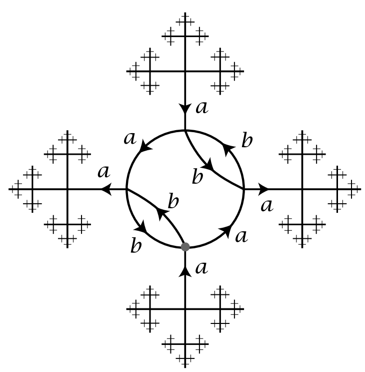
Algebraic Topology QR Exam – Jan 2024

1. (a) State the definition of a CW complex, and its topology (the weak topology).

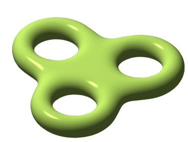
   (b) Let X be a CW complex and $A \subseteq X$ a nonempty CW subcomplex. Working directly from your definition, describe a CW complex structure on the quotient space X/A, and verify explicitly that the quotient topology on X/A agrees with the weak topology of your given CW complex structure.

2. (a) Let X be a path-connected, locally path-connected, and semi-locally simply connected space. Let $p : (\widetilde{X}, \widetilde{v}) \to (X, v)$ be the covering space associated to a subgroup $H \subseteq \pi_1(X, v)$. For an element $[\gamma] \in \pi_1(X, v)$, let $\widetilde{\gamma}$ denote the lift of γ to $\widetilde{X}$ starting at $\widetilde{v}$. Show that $[\gamma] \in \pi_1(X, v)$ is in the normalizer $N(H)$ of H if and only if the lift $\widetilde{\gamma}$ has endpoint $\widetilde{w} := \widetilde{\gamma}(1)$ in the orbit of $\widetilde{v}$ under the deck group of the cover p.

   (b) Consider the wedge $S^1 \vee S^1$ of circles a and b with wedge point v. Below is a (based) cover associated to a certain subgroup H of $\pi_1(S^1 \vee S^1, v)$. The covering map is specified by the edge labels and orientations, and a basepoint $\widetilde{v}$ is marked with a gray dot. Find a (not necessarily free) finite generating set for the normalizer $N(H)$ of H, with very brief justification.



3. Fix $g \geq 0$. The closed orientable genus-g surface Σ_g is the boundary of a compact 3-dimensional manifold $\mathbf{H}_g$ called a *genus-g handlebody*, as pictured for $g = 3$. [Image by Oleg Alexandrov]



   The *doubled handlebody* $\mathbf{D}_g$ is obtained by gluing two copies of $\mathbf{H}_g$ along their boundary via the identity map. Concretely, for $\mathbf{H} = \mathbf{H}' = \mathbf{H}_g$ and $I : \mathbf{H} \to \mathbf{H}'$ the the identity map, the space $\mathbf{D}_g$ is the quotient of the disjoint union $\mathbf{H}' \sqcup \mathbf{H}$ by the equivalence relation $I(x) \sim x$ for all $x \in \partial \mathbf{H} = \Sigma_g$.

   (a) Compute $\pi_1(\mathbf{D}_g)$.

   (b) Compute $\widetilde{H}_*(\mathbf{D}_g)$.

   For this question, you can assert descriptions of the fundamental groups and homology groups of Σ_g and $\mathbf{H}_g$ without proof. Please justify the other steps in your computation.

4. The following proposition is a step in the proof of the Five Lemma. Perform a diagram chase to prove this proposition.

   Proposition. Suppose that in the following commutative diagram of abelian groups,

   - Both rows are exact.
   - The maps β and δ are injective.
   - The map α is surjective.

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \xrightarrow[h']{} & D' \end{array}$$

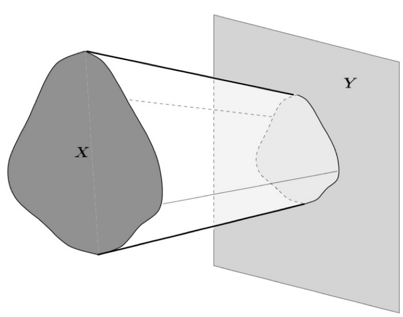
   Then the map γ is injective.

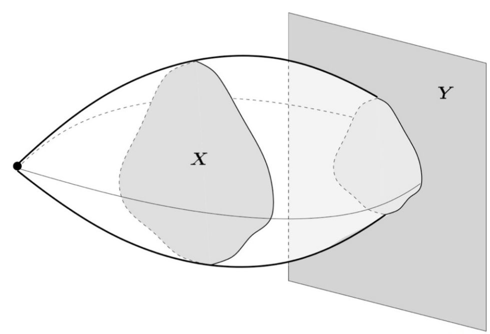
5. Let $f : X \to Y$ be a continuous map of nonempty topological spaces. Let $[0,1]$ denote the closed interval.

   The *mapping cylinder* M_f of f is obtained by gluing $X \times [0,1]$ to Y via f in the following sense: it is the quotient of the disjoint union of $X \times [0,1]$ and Y by the equivalence relation generated by $(x,1) \sim f(x)$.

   Let X_0 denote the image of $X \times \{0\}$ in M_f. The *mapping cone* C_f of f is the quotient of M_f that collapses X_0 to a point.

   The spaces M_f and C_f, respectively, are illustrated below. [Images by Fernando Muro]





   Fix $k \geq 0$ in $\mathbb{Z}$. Prove that the induced map $f_* : H_i(X) \to H_i(Y)$ is an isomorphism for $0 \leq i \leq k$ if $\widetilde{H}_i(C_f) = 0$ for $0 \leq i \leq k+1$.

   Hint: First verify that (M_f, X_0) is a good pair.

