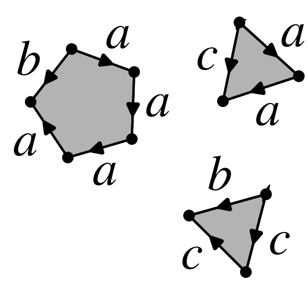
Algebraic Topology QR Exam – May 2024

1. Let $f : X \to Y$ be a map of topological spaces, and let $x_0 \in X$. Show that, if f is a homotopy equivalence, then the induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.
   (Do not assume its homotopy inverse, or the associated homotopies, respect basepoints).

2. (a) State the definition of a CW complex, and its topology (the weak topology).
   (b) Let $p : \widetilde{X} \to X$ be a degree-d covering map. If X is a CW complex, then its cover $\widetilde{X}$ naturally inherits a CW complex structure. Construct the attaching maps and characteristic maps for this CW complex structure, and verify that your construction defines a cellular decomposition of $\widetilde{X}$. Give complete statements of any properties of covering spaces you use. You do not need to check that the topology of $\widetilde{X}$ agrees with the weak topology with respect to your cell structure. Do, however, verify that $\widetilde{X}$ has d many n-cells for each n-cell of X, and that p restricts to a homeomorphism from each (open) n-cell of $\widetilde{X}$ to an (open) n-cell of X.

3. Let F_5 be the free group on 5 letters. Prove that every finite-index subgroup of F_5 is a free group with rank congruent to 1 mod 4, and conversely that every free group of rank $m \geq 5$ congruent to 1 mod 4 occurs as a finite-index subgroup of F_5.

4. Let X be the quotient space defined as the union of the polygons below, modulo the given edge identifications.



   (a) Compute the homology of X.
   (b) Let $B \subseteq X$ be the image of the loop b. Prove that B is not a retract of X.

5. Let $Y \cong S^n$ be a smooth n-sphere, and let $X \subseteq Y$ be a smoothly embedded d-sphere, for some $0 \leq d < n$.
   (a) Show that the inclusion $\iota : X \to Y$ is nullhomotopic.
   (b) Compute the reduced homology groups of the quotient space Y/X.

Solutions

1. See (for example) the proofs of Hatcher Propositions 1.5 and 1.18.

2. (a) There are multiple standard ways to define a CW complex. Here is one (following Hatcher):

   A CW complex is a (filtered) topological space X defined inductively as follows. Its *0-skeleton* $X^{(0)}$ is a discrete set of points. For each n, the n-skeleton $X^{(n)}$ is built from the $(n-1)$-skeleton by gluing a set of closed n-disks $\{D^n_\alpha\}_\alpha$ along their boundaries via continuous *attaching maps* $\phi_\alpha : \partial D^n_\alpha \to X^{(n-1)}$, as follows. We define $X^{(n)}$ to be the quotient of

   $$X^{(n-1)} \bigsqcup_\alpha D^n_\alpha$$

   via the equivalence relation that identifies a point in $X^{(n-1)}$ with all points in its preimages under ϕ_α for all α. We let $X = \bigcup_{n\geq 0} X^{(n)}$.

   We endow X with the *weak topology*: a subset $U \subseteq X$ is open in X if and only if $U \cap X^{(n)}$ is open in $X^{(n)}$ (with its inductively defined quotient topology) for every n.

   An *(open) cell* of X is the image e^n_α of $\text{int}(D^n_\alpha)$ for some n, α. It follows from the continuity of the attaching maps that the open cells are embedded in X. The *characteristic map* Φ^n_β of a cell e^n_β is the composite

   $$\Phi^n_\beta \quad : \quad D^n_\beta \quad \hookrightarrow \quad X^{(n-1)} \bigsqcup_\alpha D^n_\alpha \quad \longrightarrow \quad X^{(n)} \quad \longrightarrow \quad X.$$

   (b) We will construct the CW complex structure on $\widetilde{X}$ inductively by skeleta, in such a way that the k-skeleton of $\widetilde{X}$ is $p^{-1}\left(X^{(k)}\right)$.

   Let the 0-skeleton be $p^{-1}\left(X^{(0)}\right)$. Since $X^{(0)}$ is discrete and p is a local homeomorphism, this preimage is discrete.

   Suppose by induction that we have constructed a $(k-1)$-dimension CW complex structure on $p^{-1}\left(X^{(k-1)}\right) \subseteq \widetilde{X}$. Let $\Phi^k_\alpha : D^k_\alpha \to X$ be the characteristic map of a k-cell of X. Recall the following results on the existence and uniqueness of lifts of maps from X to $\widetilde{X}$.

