Differential Topology QR Exam – With Solutions
Monday, January 8, 2024

All manifolds are assumed to be smooth. $\Omega^k(M)$ denotes the space of smooth k-forms on the manifold M. All items will be graded independently of each other.

Problem 1. Let $f : X \to M$ be an injective immersion, where X and M are manifolds without boundary.

(a) Give an example, with proofs, where f is not an embedding.
(b) Show that if X is compact f must be an embedding.

SOLUTION: (a) Take X an open interval, $M = \mathbb{R}^2$, and f a parametrization of of a lemniscate (figure eight-see example 4.19 in Lee). $f(X)$ is compact, but X is not so f is not a homeomorphism onto its image. (b) If $F \subset X$ is closed then it is compact. Since f is continuous $f(X)$ is compact and therefore closed in M and therefore in $f(X)$. So the pull-back of closed sets under inverse map $f(X) \to X$ is closed, and therefore the inverse map is continous.

Problem 2. Let M be an n-dimensional manifold. The orientation covering of M is defined as

$$\widetilde{M} = \{(p, \mathfrak{o}) \mid p \in M \text{ and } \mathfrak{o} \text{ is an orientation of } T_pM\}.$$

$\widetilde{M}$ has a C^∞ manifold structure such that the natural projection $\pi : \widetilde{M} \to M$ is a smooth covering map (you can freely use this without proof).

(a) Show that $\widetilde{M}$ has a natural orientation.
(b) Let ω be a compactly-supported n-form on M. Show that $\int_{\widetilde{M}} \pi^*\omega = 0$.

SOLUTION: (a) Note that the natural projection induces $T_{(p,\mathfrak{o})}\widetilde{M} \cong T_pM$. Define a point-wise orientation on $\widetilde{M}$ by orienting $T_{(p,\mathfrak{o})}\widetilde{M}$ by $\mathfrak{o}$. To prove that this is a continous orientation, pick $(p, \mathfrak{o})$ and a connected chart (U, ϕ) of M where $p \in U$ and the chart orientation agrees with $\mathfrak{o}$ at p. Note that $\pi^{-1}(U) = U_+ \coprod U_-$ where $(p, \mathfrak{o}) \in U_+$ (and $(p, -\mathfrak{o}) \in U_-$). Then $\phi \circ \pi|_{U_+} : U_+ \to \mathbb{R}^n$ is a positive chart in the point-wise orientation previously defined.

(b) Using a partition of unit WOLOG assume that β is supported in the domain U of a connected chart in M. Keep the notation $\pi^{-1}(U) = U_+ \coprod U_-$ of (a), where $(p, \mathfrak{o}) \in U_+ \Leftrightarrow (p, -\mathfrak{o}) \in U_-$. Then $\int_{\pi^{-1}(U)} \pi^*\beta = \int_{U_+} \pi^*\beta + \int_{U_-} \pi^*\beta$. Now the obvious diffeomorphism $f : U_+ \to U_-$ is orientation-reversing, and $f^*\pi^*\beta = \pi^*\beta$ since $\pi \circ f = \pi$. Therefore

$$\int_{U_+} \pi^*\beta = \int_{U_+} f^*\pi^*\beta = -\int_{U_-} \pi^*\beta,$$

which implies $\int_{\pi^{-1}(U)} \pi^*\beta = 0$.

Problem 3. Let $f : X \to M$ and $g : Y \to M$ be smooth maps between manifolds, where f is a submersion. Show that

$$W := \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

is a submanifold of $X \times Y$. HINT: Consider $F := f \times g : X \times Y \to M \times M$.

SOLUTION: The strategy is to show that $F = f \times g$ intersects the diagonal $\Delta \subset M \times M$ transversely (observe that $W = F^{-1}(\Delta)$). Let $(x, y) \in X \times Y$ be such that $F(x, y) \in \Delta$, i.e. $f(x) = m = g(y)$. Let $a, b \in T_m M$; (a, b) is a generic vector in $T_{(m,m)} M \times M$. Note that $(b, b) \in T_{(m,m)}\Delta$. Since f is a submersion, $\exists u \in T_x M$ such that $df_x(u) = a - b$, and therefore

$$dF_{(x,y)}(u, 0) + (b, b) = (df_x(u) + b, b) = (a, b).$$

This shows $\operatorname{im}\left(dF_{(x,y)}\right) + T_{(m,m)}\Delta = T_{(m,m)} M \times M$.