Solutions

[Note: These solutions contain more detail than is expected on the exam.]

1. (a) There are multiple standard ways to define a CW complex. Here is one (following Hatcher):

   A CW complex is a (filtered) topological space X defined inductively as follows. Its *0-skeleton* $X^{(0)}$ is a discrete set of points. For each n, the n-skeleton $X^{(n)}$ is built from the $(n-1)$-skeleton by gluing a set of closed n-disks $\{D^n_\alpha\}_\alpha$ along their boundaries via continuous *attaching maps* $\phi_\alpha : \partial D^n_\alpha \to X^{(n-1)}$, as follows. We define $X^{(n)}$ to be the quotient of

$$X^{(n-1)} \bigsqcup_\alpha D^n_\alpha$$

   via the equivalence relation that identifies a point in $X^{(n-1)}$ with all points in its preimages under ϕ_α for all α. We let $X = \bigcup_{n\geq 0} X^{(n)}$.

   We endow X with the *weak topology*: a subset $U \subseteq X$ is open in X (respectively, closed) if and only if $U \cap X^{(n)}$ is open in $X^{(n)}$ (respectively, closed) for every n.

   A *cell* of X is the image of $\mathrm{int}(D^n_\alpha)$ for some n, α.

   (b) Let X be a CW complex, and $A \subseteq X$ a CW subcomplex, that is, A is a union of cells of X that is closed in X. Note that this implies (by definition of closure) that the closure of any cell of A is contained in A, hence for each cell of A the image of $D^n_\alpha = \mathrm{cl}(\mathrm{int}(D^n_\alpha))$ is contained in A.

   Our goal is to show that the quotient space X/A inherits a CW complex structure from the structure on X. Let $p : X \to X/A$ denote the quotient map. We first note that p is a closed map. Let $C \subseteq X$ be closed. Then $p^{-1}(p(C))$ is either C or $C \cup A$. Both sets are closed, so $p(C)$ is closed by definition of the quotient topology. Then the restriction of p to any closed subset (in particular, $X^{(n)}$) is a closed map, hence also a quotient map onto its image.

   We claim that there is a cell structure on $Y := X/A$ as follows. The 0-skeleton $Y^{(0)}$ of Y is the quotient space $X^{(0)}/A^{(0)} \subseteq X/A$. In other words, it is a discrete set with one 0-cell for each 0-cell in $X \setminus A$, and one 0-cell corresponding to A. For $n \geq 1$, there is an n-cell for every n-cell of X that is not contained in A. For $n \geq 1$ we inductively define the n-skeleton $Y^{(n)} \subseteq X/A$ as the image of the map

$$Y^{(n-1)} \bigsqcup_{\mathrm{int}(D^n_\alpha) \text{ a cell in } X\setminus A} D^n_\alpha \xrightarrow{q^Y_n} X/A$$

   where $q^Y_n\big|_{Y^{(n-1)}}$ is defined by induction, and $q^Y_n\big|_{D^n_\alpha}$ is defined as the composite

$$D^n_\alpha \hookrightarrow X^{(n-1)} \bigsqcup_\alpha D^n_\alpha \xrightarrow{q^X_n} X^{(n)} \longrightarrow X \xrightarrow{p} X/A.$$

   By construction, as a subspace of X/A, the space $Y^{(n)}$ coincides with the image of $X^{(n)}$ in the quotient X/A. This observation also implies that $X/A = \bigcup_n Y^{(n)}$. To complete the proof, we must verify that q^Y_n is a quotient map of the correct form, and that the weak topology on $Y = \bigcup Y^{(n)}$ agrees with the quotient topology on X/A.

   We will show that q^Y_n is a quotient map onto its image. Suppose a set $U \subseteq Y^{(n)}$ has open preimage $W := (q^Y_n)^{-1}(U)$; we must show that U is open. That W is open means $W \cap Y^{(n-1)}$ is open in $Y^{(n-1)}$ and $W \cap D^n_\alpha$ is open in D^n_α for all indices n, α corresponding to cells of $X \setminus A$. Consider the preimage of U in $X^{(n-1)} \bigsqcup_\alpha D^n_\alpha$ under $(p\big|_{X^{(n)}}) \circ (q^X_n)$. Its intersection with $X^{(n-1)}$ is the preimage of the open subset $W \cap Y^{(n-1)}$ of $Y^{(n-1)}$ under the continuous map $X^{(n-1)} \to Y^{(n-1)}$. For all n, α indexing cells not in A, the preimage intersects D^n_α in the open subset $W \cap D^n_\alpha$. And for all n, α indexing cells of A, the preimage intersects D^α_n in D^α_n or in $\varnothing$, depending on whether U contains the point of X/A that is the image of A. Thus the preimage of U is open in $X^{(n-1)} \bigsqcup_\alpha D^n_\alpha$. Since q^X_n and $p\big|_{X^{(n)}}$ are quotient maps, their composite is a quotient map, so we conclude that U is open in $Y^{(n)}$.

We can check moreover (by considering the fibres of q^Y_n) that it is the quotient map corresponding to the equivalence relation we obtain from the data of the attaching maps

$$D^n_\alpha \xrightarrow{\phi_\alpha} X^{(n)} \xrightarrow{p|_{X^{(n)}}} Y^{(n)},$$

and we conclude that the map q^Y_n does define a CW structure in the sense of the definition given in part (a).

