Algebraic Topology QR Exam – May 2024

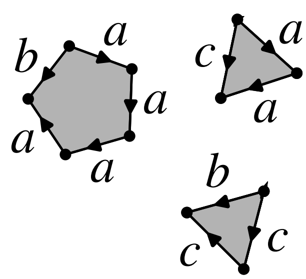
1. Let $f : X \to Y$ be a map of topological spaces, and let $x_0 \in X$. Show that, if f is a homotopy equivalence, then the induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.
(Do not assume its homotopy inverse, or the associated homotopies, respect basepoints).

2. (a) State the definition of a CW complex, and its topology (the weak topology).

   (b) Let $p : \widetilde{X} \to X$ be a degree-d covering map. If X is a CW complex, then its cover $\widetilde{X}$ naturally inherits a CW complex structure. Construct the attaching maps and characteristic maps for this CW complex structure, and verify that your construction defines a cellular decomposition of $\widetilde{X}$. Give complete statements of any properties of covering spaces you use. You do not need to check that the topology of $\widetilde{X}$ agrees with the weak topology with respect to your cell structure. Do, however, verify that $\widetilde{X}$ has d many n-cells for each n-cell of X, and that p restricts to a homeomorphism from each (open) n-cell of $\widetilde{X}$ to an (open) n-cell of X.

3. Let F_5 be the free group on 5 letters. Prove that every finite-index subgroup of F_5 is a free group with rank congruent to 1 mod 4, and conversely that every free group of rank $m \geq 5$ congruent to 1 mod 4 occurs as a finite-index subgroup of F_5.

4. Let X be the quotient space defined as the union of the polygons below, modulo the given edge identifications.



   (a) Compute the homology of X.

   (b) Let $B \subseteq X$ be the image of the loop b. Prove that B is not a retract of X.

5. Let $Y \cong S^n$ be a smooth n-sphere, and let $X \subseteq Y$ be a smoothly embedded d-sphere, for some $0 \leq d < n$.

   (a) Show that the inclusion $\iota : X \to Y$ is nullhomotopic.

   (b) Compute the reduced homology groups of the quotient space Y/X.