Problem 4. Consider $\phi_t : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\phi_t(x, y, z) = (e^t x,\ \cos(t)y - \sin(t)z,\ \sin(t)y + \cos(t)z), \quad t \in \mathbb{R}.$$

(a) Show that ϕ is a flow, and find the vector field V that generates it.
(b) Use the definition of the Lie derivative of a form to compute $\mathcal{L}_V(dx \wedge dy)$.
(c) Quote Cartan's formula, and use it to verify your answer to (b).

SOLUTION: (a) ϕ_t induces the standard rotation by t radians in the $y - z$ plane, so it is easy to check that $\phi_{t+s} = \phi_t \circ \phi_s$. Moreover

$$V_{(x,y,z)} = \frac{d}{dt}\phi_t(x, y, z)|_{t=0} = \langle x, -z, y\rangle.$$

(b) $\mathcal{L}_V(dx \wedge dy) = d/dt\, \phi_t^*\,(dx \wedge dy)\,|_{t=0}$. Computing:

$$\phi_t^*\,(dx \wedge dy) = e^t dx \wedge (\cos(t)dy - \sin(t)dz)$$

and so $\mathcal{L}_V(dx \wedge dy) = dx \wedge dy - dx \wedge dz$. (c) Cartan's formula: $\mathcal{L}_V\alpha = \iota_V d\alpha + d\iota_V\alpha$. Here $\alpha = dx \wedge dy$ is closed, so the formula reduces to

$$\mathcal{L}_V\alpha = d\iota_V\alpha = d\,(xdy + zdx) = dx \wedge dy + dz \wedge dx,$$

which agrees with what was found in (b).

Problem 5. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ that we identify with $T_I G$. Let Ω_G^k denote the space of all left-invariant forms on G of degree k.

(a) Establish a natural isomorphism $\Omega_G^k \cong \bigwedge^k \mathfrak{g}^*$.
(b) Show that the exterior differential maps Ω_G^k into Ω_G^{k+1}.
(c) Combining (a) and (b) with $k = 0, 1$, we obtain maps

$$d_0 : \wedge^0\mathfrak{g}^* \cong \mathbb{R} \to \mathfrak{g}^* \quad \text{and} \quad d_1 : \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*.$$

Show that $d_0 = 0$ and compute d_1. HINT: For d_1, use a formula for $d\alpha(V, W)$ where α is any one-form and V, W are vector fields.

SOLUTION: (a) In one direction $\Omega_G^k \to \bigwedge^k \mathfrak{g}^*$ is just evaluation at the identity I. The inverse is obtained by left-invariance,

$$\forall \alpha \in \Omega_G^k,\ g \in G \qquad \alpha_g = d\left(L_{g^{-1}}\right)_g^* \alpha_I$$

where $L_g : G \to G$ is left translation by g. (b) This follows because d commutes with the operation of pull-back by any smooth map, so $\forall \alpha \in \Omega_G^k\ L_g^* d\alpha = dL_g^* \alpha = d\alpha$ which shows that $d\alpha$ is left-invariant. (c) $k = 0$: An invariant function is constant, so its differential is zero. $k = 1$: Use

$$d\alpha(V, W) = V\alpha(W) - W\alpha(V) - \alpha([V, W]).$$

We want to compute $d\alpha_I$ for a given $\alpha \in \Omega_G^k$. The idea is to take V, W to be left-invariant fields, in which case the first two terms vanish (because $\alpha(V), \alpha(W)$ are constant functions), and the commutator $[V, W]$ corresponds to the Lie algebra bracket of $\mathfrak{g}$. The conclusion is that

$$\forall a \in \mathfrak{g}^*,\ v, w \in \mathfrak{g} \qquad d_1(a)(v, w) = -a([v, w]).$$