   > **Theorem (Existence of lifts).** Suppose $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering space map and suppose $f : (Y, y_0) \to (X, x_0)$ is a map from a path-connected and locally path-connected space Y. Then a lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

   > **Theorem (Uniqueness of lifts).** Suppose $p : \widetilde{X} \to X$ is a covering space map and $f : Y \to X$ is a map from a connected space Y. If two lifts $\widetilde{f}_1, \widetilde{f}_2 : Y \to \widetilde{X}$ of f agree at one point of Y, then $\widetilde{f}_1$ and $\widetilde{f}_2$ agree on all of Y.

Choose a basepoint $y_0 \in D^k_\alpha$. Since the disk D^k_α is a contractible manifold—in particular, it is locally path-connected and simply connected—the Existence Theorem implies that there exists a lift $\widetilde{\Phi}^k_\alpha : D^k_\alpha \to \widetilde{X}$ for each p-preimage of $\Phi^k_\alpha(y_0)$. We claim that these d lifts $\widetilde{\Phi}^k_\alpha$ define the characteristic maps for the k-cells of a CW structure on $\widetilde{X}$. The attaching maps are the restrictions of the lifts $\widetilde{\Phi}^k_\alpha$ to ∂D^k_α; by construction their image is contained in

$$p^{-1}\left(\Phi^k_\alpha\left(\partial D^k_\alpha\right)\right) \subseteq p^{-1}\left(X^{(k-1)}\right) = \widetilde{X}^{(k-1)}.$$

We must verify that every point in $\widetilde{X}$ lies in precisely one open cell, and that the characteristic maps $\widetilde{\Phi}^k_\alpha$ restrict to homeomorphisms on $\text{int}(D^k_\alpha)$. Let $\widetilde{x} \in \widetilde{X}$. Then $p(\widetilde{x})$ is contained in precisely one open cell

e^n_α of X. Let $\Phi^n_\alpha : D^k_\alpha \to X$ be the associated characteristic map, and $y_0 \in D^k_\alpha$ the chosen basepoint. Let $y \in \operatorname{int}(D^n_\alpha)$ be the unique preimage of $p(\widetilde{x})$. To show that $\widetilde{x}$ lies in a unique open cell, it is necessary and sufficient to show that precisely one of our d lifts of Φ^n_α maps y to $\widetilde{x}$. By the Existence Theorem, there is a lift $\widetilde{\Phi}^n_\alpha$ mapping y to $\widetilde{x}$. By the Uniqueness Theorem, this lift is unique, and coincides with the unique lift constructed above that maps the basepoint $y_0 \in D^k_\alpha$ to its preimage point $\widetilde{\Phi}^n_\alpha(y_0) \in \widetilde{X}$. Thus $\widetilde{x}$ is contained in a unique open cell of $\widetilde{X}$.

Finally, observe that for each n, α and lift $\widetilde{\Phi}^n_\alpha$, the composite

$$\Phi^n_\alpha\Big|_{\operatorname{int}(D^n_\alpha)} \quad = \quad p \circ \widetilde{\Phi}^n_\alpha\Big|_{\operatorname{int}(D^n_\alpha)} \quad : \quad \operatorname{int}(D^n_\alpha) \xrightarrow{\widetilde{\Phi}^n_\alpha} \widetilde{X} \xrightarrow{p} X$$

is a homeomorphism onto its image. This implies that both the restriction of $\widetilde{\Phi}^n_\alpha$ to $\operatorname{int}(D^n_\alpha)$ and the restriction of p to $\widetilde{\Phi}^n_\alpha(\operatorname{int}(D^n_\alpha))$ must be injective. Moreover, both restrictions must be homeomorphisms onto their image (neither map can make the topology coarser). This concludes the construction of the CW complex structure on $\widetilde{X}$, and establishes that p restricts to a homeomorphism on any open n-cell of $\widetilde{X}$ to an open n-cell of X.

Per the question statement, we will not verify that the topology on $\widetilde{X}$ agrees with the weak topology for this cell structure. See (for example) Hatcher Proposition A.2 for point-set conditions that ensure that a family of maps to a space $\tilde{X}$ are the characteristics maps for a CW complex structure on $\tilde{X}$.