Finally, we show the quotient topology on X/A agrees with the weak topology. Since p is a closed map, $Y^{(n)} = p(X^{(n)})$ is closed in the quotient topology on X/A. Thus for any subset $C \in X/A$ that is closed in the quotient topology, the intersection $C \cap Y^{(n)}$ is closed for all n, so C is closed in the weak topology. Suppose conversely that $C \subseteq X/A$ is a subset with the property that $C \cap Y^{(n)}$ is closed in $Y^{(n)}$ for all n. Since the restriction $p\big|_{X^{(n)}}$ is continuous, it follows that $\left(p\big|_{X^{(n)}}\right)^{-1}(C \cap Y^{(n)})$ is closed in $X^{(n)}$ for all n. But

$$p^{-1}(C) \cap X^{(n)} = \left(p\big|_{X^{(n)}}\right)^{-1}(C) = \left(p\big|_{X^{(n)}}\right)^{-1}(C \cap Y^{(n)})$$

hence $p^{-1}(C)$ is closed in X by definition of the weak topology on X. Therefore C is closed in X/A by definition of the quotient topology. This concludes the proof.

2. (a) Let $\widetilde{w}$ be a point in the fibre above v, and suppose there is a deck transformation f mapping $\widetilde{w}$ to $\widetilde{v}$. Recall that a *deck transformation* of the cover $p : \widetilde{X} \to X$ is a homeomorphism $f : \widetilde{X} \to X$ that is an automorphism of the cover in the sense that $p \circ f = p$. The inverse homeomorphism f^{-1} of f automatically also satisfies the condition $p \circ f^{-1} = p$.
We may view f as a lift of $p : (\widetilde{X}, \widetilde{w}) \to (X, v)$ to the cover $p : (\widetilde{X}, \widetilde{v}) \to (X, v)$, and its inverse f^{-1} as a lift of the map $p : (\widetilde{X}, \widetilde{v}) \to (X, v)$ to the cover $p : (\widetilde{X}, \widetilde{w}) \to (X, v)$.

$$\begin{array}{ccc} & & (\widetilde{X}, \widetilde{v}) \\ & \overset{f}{\nearrow} & \downarrow p \\ (\widetilde{X}, \widetilde{w}) & \xrightarrow{p} & (X, v) \end{array} \qquad \begin{array}{ccc} & & (\widetilde{X}, \widetilde{w}) \\ & \overset{f^{-1}}{\nearrow} & \downarrow p \\ (\widetilde{X}, \widetilde{v}) & \xrightarrow{p} & (X, v) \end{array}$$

The lifting criterion for covering spaces states:

> **Theorem.** Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering space map and $f : (Y, y_0) \to (X, x_0)$ any continuous map. Assume Y is path-connected and locally path-connected. Then a lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

The cover $\widetilde{X}$ is path-connected by assumption (the cover of a path-connected, locally path-connected, semi-locally simply connected space associated to a subgroup $H \subseteq \pi_1(X, v)$ is path-connected by the classification theorem for covers). The lifting criterion therefore applies. Thus the deck map f and its inverse f^{-1} exist if and only if

$$p_*(\pi_1(\widetilde{X}, \widetilde{w})) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{v})) \qquad \text{and} \qquad p_*(\pi_1(\widetilde{X}, \widetilde{v})) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{w})),$$

that is, the lifts f and f^{-1} exist if and only if $p_*(\pi_1(\widetilde{X}, \widetilde{v})) = p_*(\pi_1(\widetilde{X}, \widetilde{w}))$. Note that if both lifts exist then they are automatically inverses, since their composites $f \circ f^{-1}$ and $f^{-1} \circ f$ are each lifts of maps from a connected space that fix a point ($\widetilde{v}$ and $\widetilde{w}$, respectively) and hence are the identity maps.

Now consider a loop γ in X based at v that lifts to a path $\widetilde{\gamma}$ from $\widetilde{v}$ to $\widetilde{w}$. We may use this lift to define a change-of-basepoint map, giving an isomorphism between the fundamental group of $\widetilde{X}$ based at $\widetilde{v}$ and at $\widetilde{w}$.

$$\pi_1(\widetilde{X},\widetilde{w}) \xrightarrow{\cong} \pi_1(\widetilde{X},\widetilde{v})$$
$$[\alpha] \longmapsto [\widetilde{\gamma}\cdot\alpha\cdot\widetilde{\gamma}^{-1}]$$

But the induced map p_* on paths is compatible with concatenation of paths, so

$$p_*(\widetilde{\gamma}\cdot\alpha\cdot\widetilde{\gamma}^{-1}) = p_*(\widetilde{\gamma})\cdot p_*(\alpha)\cdot p_*(\widetilde{\gamma}^{-1}) = \gamma\cdot p_*(\alpha)\cdot\gamma^{-1}$$

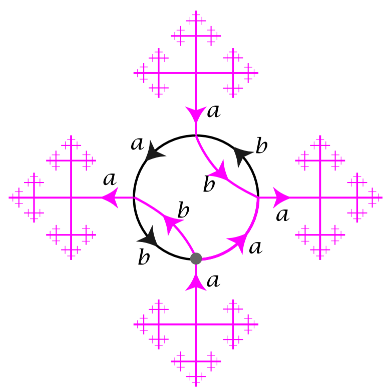
Thus

$$p_*(\pi_1(\widetilde{X},\widetilde{w})) = \gamma\cdot\Big(p_*(\pi_1(\widetilde{X},\widetilde{v}))\Big)\cdot\gamma^{-1},$$

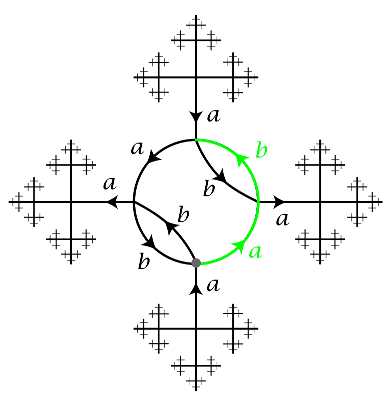
and we have equality $p_*(\pi_1(\widetilde{X},\widetilde{v})) = p_*(\pi_1(\widetilde{X},\widetilde{w}))$ if and only if γ is in the normalizer of $H = p_*(\pi_1(\widetilde{X},\widetilde{v}))$. The claim follows.

