Differential Topology QR Exam – Tuesday, January 4, 2022

All manifolds are assumed to be smooth. $\Omega^k(M)$ denotes the space of smooth k-forms and $\mathfrak{X}(M)$ the space of smooth vector fields on the manifold M.

All items will be graded independently of each other.

Problem 1. Define $F : S^2 \to \mathbb{R}^4$ by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Show that F induces a smooth embedding $G : \mathbb{RP}^2 \to \mathbb{R}^4$. **Note:** After you explain how a map G is obtained, to save time, you do not have to prove in detail that it is injective.

Problem 2. Let $\pi : M \to B$ be a surjective submersion.

1. Let us call a vector field $V \in \mathfrak{X}(M)$ *vertical* if and only if $d\pi_p(V_p) = 0$ for all $p \in M$. Show that if a given $X \in \mathfrak{X}(M)$ is π-related to some field $Y \in \mathfrak{X}(B)$, then for all vertical fields V the commutator $[X, V]$ is vertical.
2. Show that if $X \in \mathfrak{X}(M)$ has the following property:

$$\forall b \in B,\ \forall p, q \in \pi^{-1}(b) \qquad d\pi_p(X_p) = d\pi_q(X_q) \tag{$\heartsuit$}$$

then X is π-related to a unique *smooth* field $Y \in \mathfrak{X}(B)$.

Problem 3. Let $P = \left\{ p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a > 0,\ b \in \mathbb{R} \right\}$.

1. Show that P is a Lie subgroup of $\mathrm{GL}(2, \mathbb{R})$, and identify its Lie algebra $T_I P$ (where I is the identity matrix).
2. Let $F : \mathrm{SO}(2) \times P \to \mathrm{SL}(2, \mathbb{R})$ be given by $F(k, p) = kp$ (matrix multiplication). Obtain a description of $dF_{(k,p)}$ that allows you to show that F is a local diffeomorphism. (F is in fact bijective and therefore a diffeomorphism, but you do not have to prove that.)

Problem 4. Let $X \in \mathfrak{X}(M)$ be a complete vector field, and $\forall t \in \mathbb{R}$ let $\theta_t : M \to M$ be the time t map of its flow. Let $\omega \in \Omega^k(M)$.

1. Recall the definition of $\mathcal{L}_X \omega$, and show that

$$\forall t \in \mathbb{R} \qquad \theta_t^* \omega = \omega \tag{$\diamondsuit$}$$

is equivalent to $\mathcal{L}_X \omega = 0$.

2. Take now $M = \mathbb{R}^n$, $\omega = dx^1 \wedge \cdots \wedge dx^n$ the standard volume form, and $X = \nabla f$ for some $f \in C^\infty(\mathbb{R}^n)$ (the usual gradient field of f). Derive a condition on f equivalent to ($\diamondsuit$).

–over please–

Problem 5. Let $F : M \to N$ be a smooth map between compact, connected, oriented manifolds without boundary, of the same dimension n.

1. Let $q \in N$ be a regular value of F. Show that $\exists V \subset N$ neighborhood of q and $\forall p \in F^{-1}(q)\ \exists\, U_p \subset M$ neighborhood of p such that (i) $F^{-1}(V) = \coprod_{p \in F^{-1}(q)} U_p$ (disjoint union) and (ii) $\forall p \in F^{-1}(q)$ the restriction $F|_{U_p}$ is a diffeomorphism from U_p onto V.

2. Define $\forall p \in F^{-1}(q)$

$$(-1)^p := \begin{cases} +1 & \text{if } dF_p \text{ is orientation preserving,} \\ -1 & \text{if } dF_p \text{ is orientation reversing,} \end{cases}$$

   and let $\delta(F) = \sum_{p \in F^{-1}(q)} (-1)^p \in \mathbb{Z}$.

   Construct $\nu \in \Omega^n(N)$ supported in the neighborhood V of part (1) and such that $\int_N \nu = 1$, and prove that

$$\int_M F^* \nu = \delta(F).$$

3. Given that $H^n(M) \cong \mathbb{R} \cong H^n(N)$, deduce from (2) that the integer $\delta(F)$ is independent of the choice of q.

SOLUTIONS

Problem 1. Let $\pi : S^2 \to \mathbb{RP}^2$ be the quotient map of the equivalence relation that identifies antipodal points of S^2. Since $\forall p \in S^2\ F(-p) = F(p)$, there is a unique map $G : \mathbb{RP}^2 \to \mathbb{R}^4$ such that $F = G \circ \pi$. By the universal property of quotient maps G is continuous. Moreover, the differentiable structure of $\mathbb{RP}^2$ is such that π is a local diffeomorphism. Therefore G is smooth because F is, as F is the restriction to S^2 of the smooth map $\widetilde{F} : \mathbb{R}^3 \to \mathbb{R}^4$ given by the same expression as F.

