General and Differential Topology QR Exam – May 4, 2023

All manifolds, vector fields, and differential forms are assumed to be smooth (C^∞).

Problem 1. Let $M = \{(w, x, y, z) \in \mathbb{R}^4 \mid w^2 + x^2 = y^2 + z^2 = 1\}$.

(a) Show that M is a submanifold of $\mathbb{R}^4$.

(b) Define a diffeomorphism $\pi : M \to M$ by $\pi(w, x, y, z) = (-y, -z, w, x)$. Let G be the group generated by this diffeomorphism. Show that the orbit space M/G is a manifold.

(c) Is M/G orientable?

Problem 2. Let $n \geq 2$. Let X be the set of real $n \times n$ matrices A satisfying $A + A^t = 0$, where A^t is the transpose of A.

(a) Is X a Lie algebra?

(b) Let $\mathrm{GL}(n)$ be the group of invertible $n \times n$ matrices. Is $X \cap \mathrm{GL}(n)$ a Lie group?

(c) Let $M(n)$ be the set of all real $n \times n$ matrices. Define a function $f : X \to M(n)$ by $f(A) = e^A - e^{-A}$. Describe the image under f of a small open neighborhood of the zero matrix.

Problem 3. Let α be a nonvanishing 1-form on a manifold M, so for any point $q \in M$, $\ker \alpha_q$ is a codimension 1 subspace of the tangent space T_qM. Assume that f is a nonvanishing smooth function on M such that

$$d(\alpha) = \frac{df}{f} \wedge \alpha.$$

Prove that for any $p \in M$, there is a regular submanifold S of M such that $p \in S$ and $T_qS = \ker \alpha_q$ for all $q \in S$.

Problem 4. Let X be a complete vector field on a manifold M, and let $\alpha \in \Omega^k(M)$ be a k-form.

(a) Show that the following two conditions on the pair (X, α) are equivalent:
- the Lie derivative $\mathcal{L}_X\alpha$ is identically zero;
- for all $t \in \mathbb{R}$, $\theta_t^*\alpha = \alpha$, where $\theta_t : M \to M$ is the time t map of the flow along X.

(b) Suppose that $M = \mathbb{R}^3$, $\alpha = dx \wedge dy \wedge dz$, and

$$X = ax(y - z)\frac{\partial}{\partial x} + by(z - x)\frac{\partial}{\partial y} + cz(x - y)\frac{\partial}{\partial z}$$

for some $a, b, c \in \mathbb{R}$. For which a, b, c is it the case that (X, α) satisfies the conditions of the previous part?

Problem 5. Let M be a compact manifold of positive dimension. Prove that there exists a vector field X on M such that for every nonempty open set U of M, X is not identically zero on U.