(b) By part (a), the normalizer $N(H)$ of $\pi_1(S^1\vee S^1, v)$ consists of all loops whose lift to $\widetilde{v}$ has endpoint $\widetilde{w}$ in the orbit of $\widetilde{v}$ under the deck action. For a given vertex $\widetilde{w}\in p^{-1}(v)$, the set of loops that lift to a path from $\widetilde{v}$ to $\widetilde{w}$ are a right coset of H in $\pi_1(S^1\vee S^1, v)$. Thus $N(H)$ is a union of right cosets of H, and it is generated by H plus a representative of each coset.

To find generators for H, we choose a maximal tree in the cover. One such choice is shown in pink, which corresponds to (free) generating set b^2, $ab^{-1}ab^{-1}$, ab^2a^{-1} for H.



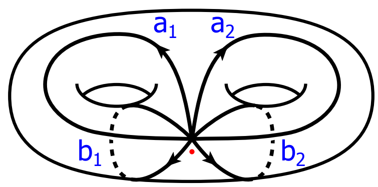
Now we consider the deck action on the cover. By visual inspection, the only non-identity graph automorphism that respects the labels and orientations of the edges is 180° rotation of the graph in the plane of the page. Hence this is the only non-identity deck map. We choose a representative of the associated right coset for H by choosing a path from $\widetilde{v}$ to its image $\widetilde{w}$ under the deck map; one such choice of path is shown, corresponding to the loop $ab\in\pi_1(S^1\vee S^1, v)$.



Thus one possible set of generators for the normalizer $N(H)$ is

$$b^2, \quad ab^{-1}ab^{-1}, \quad ab^2a^{-1}, \quad ab.$$

3. Per the question, we can use the following facts about Σ_g and $\mathbf{H}_g$ without proof. We know that $\pi_1(\Sigma_g)$ is generated by g longitudinal loops $a_1, \ldots a_g$ and g meridian loops $b_1, \ldots, b_g$, as shown for $g = 2$.



The fundamental group is given by the presentation

$$\pi_1(\Sigma_g) = \left\langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \;\middle|\; [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \right\rangle$$

where $[a, b]$ denotes the commutator $[a, b] = aba^{-1}b^{-1}$.

By abuse of notation, we also write a_i and b_i to denote the corresponding homology classes. Then

$$\widetilde{H}_k(\Sigma_g) = \begin{cases} 0, & k = 0 \\ \mathbb{Z}\{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\} \cong \mathbb{Z}^{2g}, & k = 1 \\ \mathbb{Z}, & k = 2 \\ 0, & k \geq 3 \end{cases}$$

By inspection, the handlebody $\mathbf{H}_g$ deformation retracts onto a wedge of g circles. Its fundamental group is thus a free group on g generators $\overline{a_1}, \ldots \overline{a_g}$ again corresponding to the g longitudinal loops.

$$\pi_1(\Sigma_g) = \left\langle \overline{a_1}, \ldots, \overline{a_g} \right\rangle$$

and its homology is

$$\widetilde{H}_k(\mathbf{H}_g) = \begin{cases} 0, & k = 0 \\ \mathbb{Z}\{\overline{a_1}, \ldots, \overline{a_g}\} \cong \mathbb{Z}^g, & k = 1 \\ 0, & k \geq 2 \end{cases}$$

The loops b_i are contractible in $\mathbf{H}_g$, as we see in the picture for $g = 2$. The inclusion of $\iota : \Sigma_g \to \mathbf{H}_g$ induces the maps

$$\begin{aligned} \pi_1(\Sigma_g) &\longrightarrow \pi_1(\mathbf{H}_g) \\ a_i &\longmapsto \overline{a_i} \\ b_i &\longmapsto 0 \end{aligned}$$

$$\begin{aligned} H_1(\Sigma_g) &\longrightarrow H_1(\mathbf{H}_g) \\ a_i &\longmapsto \overline{a_i} \\ b_i &\longmapsto 0 \end{aligned}$$

$$H_2(\Sigma_g) \xrightarrow{0} H_2(\mathbf{H}_g)$$

We will proceed by decomposing $\mathbf{D}_g$ into the union of two open subsets, and then applying the van Kampen and Mayer–Vietoris theorems.

Remark: Another approach to this problem is to argue that $\mathbf{D}_g$ is homeomorphic to a connected sum of g copies of $S^1 \times S^2$.

