales.

18. A BOOK

All this, and a lot more, is developed in a book in preparation, being written in collaboration with Constantinos KARDARAS at LSE.

SOME REFERENCES

- FERNHOLZ, E.R. (2002). *Stochastic Portfolio Theory.* Springer-Verlag, New York.
- FERNHOLZ, E.R., KARATZAS, I. & KARDARAS, C. (2005). Diversity and arbitrage in equity markets. *Finance & Stochastics* **9**, 1-27.
- FERNHOLZ, E.R. & KARATZAS, I. (2005). Relative arbitrage in volatility-stabilized markets. *Annals of Finance* **1**, 149-177.
- KARATZAS, I. & KARDARAS, C. (2007). The numéraire portfolio and arbitrage in semimartingale markets. *Finance & Stochastics* **11**, 447-493.
- KARATZAS, I. & KARDARAS, C. (2020+). *Arbitrage Theory via Numéraires.* Working Title of Book in preparation.