The main point is that, since $\mathbb{RP}^2$ is compact, to prove that G is an embedding it suffices to prove that G is an injective immersion.

–*Injective:* The first two components of F determine $x^2 - y^2 + 2\sqrt{-1}xy = (x + \sqrt{-1}y)^2$, and therefore determine $\pm(x, y)$. Given this information, the last two components of F determine $\pm(x, y, z)$, which correspond to a single point in $\mathbb{RP}^2$.

– *Immersion:* Using again that π is a local diffeomorphism, it suffices to show that $\forall p \in S^2\ dF_p : T_pS^2 \to \mathbb{R}^4$ is injective. If $p = (x, y, z)$, the Jacobian matrix of $\widetilde{F}$ at p is $J = \begin{pmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix}$. Since T_pS^2 is the orthogonal complement of the line $\mathbb{R}p$, the kernel of dF_p is the kernel of the augmented matrix

$$\tilde{J} = \begin{pmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \\ x & y & z \end{pmatrix}$$

The 3×3 principal minor of J is $x(2x^2 + 2y^2)$, which is non-zero if $x \neq 0$, and therefore J (and therefore dF_p) has zero kernel if $x \neq 0$. One can check by inspection that if $x = 0$ and $y \neq 0$, the rank of J is still three. Finally, if $x = 0 = y$, then $z = \pm 1$ and one can check that the kernel of $\tilde{J}$ is zero. Thus in every case dF_p is injective.

Problem 2. (1) Let $Y \in \mathfrak{X}(B)$ that is π-related to X, and let V be a vertical field. The definition of vecrtical field is equivalent to saying that V is π-related to the zero field on B. Therefore $[X, V]$ is related to $[Y, 0] = 0$, i.e. $[X, V]$ is vertical.

There is a more direct argument using normal form coordinates, in which $\pi(x^1, \ldots, x^n) = (x^1, \ldots, x^\ell)$. Then if $Y = \sum_{j=1}^{\ell} a^j \partial_{x^j}$ X is necessarily of the form $X = \sum_{j=1}^{\ell} a^j \partial_{x^j} + \sum_{i=\ell+1}^{n} b^i \partial_{x^i}$, and $V = \sum_{i=\ell+1}^{n} v^i \partial_{x^i}$. The commutator $[X, V]$ is vertical because the functions a^j do not depend on $(x^{\ell+1}, \ldots, x^n)$.

(2) Since π is surjective, one can define a (possibly rough) field Y on B by: $\forall b \in B,\ Y_b = d\pi_p(X_p)$ for some (any) $p \in \pi^{-1}(b)$. To prove that Y is smooth near $b \in B$, introduce normal form coordinates in a neighborhood U of some $p \in \pi^{-1}(b)$ and in a neighborhood of b, that is, coordinates in which π takes the form

$$\pi(x^1, \ldots, x^n) = (x^1, \ldots, x^\ell),$$

$n = \dim(M)$ and $\ell = \dim(B)$. Let $X = \sum_{i=1}^{n} a^i(x) \partial_{x^i}$. Then, at each point in U, $d\pi(X) = \sum_{i=1}^{\ell} a^i(x) \partial_{x^i}$, and condition ($\heartsuit$) means that the coefficient functions a^i for $1 \leq i \leq \ell$ are independent of $(x^{\ell+1}, \ldots, x^n)$. Thus in these coordinates $Y = \sum_{i=1}^{\ell} a^i(x^1, \ldots, x^\ell) \partial_{x^i}$, which shows that Y is smooth because the a^i are smooth.

Problem 3. (1) Identify $\mathrm{GL}(2, \mathbb{R})$ with an open set of $\mathbb{R}^4$ by

$$\mathrm{GL}(2, \mathbb{R}) \ni \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto (x, y, z, w) \in \mathbb{R}^4.$$

Then $P = F^{-1}(0)$ where $F(x, y, z, w) = (z, xw - 1)$. The Jacobian of F is

$$F'(x, y, z, w) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ w & 0 & 0 & x \end{pmatrix}$$

which has rank two if $xw = 1$. By the regular value theorem P is a submanifold, and $T_I P$ is the kernel of the matrix $F'(1, 0, 0, 1)$, which corresponds to the matrices

$$T_I P = \left\{ \xi = \begin{pmatrix} u & v \\ 0 & -u \end{pmatrix}, \middle| u, v \in \mathbb{R} \right\}.$$

In particular P has dimension two.