To construct this open cover, first observe that (as with all manifolds with compact boundary) we can find a collar neighbourhood of $\partial \mathbf{H}_g$ in $\mathbf{H}_g$ that derformation retracts back to $\partial \mathbf{H}_g$. Let A_1 be the union of the first handlebody in $\mathbf{D}_g$ and this collar neighbourhood in the second. Similarly let A_2 be the corresponding neighbourhood of the second handlebody in $\mathbf{D}_g$. Then

$$A_1 \simeq \mathbf{H}_g \qquad A_2 \simeq \mathbf{H}_g \qquad A_1 \cap A_2 \simeq \Sigma_g.$$

(a) We will use van Kampen's theorem to prove that $\pi_1(\mathbf{D}_g)$ is a rank-g free group. Van Kampen states,

> **Theorem (Van Kampen).** Suppose a space (X, x_0) is a union of path-connected open subsets A_1, A_2, each containing the basepoint x_0, and with path-connected intersection $A_1 \cap A_2$. Then $\pi_1(X, x_0) \cong \pi_1(A_1, x_0) *_{\pi_1(A_1 \cap A_2, x_0)} \pi_1(A_2, x_0)$.
>
> Here, $\pi_1(A_1, x_0) *_{\pi_1(A_1 \cap A_2, x_0)} \pi_1(A_2, x_0)$ is the *free product with amalgamation*, defined as follows. Define $\iota_i : \pi_1(A_1 \cap A_2, x_0) \to \pi_1(A_i, x_0)$ for $i = 1, 2$ to be the maps induced by the inclusions $A_1 \cap A_2 \hookrightarrow A_i$. Then the amalgamated free product is the quotient of the free product $\pi_1(A_1, x_0) * \pi_1(A_1, x_0)$ by the subgroup normally generated by the identifications $\iota_1(\alpha) \sim \iota_2(\alpha)$ for all $\alpha \in \pi_1(A_1 \cap A_2, x_0)$.

Since our chosen sets A_1, A_2, and $A_1 \cap A_2$ are open and path-connected, we can apply van Kampen's theorem to the cover $\mathbf{D}_g = A_1 \cup A_2$. Observe,

$$\begin{aligned} \pi_1(A_1 \cap A_2) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid \sim \rangle &\longrightarrow \pi_1(A_1) \cong \langle \overline{a_1}, \ldots, \overline{a_g} \rangle \\ a_i &\longmapsto \overline{a_i} \\ b_i &\longmapsto 0 \end{aligned}$$

$$\begin{aligned} \pi_1(A_1 \cap A_2) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid \sim \rangle &\longrightarrow \pi_1(A_2) \cong \langle \overline{a_1}', \ldots, \overline{a_g}' \rangle \\ a_i &\longmapsto \overline{a_i}' \\ b_i &\longmapsto 0 \end{aligned}$$

Thus there is an isomorphism

$$\pi_1(\mathbf{D}_g) \;\cong\; \langle \overline{a_1}, \ldots, \overline{a_g} \rangle * \langle \overline{a_1}', \ldots, \overline{a_g}' \rangle / \text{normal closure of } (\overline{a_i}(\overline{a_i}')^{-1})_i$$

We claim that this group is isomorphic to the free group $F_g = \langle x_1, \ldots, x_g \rangle$. To verify this rigorously, consider the homomorphism

$$\begin{aligned} \langle \overline{a_1}, \ldots, \overline{a_g} \rangle * \langle \overline{a_1}', \ldots, \overline{a_g}' \rangle &\longrightarrow F_g \\ \overline{a_i},\ \overline{a_i}' &\longmapsto x_i \end{aligned}$$

It vanishes on the relations $\overline{a_i}(\overline{a_i}')^{-1}$, thus factors through a homomorphism $\pi_1(\mathbf{D}_g) \to F_g$. We can see this map is an isomorphism by checking directly that the following composites are the identity maps.

$$\begin{array}{ccccc} \langle \overline{a_1}, \ldots, \overline{a_g} \rangle * \langle \overline{a_1}', \ldots, \overline{a_g}' \rangle / \sim & \longrightarrow & F_g & \longrightarrow & \langle \overline{a_1}, \ldots, \overline{a_g} \rangle * \langle \overline{a_1}', \ldots, \overline{a_g}' \rangle / \sim \\ & \overline{a_i},\ \overline{a_i}' \longmapsto x_i, & & x_i \longmapsto \overline{a_i} & \end{array}$$

$$\begin{array}{ccccc} F_g & \longrightarrow & \langle \overline{a_1}, \ldots, \overline{a_g} \rangle * \langle \overline{a_1}', \ldots, \overline{a_g}' \rangle / \sim & \longrightarrow & F_g \\ x_i \longmapsto \overline{a_i}, & & & \overline{a_i},\ \overline{a_i}' \longmapsto x_i & \end{array}$$

We conclude that $\pi_1(\mathbf{D}_g)$ is a free group on g free generators, corresponding to the g longitudinal loops in the first copy of $\mathbf{H}_g$ (equivalently, in the second copy of $\mathbf{H}_g$).

(b) We will prove

$$\widetilde{H}_k(\mathbf{D}_g) = \begin{cases} 0, & k = 0 \\ \mathbb{Z}^g, & k = 1 \\ \mathbb{Z}^g, & k = 2, \\ \mathbb{Z}, & k = 3 \\ 0, & k \geq 4 \end{cases}$$

Consider the Mayer–Vietoris long exact sequence associated to the open cover $\mathbf{D}_g = A_1 \cup A_2$.

$$\cdots \longrightarrow \widetilde{H}_n(A_1 \cap A_2) \longrightarrow \widetilde{H}_n(A_1) \oplus \widetilde{H}_n(A_2) \longrightarrow \widetilde{H}_n(\mathbf{D}_g) \xrightarrow{\delta} \widetilde{H}_{n-1}(A_1 \cap A_2) \longrightarrow \cdots$$

Given our descriptions of A_1, A_2, and their intersection, this long exact sequence is as follows (all terms not shown vanish).