3. We can identify F_5 with the fundamental group of the wedge $X = S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1$. The space X is path-connected, and since it is a graph (a 1-dimensional CW complex) it is locally path-connected and semi-locally simply-connected. Hence the classification of covering spaces implies that each subgroup G of F_5 is isomorphic to the fundamental group of a path-connected cover of X. The index of the subgroup equals the number of sheets of the cover, so the finite-index subgroups G of F_5 correspond exactly to finite-sheeted covers of X.

   Let $\widetilde{X} \to X$ be a d-sheeted cover of X. By Problem 2, a cover $\widetilde{X}$ of a graph X is itself a graph, and since X has 1 vertex and 5 edges, the cover $\widetilde{X}$ has d vertices and $5d$ edges. A spanning tree in $\widetilde{X}$ will contain all d vertices, and $(d-1)$ edges. Because the spanning tree is a contractible CW subcomplex of $\widetilde{X}$, the space $\widetilde{X}$ is homotopy equivalent to its quotient by the spanning tree. This quotient is a wedge of

$$5d - (d-1) = 4d+1$$

   circles. Thus $\pi_1(\widetilde{X})$ is a free group of rank $4d+1$. This rank is always congruent to 1 mod 4, as claimed.

   To complete the problem, we must show that X has a d–sheeted cover for every $d \geq 1$. The identity map $X \to X$ is a one-sheeted cover. For $d > 1$, consider a surjective homomorphism ϕ_d from F_5 to the cyclic group C_d of order d (say, a map sending all five free generators of F_5 to a distinguished generator of C_d.) Then the kernel G of ϕ_d is an index-d subgroup, and so corresponds to a d-sheeted cover, and we conclude that $G \cong F_{4d+1}$.

4. (a) When we trace through the identifications of the vertices induced by the identifications of the edges, we see that all the vertices of all the polygons are identified to a single point v. Thus the space X has a CW complex structure with 0-skeleton a single vertex v, 1-skeleton a wedge of three circles a, b, c, and three 2-cells we'll call A, B, C. The 2-cell A is glued along its boundary via the word a^4b^{-1}, the 2-cell B is glued along the word a^2c^{-1}, and the 2-cell C is glued along the word c^2b^{-1}.

   We therefore obtain the cellular chain complex

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_2(X) & \xrightarrow{\partial_2} & C_1(X) & \xrightarrow{\partial_1} & C_0(X) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
 & & \mathbb{Z}\{A,B,C\} & & \mathbb{Z}\{a,b,c\} & & \mathbb{Z}\{v\} & &
\end{array}$$

$$\begin{aligned}
A &\longmapsto 4a-b \\
B &\longmapsto 2a-c \\
C &\longmapsto 2c-b
\end{aligned}$$

$$\begin{aligned}
a &\longmapsto v-v=0 \\
b &\longmapsto v-v=0 \\
c &\longmapsto v-v=0
\end{aligned}$$

Explicitly,

$$\partial_2 = \begin{bmatrix} 4 & 2 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad \text{and} \qquad \partial_1 = 0.$$

By direct calculation (say, writing ∂_2 in RREF) we see that the kernel of ∂_2 is the rank-1 subgroup spanned by $-A+2B+C$. Thus,

$$H_0(X) = \frac{C_0(X)}{\operatorname{im}(\partial_1)} = \frac{\mathbb{Z}\{v\}}{\langle 0 \rangle} \cong \mathbb{Z}$$

$$H_1(X) = \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = \frac{\mathbb{Z}\{a,b,c\}}{\langle 4a-b, 2a-c, 2c-b \rangle} \cong \mathbb{Z}$$

$$H_2(X) = \ker(\partial_2) = \langle -A+2B+C \rangle \cong \mathbb{Z}$$

In the calculation of H_1, we note that the matrix ∂_2 has rank 2 and image spanned by $4a-b$ and $2a-c$. Thus, $H_1(X)$ is the infinite cyclic group generated by the class of cycle a. The cycle b is homologous to $4a$ and the cycle c is homologous to $2a$.

(b) We will use the calculation of $H_1(X)$ to show that B is not a retract of X. Suppose for the sake of contradiction that it is. By definition, this means that there exists a retraction, a map $r : X \to B$ satisfying $r \circ \iota = id_B$, where $\iota : B \hookrightarrow X$ is the inclusion of B.

