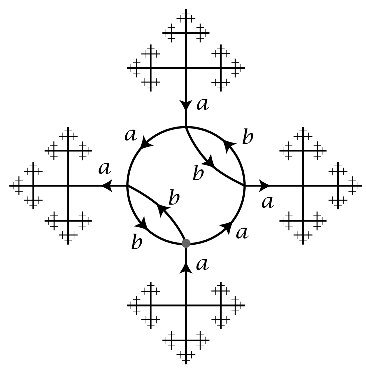
Algebraic Topology QR Exam – Jan 2024

1. (a) State the definition of a CW complex, and its topology (the weak topology).

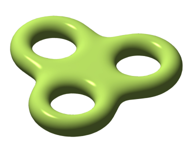
   (b) Let X be a CW complex and $A \subseteq X$ a nonempty CW subcomplex. Working directly from your definition, describe a CW complex structure on the quotient space X/A, and verify explicitly that the quotient topology on X/A agrees with the weak topology of your given CW complex structure.

2. (a) Let X be a path-connected, locally path-connected, and semi-locally simply connected space. Let $p : (\widetilde{X}, \widetilde{v}) \to (X, v)$ be the covering space associated to a subgroup $H \subseteq \pi_1(X, v)$. For an element $[\gamma] \in \pi_1(X, v)$, let $\widetilde{\gamma}$ denote the lift of γ to $\widetilde{X}$ starting at $\widetilde{v}$. Show that $[\gamma] \in \pi_1(X, v)$ is in the normalizer $N(H)$ of H if and only if the lift $\widetilde{\gamma}$ has endpoint $\widetilde{w} := \widetilde{\gamma}(1)$ in the orbit of $\widetilde{v}$ under the deck group of the cover p.

   (b) Consider the wedge $S^1 \vee S^1$ of circles a and b with wedge point v. Below is a (based) cover associated to a certain subgroup H of $\pi_1(S^1 \vee S^1, v)$. The covering map is specified by the edge labels and orientations, and a basepoint $\widetilde{v}$ is marked with a gray dot. Find a (not necessarily free) finite generating set for the normalizer $N(H)$ of H, with very brief justification.



3. Fix $g \geq 0$. The closed orientable genus-g surface Σ_g is the boundary of a compact 3-dimensional manifold $\mathbf{H}_g$ called a *genus-g handlebody*, as pictured for $g = 3$. [Image by Oleg Alexandrov]



   The *doubled handlebody* $\mathbf{D}_g$ is obtained by gluing two copies of $\mathbf{H}_g$ along their boundary via the identity map. Concretely, for $\mathbf{H} = \mathbf{H}' = \mathbf{H}_g$ and $I : \mathbf{H} \to \mathbf{H}'$ the the identity map, the space $\mathbf{D}_g$ is the quotient of the disjoint union $\mathbf{H}' \sqcup \mathbf{H}$ by the equivalence relation $I(x) \sim x$ for all $x \in \partial \mathbf{H} = \Sigma_g$.

   (a) Compute $\pi_1(\mathbf{D}_g)$.

   (b) Compute $\widetilde{H}_*(\mathbf{D}_g)$.

   For this question, you can assert descriptions of the fundamental groups and homology groups of Σ_g and $\mathbf{H}_g$ without proof. Please justify the other steps in your computation.

4. The following proposition is a step in the proof of the Five Lemma. Perform a diagram chase to prove this proposition.

   Proposition. Suppose that in the following commutative diagram of abelian groups,

   - Both rows are exact.
   - The maps β and δ are injective.
   - The map α is surjective.

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \xrightarrow[h']{} & D' \end{array}$$

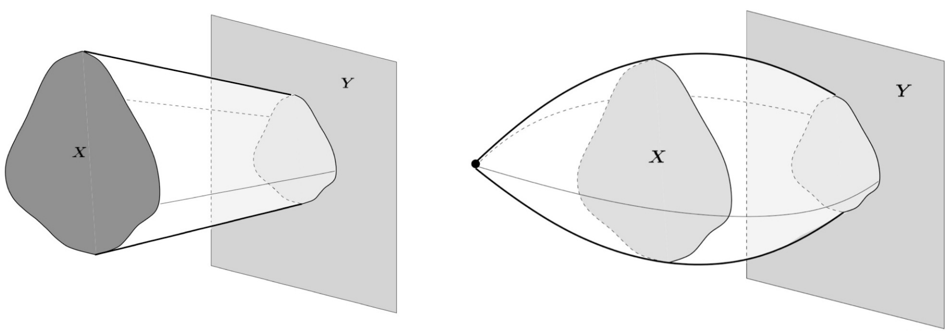
   Then the map γ is injective.

5. Let $f : X \to Y$ be a continuous map of nonempty topological spaces. Let $[0,1]$ denote the closed interval.

   The *mapping cylinder* M_f of f is obtained by gluing $X \times [0,1]$ to Y via f in the following sense: it is the quotient of the disjoint union of $X \times [0,1]$ and Y by the equivalence relation generated by $(x,1) \sim f(x)$.

   Let X_0 denote the image of $X \times \{0\}$ in M_f. The *mapping cone* C_f of f is the quotient of M_f that collapses X_0 to a point.

   The spaces M_f and C_f, respectively, are illustrated below. [Images by Fernando Muro]



   Fix $k \geq 0$ in $\mathbb{Z}$. Prove that the induced map $f_* : H_i(X) \to H_i(Y)$ is an isomorphism for $0 \leq i \leq k$ if $\widetilde{H}_i(C_f) = 0$ for $0 \leq i \leq k+1$.

   Hint: First verify that (M_f, X_0) is a good pair.