$$\begin{array}{ccccccc} \cdots \longrightarrow \widetilde{H}_3(\mathbf{H}_g) \oplus \widetilde{H}_3(\mathbf{H}_g) & \longrightarrow & \widetilde{H}_3(\mathbf{D}_g) & \longrightarrow & \widetilde{H}_2(\Sigma_g) & \longrightarrow & \widetilde{H}_2(\mathbf{H}_g) \oplus \widetilde{H}_2(\mathbf{H}_g) \\ \| & & & & \| & & \| \\ 0 & & & & \mathbb{Z} & & 0 \end{array}$$

$$\begin{array}{ccccccc} \longrightarrow \widetilde{H}_2(\mathbf{D}_g) & \longrightarrow & \widetilde{H}_1(\Sigma_g) & \longrightarrow & \widetilde{H}_1(\mathbf{H}_g) \oplus \widetilde{H}_1(\mathbf{H}_g) & \longrightarrow & \widetilde{H}_1(\mathbf{D}_g) \\ & & \| & & \| & & \\ & & \mathbb{Z}\{a_1, b_1, \ldots, a_g, b_g\} & & Z\{\overline{a_1}, \ldots, \overline{a_g}\} \oplus Z\{\overline{a_1}', \ldots, \overline{a_g}'\} & & \end{array}$$

$$\begin{array}{ccccccc} \longrightarrow \widetilde{H}_0(\Sigma_g) & \longrightarrow & \widetilde{H}_0(\mathbf{H}_g) \oplus \widetilde{H}_0(\mathbf{H}_g) & \longrightarrow & \widetilde{H}_0(\mathbf{D}_g) & \longrightarrow & \widetilde{H}_{-1}(\Sigma_g) \longrightarrow \cdots \\ \| & & \| & & & & \| \\ 0 & & 0 & & & & 0 \end{array}$$

By exactness, we see $\widetilde{H}_0(\mathbf{D}_g) \cong 0$, we see $\widetilde{H}_3(\mathbf{D}_g) \cong \mathbb{Z}$, and we see $\widetilde{H}_k(\mathbf{D}_g)$ vanishes for $k \geq 4$. Moreover, we see that $\widetilde{H}_2(\mathbf{D}_g)$ and $\widetilde{H}_1(\mathbf{D}_g)$ are isomorphic to the kernel and cokernel, respectively, of the map

$$\begin{array}{ccc} \widetilde{H}_1(\Sigma_g) & \xrightarrow{\quad\phi\quad} & \widetilde{H}_1(\mathbf{H}_g) \oplus \widetilde{H}_1(\mathbf{H}_g) \\ \| & & \| \\ \mathbb{Z}\{a_1, b_1, \ldots, a_g, b_g\} & & \mathbb{Z}\{\overline{a_1}, \ldots, \overline{a_g}\} \oplus Z\{\overline{a_1}', \ldots, \overline{a_g}'\} \end{array}$$

$$a_i \longmapsto \overline{a_i} - \overline{a_i}'$$

$$b_i \longmapsto 0$$

The kernel of this map is $\mathbb{Z}\{b_1, \ldots, b_g\} \cong \mathbb{Z}^g$. (We could check this rigorously by expanding the expression $\phi(c_1^a a_1 + c_1^b b_1 + \ldots) = 0$ for $c_i^a, c_i^b \in \mathbb{Z}$). Its cokernel is also isomorphic to $\mathbb{Z}^g$, which we could verify by the same line of argument as in part (a). (By the Hurewicz theorem, we could also calculate $\widetilde{H}_1(\mathbf{D}_g)$ directly from part (a) by abelianizing $\pi_1(\mathbf{D}_g)$). This concludes the calculation.

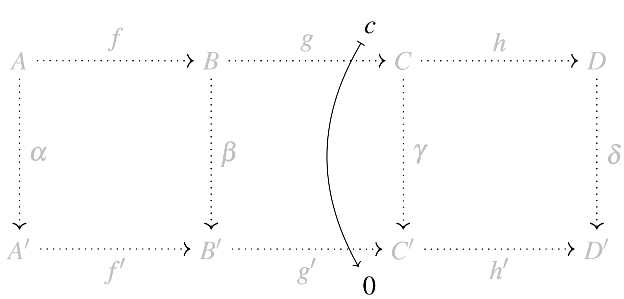
4. For the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \xrightarrow[h']{} & D' \end{array}$$

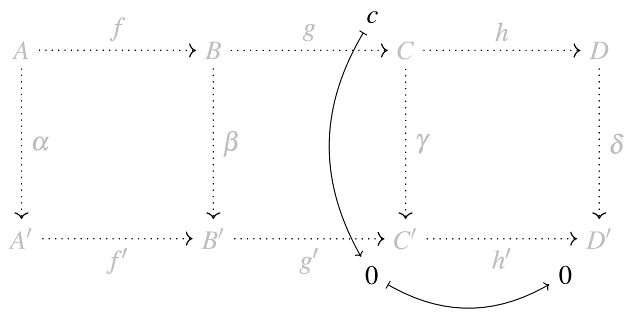
we assume

- both rows are exact,
- β and δ are injective,
- α is surjective.

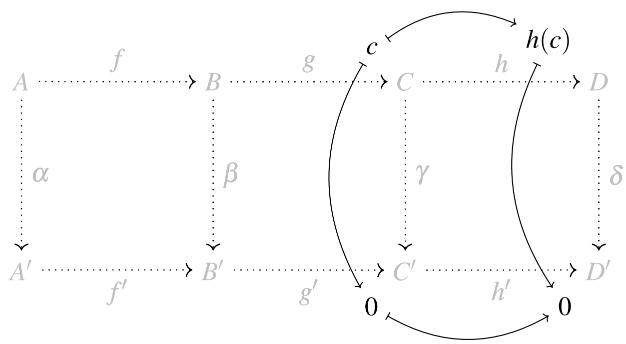
We wish to show γ is injective. To do this, choose an element c in $\ker(\gamma)$. The goal of our diagram chase is to show that $c = 0$.