By functoriality of H_1, this implies that $r_* \circ \iota_* = id_{H_1(B)}$. Now, B is a circle, with $H_1(B) \cong \mathbb{Z}$ generated by the loop b. The inclusion $\iota : B \to X$ induces on H_1 the inclusion of the subgroup generated by b, i.e., the inclusion of the subgroup $4\mathbb{Z} \subseteq \mathbb{Z} \cong H_1(X)$. We have a commuting diagram,

$$\begin{array}{ccccc}
 & & \overset{id}{\frown} & & \\
H_1(B) & \xrightarrow{\iota_*} & H_1(X) & \xrightarrow{r_*} & H_1(B) \\
\| & & \| & & \| \\
\langle b \rangle & & \langle a \rangle & & \langle b \rangle \\
b & \longmapsto & b = 4a & & \\
 & & a & \longmapsto & r_*(a)
\end{array}
\qquad
\begin{array}{ccccc}
 & & \overset{1}{\frown} & & \\
\mathbb{Z} & \xrightarrow{4} & \mathbb{Z} & \xrightarrow{?} & \mathbb{Z} \\
1 & \longmapsto & 4 & & \\
 & & 1 & \longmapsto & ?
\end{array}$$

The crux of our contradiction is that the inclusion $4\mathbb{Z} \hookrightarrow \mathbb{Z}$ does not have a left inverse r_*. To see this, we consider the possibilities for the element $r_*(a)$. We know

$$\begin{aligned} r_*(\iota_*(b)) &= id(b) = b \\ r_*(4a) &= b \\ 4r_*(a) &= b \end{aligned}$$

But there is no element x in $H_1(B) = \langle b \rangle$ satisfying $4x = b$. We conclude that the retraction map r cannot exist, and B is not a retract of X.

5. (a) Let $f : S^d \to S^n$ be any continuous map. There exist CW complex structures on S^d (respectively, S^n) with one 0-cell and one d-cell (respectively, one 0-cell and one n-cell). By the cellular approximation theorem, the map $f : S^d \to S^n$ is homotopic to a cellular map g. But then g maps the d-skeleton of S^d (which is all of S^d) to the d-skeleton of S^n (which, since $d < n$, is a point). Hence g is a constant map, and we conclude that an arbitrary continuous map $f : S^d \to S^n$ is nullhomotopic.

   (b) The sphere S^p has reduced homology

$$\widetilde{H}_i(S^p) \cong \begin{cases} \mathbb{Z}, & i = p \\ 0, & i \neq p. \end{cases}$$

Since $X \subseteq Y$ is a smooth submanifold, the pair (Y, X) is a good pair; this follows (for example) from the tubular neighbourhood theorem. Thus $\widetilde{H}_i(X/Y) \cong H(Y, X)$, and we apply the long exact sequence of a pair

$$\cdots \longrightarrow \widetilde{H}_i(S^d) \xrightarrow{\iota_*} \widetilde{H}_i(S^n) \longrightarrow \widetilde{H}_i(Y/X) \longrightarrow \widetilde{H}_{i-1}(S^d) \xrightarrow{\iota_*} \widetilde{H}_{i-1}(S^n) \longrightarrow \cdots$$

By part (a), the map ι is nullhomotopic, hence $\iota_* = 0$, and the long exact sequence decomposes into a sequence of short exact sequences

$$0 \longrightarrow \widetilde{H}_i(S^n) \longrightarrow \widetilde{H}_i(Y/X) \longrightarrow \widetilde{H}_{i-1}(S^d) \longrightarrow 0$$

Thus if $d < n - 1$, we find

$$\widetilde{H}_i(Y/X) \cong \begin{cases} \mathbb{Z}, & i = n \\ \mathbb{Z}, & i = d+1 \\ 0, & \text{otherwise.} \end{cases}$$

If $d = n - 1$, then the nonzero reduced homology groups of Y/X are determined by the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{H}_n(Y/X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

We will use the short exact sequence to prove that $\widetilde{H}_n(Y/X) \cong \mathbb{Z}^2$. This follows directly from the observation that the quotient group of the short exact sequence is free abelian, hence the short exact sequence must split. We also outline an argument to check this isomorphism by hand: Because $\widetilde{H}_n(Y/X)$ is an extension of finitely generated abelian groups, it is itself a finitely generated abelian group, and so is determined by its rank and torsion subgroup. The torsion subgroup must be in the kernel of the surjective map, hence by exactness is zero. The rank-nullity theorem implies that the rank of $\widetilde{H}_n(Y/X)$ is 2.
We conclude, when $d = n - 1$, that

$$\widetilde{H}_i(Y/X) \cong \begin{cases} \mathbb{Z}^2, & i = n \\ 0, & \text{otherwise.} \end{cases}$$

Remark: For an alternate proof, see Hatcher Example 0.14 to argue that $Y/X \simeq S^n \vee S^{d+1}$.