THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Algebraic Topology

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1. The *unreduced suspension* of a topological space X is the quotient of the space $X \times [0,1]$ by the smallest equivalence relation $\sim$ which has $(x,0) \sim (y,0)$ and $(x,1) \sim (y,1)$ for all $x, y \in X$, with the quotient topology. For which $n \in \mathbb{N}$ is the unreduced suspension Z_n of the real projective space $\mathbb{R}P^n$ a topological manifold without boundary (i.e. has the property that every point $u \in Z_n$ has a neighborhood homeomorphic to $\mathbb{R}^k$ for some k)?

2. Give an example of a subgroup $H \subset F(a,b)$ of the free group on two generators a, b which has finite index but is not normal. Recalling that H is necessarily also free, give a set of free generators of H.

3. Let X be the quotient of the space $S^1 \times S^1$ obtained by identifying two different chosen points. Is the universal covering space of X contractible? Explain.

4. Let X be a CW-complex with exactly four cells, of dimensions $0, n, n+1, n+2$, where $n > 0$. Assume further that the attaching map of the $(n+1)$-cell is not homotopic to a constant map. Denoting by X_n the n-skeleton of X, prove that the quotient space X/X_n is homotopy equivalent to $S^{n+1} \vee S^{n+2}$ where $Y \vee Z$ denotes the one-point union, i.e. the quotient of the disjoint union by identifying one point of Y with one point of Z. [Hint: Use the definition of cellular homology.]

5. Let $X = (S^1 \times S^1)/(\{1, -1\} \times S^1)$ where $S^1 \subset \mathbb{C}$ is the unit circle. Calculate the homology groups of X.