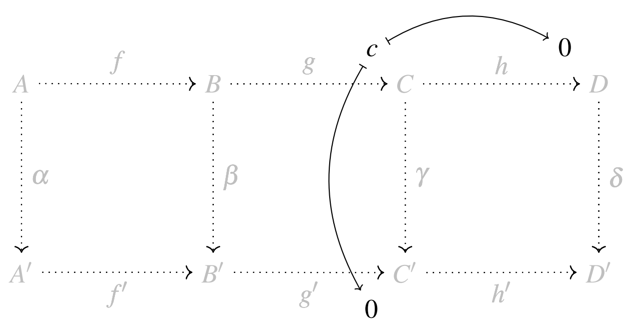
Necessarily $h'(0) = 0$.



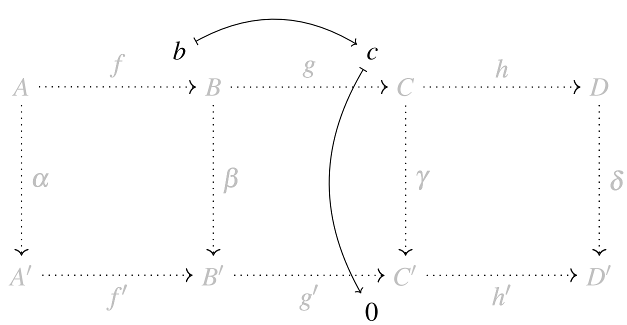
By commutativity of the diagram, $\delta(h(c)) = 0$.



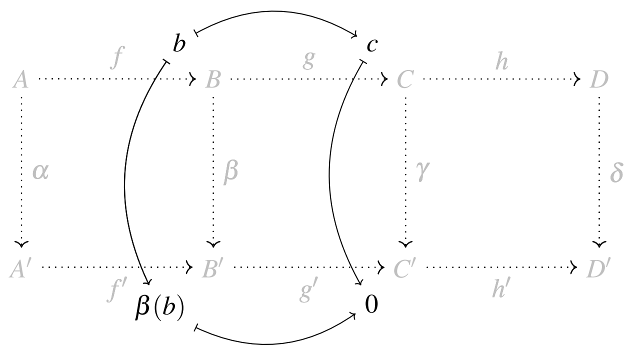
But δ is injective by assumption, which implies $h(c) = 0$.



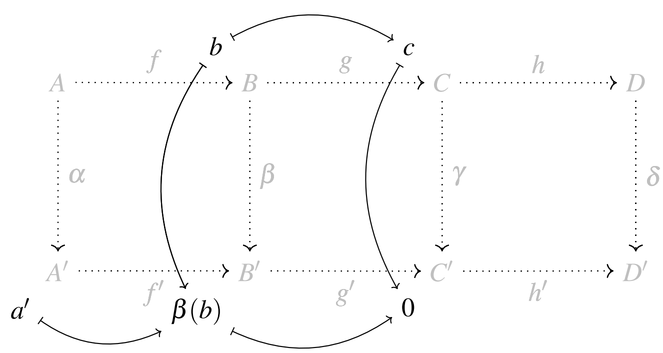
Then $c \in \ker(h)$, and $\ker(h) = \operatorname{im}(g)$ by exactness at C. There therefore exists some $b \in B$ with $g(b) = c$.



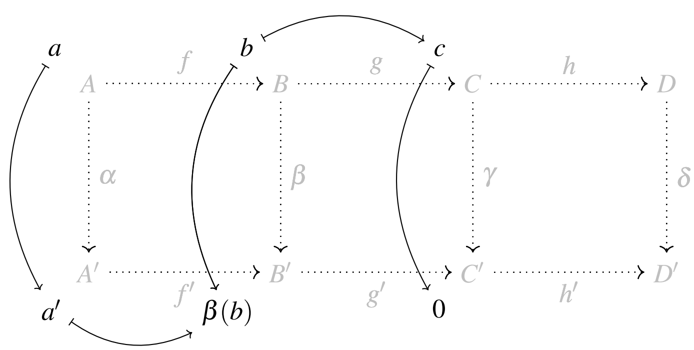
By commutativity, $g'(\beta(b)) = 0$.



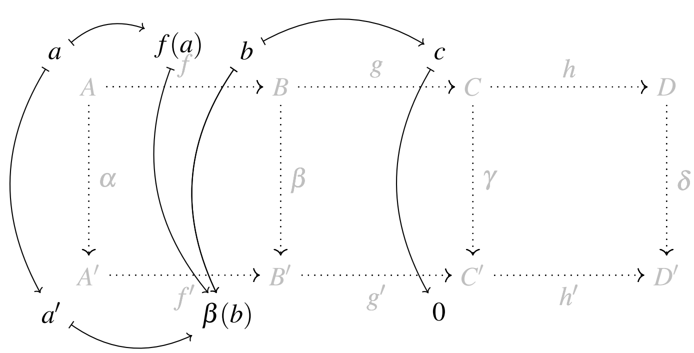
Thus $\beta(b) \in \ker(g')$. By exactness at B', $\ker(g') = \operatorname{im}(f')$. Thus there exists some $a' \in A$ with $f'(a) = \beta(b)$.