(2) Fix $(k, p) \in \mathrm{SO}(2, \mathbb{R}) \times P$ and let $\gamma : (-\epsilon, \epsilon) \to \mathrm{SO}(2, \mathbb{R}) \times P$ be a smooth curve such that $\gamma(0) = (k, p)$. If we write $\gamma = (\gamma_1, \gamma_2)$ for the components of γ, then we want to find an expression for

$$dF_{(k,p)}(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) = \frac{d}{dt} \gamma_1(t) \gamma_2(t)|_{t=0} = \dot{\gamma}_1(0) p + k \dot{\gamma}_2(0).$$

Using part (1), the fact that the Lie algebra of SO(2) consists of the 2×2 skew-symmetric matrices, and using left and right translations, we can write

$$\dot\gamma_1(0) = k \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix} \quad \text{and} \quad \dot\gamma_2(0) = \begin{pmatrix} u & v \\ 0 & -u \end{pmatrix} p$$

for some unique $u, v, w \in \mathbb{R}$. In conclusion

$$dF_{(k,p)}(\dot\gamma_1(0), \dot\gamma_2(0)) = k \begin{pmatrix} u & v+w \\ -w & -u \end{pmatrix} p.$$

Conversely, given $u, v, w \in \mathbb{R}$ arbitrary there exists a curve γ such that this holds. From this expression we see that $dF_{(k,p)}$ has zero kernel, and since the dimension of $\mathrm{SL}(2,\mathbb{R})$ is three $dF_{(k,p)}$ is an isomorphism, and by the implicit function theorem F is a local diffeomorphism.

Problem 4. (1) $\mathcal{L}_X\omega = \frac{d}{dt}\theta_t^*\omega|_{t=0}$. From this it follows that ($\diamondsuit$) implies that $\mathcal{L}_X\omega = 0$.

On the other hand, $\forall t \in \mathbb{R}$

$$\frac{d}{dt}\theta_t^*\omega = \frac{d}{ds}\theta_{s+t}^*\omega|_{s=0} = \frac{d}{ds}\theta_s^*\left(\theta_t^*\omega\right)|_{s=0} = \mathcal{L}_X\theta_t^*\omega = \theta_t^*\mathcal{L}_X\omega$$

where we have used the group law plus the fact that $(\theta_t)_*X = X$. So if $\mathcal{L}_X\omega = 0$ then $\theta_t^*\omega$ is constant with respect to t, and since it equals ω at $t = 0$ we are done.

(2) Using Cartan's M-formula and since $d\omega = 0$, we obtain for any field X that $\mathcal{L}_X\omega = d\iota_X\omega$. One computes that

$$\iota_{\nabla f}dx^1 \wedge \cdots \wedge dx^n = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(-1)^{i+1}dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

The condition on f is that the exterior derivative of this $(n-1)$ form is zero. But

$$d\left(\iota_{\nabla f}dx^1 \wedge \cdots \wedge dx^n\right) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i}(-1)^{i+1}dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

$$= \Delta f\, dx^1 \wedge \cdots \wedge dx^n, \qquad \Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial (x^i)^2}.$$

So the condition is $\Delta f = 0$.

Problem 5. (1) Since q is a regular value, $F^{-1}(q)$ is a submanifold of dimension zero. Since M is compact, $F^{-1}(q)$ is a finite set of points, $\{p_1, \ldots, p_K\}$. By the implicit function theorem, for each $i = 1, \ldots, K$ there exist neighborhoods W_i of p_i and V_i of q such that $F|_{W_i} : W_i \to V_i$ is a diffeomorphism. Shrinking these neighborhoods if necessary, WOLOG the W_i are pair-wise disjoint. Now let $V = \cap_{i=1}^{K} V_i$, and let U_i the inverse image of V under $F|_{W_i} : W_i \to V_i$.

(2) Let $(y^1, \ldots y^n)$ be a positive coordinate system of N defined in a neighborhood V_0 of q contained in V, and let $\chi \in C_0^\infty(V_0)$ be a bump function. Note that

$\int_{V_0} \chi dy^1 \wedge \cdots \wedge dy^n > 0$, so there exists $c \in \mathbb{R}$ such that $\int_{V_0} c\chi dy^1 \wedge \cdots \wedge dy^n = 1$. Now let $\nu \in \Omega^n$ be identically zero outside of V_0 and equal to $c\chi dy^1 \wedge \cdots \wedge dy^n$ on V_0.

Since for each $i = 1, \ldots, K$ F restricts to a diffeomorphism $U_i \to V$, taking into account orientations one has

$$\int_{U_i} F^*\nu = (-1)^{p_i} \int_{V_0} \nu = (-1)^{p_i}.$$

Summing over i we get $\int_M F^*\nu = \delta(F)$, where we have used that $F^*\nu$ is supported in $F^{-1}(V) = \coprod U_i$.

(3) If q' is another regular value of F, let V', U_i' and ν' be as above but associated with q'. Since the top-degree cohomology of N is one-dimensional, $\exists c \in \mathbb{R}$ and $\omega \in \Omega^{n-1}(N)$ such that $\nu' = c\nu + d\omega$. (The class $[\nu]$ cannot be zero since $\int \nu = 1$.) Integrating both sides and using Stokes' theorem, we see that $c = 1$ and

$$\int_M F^*\nu' = \int_M F^*\nu + d\left(F^*\omega\right) = \int_M F^*\nu.$$