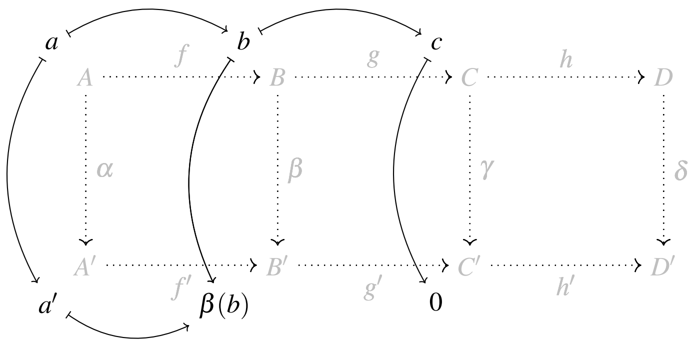
The map α surjects by assumption, so there exists some $a \in A$ with $\alpha(a) = a'$.



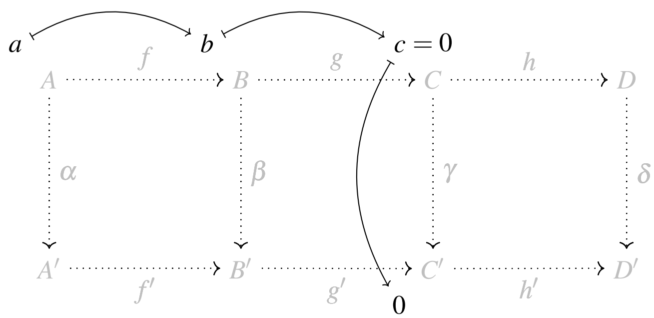
By commutativity of the diagram, $\beta(f(a)) = \beta(b)$.



But β is injective by assumption, so this implies $f(a) = b$.



But then $b \in \mathrm{im}(f)$. By exactness at B, $\mathrm{im}(f) = \ker(g)$. Hence $c = g(b) = 0$.



This concludes the proof.

5. We note that the restriction $q|_{X\times\{0\}}$ is injective and (since $(X \times \{0\})$ is a closed saturated subset of the domain) is a quotient map, hence X_0 is homeomorphic to $X \times \{0\} \cong X$. Similarly $q|_Y$ is injective, and we can check it is a homeomorphism: Consider a closed subset $C \subseteq Y$. The preimage $f^{-1}(C)$ is closed in X, so $q^{-1}(q(C)) = C \sqcup (f^{-1}(C) \times \{1\})$ is closed in $Y \sqcup (X \times I)$. By the definition of the quotient topology, this implies $q(C) \subseteq q(Y)$ is closed in M_f, which implies $q|_Y$ is a homeomorphism as claimed.

   As in the hint, we first verify that (M_f, X_0) is a good pair. This means we must check that X_0 is a nonempty closed subspace that is a deformation retract of some neighborhood U in M_f. By construction, the preimage of X_0 under the quotient map $q : (X \times [0,1]) \sqcup Y \to M_f$ is $X \times \{0\}$, which is nonempty and closed. Thus X_0 is nonempty, and it is closed by definition of the quotient topology. Now, consider $U = q(X \times \left[0, \frac{1}{2}\right))$ in M_f. Its full preimage under q is the open set $X \times \left[0, \frac{1}{2}\right)$, hence it is open in M_f. Consider the deformation retraction of $X \times \left[0, \frac{1}{2}\right)$ onto $X \times \{0\}$

$$F_t : X \times \left[0, \frac{1}{2}\right) \longrightarrow X \times \left[0, \frac{1}{2}\right)$$
$$(x, s) \longmapsto (x, s(1-t))$$

   At each time t, the composition $q \circ F_t : X \times \left[0, \frac{1}{2}\right) \to U$ is constant on fibres of q, hence this homotopy factors continuously through a map from U. Thus there is an induced homotopy $U \to U$ that deformation retracts U to X_0.

   Next, we show that there is a homotopy equivalence $M_f \to Y$. Consider the deformation retraction of $X \times [0,1]$ onto $X \times \{1\}$,

$$G_t : X \times [0,1] \longrightarrow X \times [0,1]$$
$$(x, s) \longmapsto (x, s(1-t) + t)$$

and extend G_t to a homotopy $(X \times [0,1]) \sqcup Y \to (X \times [0,1]) \sqcup Y$ by defining it to be the identity on Y at all times t. For each t, the homotopy is constant on equivalence classes of q. Hence it induces a deformation retraction G'_t of M_f onto Y.

Now, observe that the following composite is the map f:

$$\begin{array}{ccccccc} X & \xrightarrow[\cong]{} & X_0 & \overset{\iota}{\hookrightarrow} & M_f & \xrightarrow[\simeq]{G'_1} & Y \\ x & \longmapsto & q(x,0) & \longmapsto & q(x,0) & \longmapsto & q(x,1) \sim f(x) \end{array}$$

Since the first and third maps are homotopy equivalences, they induce isomorphisms on homology. Hence the map f induces an isomorphism on degree-i homology if and only if the inclusion ι of X_0 into M_f does.

Finally, we consider the long exact sequence of the pair (M_f, X_0). Because this is a good pair, for all i we have isomorphisms

$$H_i(M_f, X_0) \cong \widetilde{H}_i(M_f/X_0) = \widetilde{H}_i(C_f).$$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_{i+1}(M_f, X_0) & \longrightarrow & \widetilde{H}_i(X_0) & \xrightarrow{\iota_*} & \widetilde{H}_i(M_f) & \longrightarrow & H_i(M_f, X_0) & \longrightarrow \cdots \\ & \downarrow\cong & & & & & & \downarrow\cong & \\ & \widetilde{H}_{i+1}(C_f) & & & & & & \widetilde{H}_i(C_f) & \end{array}$$

By exactness, the map ι_* is an isomorphism whenever $\widetilde{H}_{i+1}(C_f)$ and $\widetilde{H}_i(C_f)$ vanish. The result follows.